Analytical long-profile models of coupled glacier-fluvial systems

Eric Deal - Earth Surface Dynamics, Geological Institute, ETH Zürich
Günther Prasicek - Interdisciplinary Center of Mountain Research, Univ. of Lausanne

EGU 2020

Introduction

Simple, analytical long-profile models of fluvial incision are fundamental in tectonic geomorphology, however, analogous models are missing for glaciers, despite the fact that glaciers are an important and widespread agent of physical erosion with implications for the interactions of climate, topography and tectonics. Using recently validated models of erosion due to ice sliding [e.g. Cook et al. 2020], we introduce several simplifications to the relevant equations that lead to an analytical solution for coupled glacial-fluvial systems under steady state climate and block uplift.

Our main result is that we can rewrite the empirically validated ice-sliding erosion law in terms of ice flux and ice surface slope. In this form, strong similarities with the classic stream power incision model (SPIM) emerge, allowing us, for example, to extract the response of average glacial slope to uplift rate at steady state. There are also differences to the SPIM. The fact that glacial ice accumulation depends on elevation leads to a stronger erosional response to increased slope that fluvial models exhibit.

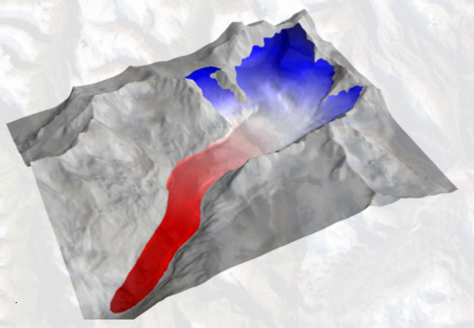

Figure 1: Visualization of 2D glacial network

Analytical solution

Our approach is similar to the standard fluvial long-profile models, where we use a Hack-like law to reduce 2D glacial networks to 1D. The solution for the bedrock surface at steady state is:

$$z(x) = \begin{cases} h(x) - H(x) & : x < x_t \\ k_s ln(L/x) & : x \geq x_t \end{cases}$$

Where the ice surface is

$$h(x) = ELA + k_g x_t^r \left[\kappa - I_{(x/x_t)}(a,b)\right]^*$$

And the ice thickness is

$$H(x) = k_H (x^\lambda x_t^r (1 - x/x_t))^\gamma$$

where the glacier length is

$$x_t = \left(\tau W\left[(L^r/\tau)\, e^{-\frac{rELA}{k_s}}\right]\right)^r$$

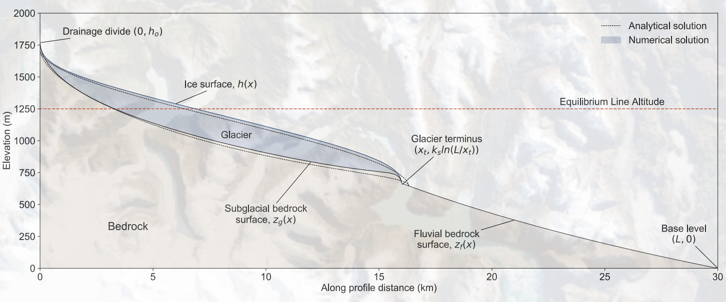

Figure 2: Schematic of glacial-fluvial profile model

* $I_{(x/x_t)}(a,b)$ is the regularized incomplete beta function, $W(\cdot)$ is the lambert-w function, and the other variables are described in table 1 at the end of this presentation.

Approximations

The first approximation is a simplification to the ice-flow law with a single power-law that captures mixed sliding and deformation dominated ice flux. This works because alpine glacial ice tends to be within 100 - 1000 m thick, so we can ignore end-case ice flux due to pure sliding or pure deformation. Figure 3 shows a comparison of full shallow ice approx. ice-flow law (black line), with pure deform. approx. (red line), pure sliding approx. (yellow line), and our mixed approx. (cyan, and rescaled - dashed blue).

$$H \propto \left(\frac{q}{u_s}\right)^\gamma$$

$$\gamma = \begin{cases} 1/3 & : \text{deform.} \\ 2/3 & : \text{mixed} \\ 3/3 & : \text{sliding} \end{cases}$$

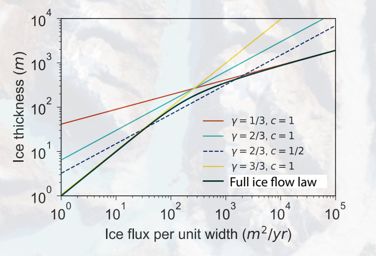

Figure 3: Comparison of different ice-flow laws

Stream power

$$\frac{\partial z}{\partial t} = U - Kq^m S^n$$

New ice flux erosion model

$$\frac{\partial z}{\partial t} = U - Kq^\mu S^\nu$$

Old ice-sliding velocity erosion model:

$$\frac{\partial z}{\partial t} = U - Ku_s^\ell$$

The ice-flow law approximation leads to an erosion law in terms of glacial ice flux (q) and surface slope (S), which is analogous to the classic SPIM (where q is water flux and S is water surface slope). This is compared to the older ice sliding erosion model, which is a function of the ice sliding velocity, u_s, at the bedrock-ice interface. We predict $\nu = 9/7\ell$, and $\mu = 4/7\ell$.

The second approximation is to solve for the ice flux at a point assuming a Hack-like upstream area distribution, a standard glaciology rainfall-to-ice lapse rate and an assumption of self-similarity in the glacial network at steady state. This leads to a simple function of ice flux where the ice flux is low near the upper reaches of the catchment, increases downstream to a maxiumum due to ice flow accumulation, and then decreases towards the lower reaches of the glacial network due to melting.

$$q(x) = k_q x^\lambda (x_t - x) S_{\text{avg}}$$

Ice flux is a function of channel position, x, glacial length, x_t, and average glacial slope S_{avg}. This is a major difference from the SPIM, where water flux doesn't depend on slope

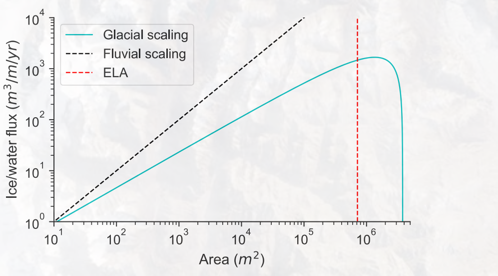

Figure 4: Ice flux as a function of channel position

We find that above the ELA, where the ice is still accumulating, the ice surface slope and contributing area exhibit a scaling that is reminiscent of the fluvial slope-area scaling. In the glacial case, we find

$$S \propto A^{-\theta}; \quad \theta = \frac{\mu}{\nu + \mu} \approx 0.3$$

However, the 'concavity' index has a different form than the flvuial version, because the ice flux depends not just on area, as in the fluvial case where q = PA, but also on elevation, leading to an extra dependence on slope ($q_{ice} \propto SA$). Therefore at steady state we find

$$U/K \propto (AS)^\mu S^\nu = A^\mu S^{\nu+\mu}$$

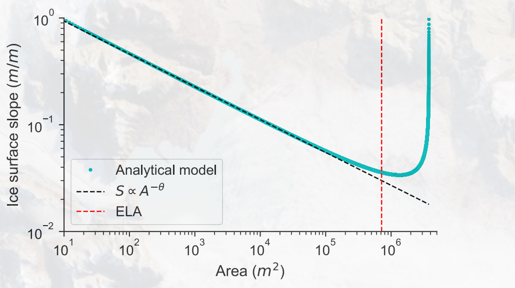

Figure 5: Ice surface slope-area scaling above the ELA

Ice flux erosion model with Hack-like ice flux approximation

$$\frac{\partial z}{\partial t} = U - K'x^{\lambda\mu}(x_t - x)^\mu S_{\text{avg}}^\mu S^\nu$$

SPIM with Hack's law for water flux

$$\frac{\partial z}{\partial t} = U - K'x^{hm} S^n$$

Solving for water flux as a function of channel position comes straight from Hack's law in the fluvial case. For glaciers, the calculation is complicated due to both ice accumulation and melting - therefore ice flux is a function of elevation and the equilibrium line altitude (ELA). We found an approximation, described above, that solves ice flux as a function of channel position. The resulting erosion law is more complex, depending on total glacial length and with a double dependency on glacial slope (because it also influences ice accumulation rate).

Scaling of glacier slope with tectonic uplift rate

An important question is how the relief of an orogen responds to changes in erosion efficiency and tectonic uplift rates. It is known that when fluvial erosion dominates, the slope exponent, n, influences the sensitivity of steady-state orogen relief to uplift rates ($h_o \propto U^{\frac{1}{n}}$). In the mixed glacial-fluvial regime both n and the glacial sliding exponent, ℓ, influence the sensitivity, but overall the scaling follows that of the purely glacial regime. The scaling can be found by noting that over much of the glacier, the local slope will be similar to the average slope, leading to the relation shown below. The value of ℓ is debated, but observations place it in the range of 1 to 2 [e.g. Herman et al., 2015, Cook et al., 2020]. Therefore, relief in glacially eroded orogens should scale similarly to fluvial orogens with n in the range of 2 to 4.

$$S_{\text{avg}} \propto \left(\frac{U}{K'}\right)^{\frac{1}{\mu+\nu}} \propto U^{\frac{1}{2\ell}}$$

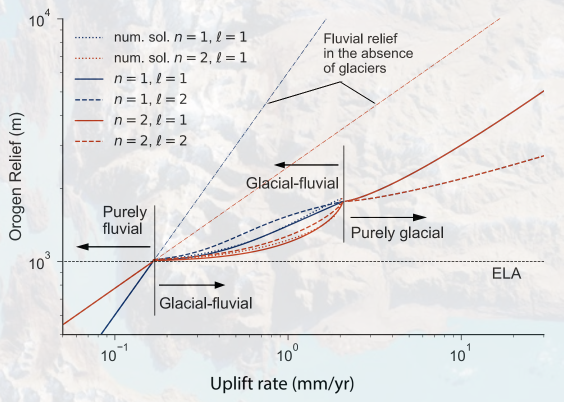

Figure 6: In general, glacial relief is expected to increase more slowly than fluvial relief in response to increasing uplift rates. This has implications for the interaction of climate and tectonics.

Table 1. Standard values and references for model parameters*

Variable	Range†	Units	Description	Reference‡
K_f	10^{-7} to 10^{-5} ($n = 1$)	$m^{1-n}\ yr^{-1}$	Erodability coefficient in SPIM	(e.g., Lague, 2014)
n	1/2 to 4	-	Slope exponent in SPIM	(e.g., Lague, 2014)
m	$\sim$ n/2	-	Flux exponent in SPIM	(e.g., Lague, 2014)
c_f	0.15	m^{2-h}	Fluvial Hack's coefficient	(Hack, 1957)
h	2	-	Fluvial Hack's exponent	(Hack, 1957)
k_s	-	$m^{h/2}$	Fluvial steepness index	(e.g., Lague, 2014)
K_g	10^{-4} ($\ell = 1$)	$m^{1-\ell}\ yr^{\ell-1}$	Glacial sliding erosion coefficient	(F. Herman et al., 2015)
f_s	4	$m^{1-(\rho-1)}\ yr^{-1}$	Ice sliding parameter	(e.g., Lague, 2014)
f_d	5x10^{-5}	$m^{1-2(\rho-1)}\ yr^{-1}$	Ice deformation parameter	(Cuffey & Paterson, 2010)
k_t	2 to 5	$m^{-\ell}\ yr^{\mu-\ell}$	Ice erosion parameter	$\left(c f_s^{\frac{2-\gamma}{\rho-1}} f_d^{\frac{\gamma-1}{\rho-1}}\right)^{\frac{\mu}{\gamma}}$
k_q	10^{-3}	$m^{1-\lambda}yr^{-1}$	Ice flux parameter	$\kappa\beta c_g/\omega$
k_H	-		Ice thickness parameter	$\frac{\sigma^{\gamma/\ell}}{k_g^{\mu/\ell}}\left(\frac{k_t k_q^\mu}{f_s^\ell}\right)^{\frac{1}{\ell(\rho-1)}}$
K	10^{-5}	$m^{1-\mu(1-\lambda)}\ yr^{-1}$	Erodability coefficient in SIIM	$K_g k_t k_q^\mu$
k_g	-	$m^{\frac{\mu(1-\lambda)}{\nu+\mu}}$	Glacial steepness index	$(\sigma\frac{U}{K})^{\frac{1}{\mu+\nu}}$
c_g	0.15	$m^{2-\frac{1}{\eta}}$	Glacial Hack's coefficient	(Prasicek et al., 2019)
η	1/2 to 2/3	-	Glacial Hack's exponent	(Prasicek et al., 2019)
ϕ	1/2	-	Network branching parameter	(Prasicek et al., 2019)
D	3/4 to 1	-	Divergence operator	$(1-\phi)/\eta$
κ	4/7 to 1	-	Ratio of relief above ELA to glacial relief	$D/(1+D)$
β	$\sim 10^{-3}$	yr^{-1}	Mass balance gradient over z	(Hagen et al., 2003)
ω	2	$m^{1-\epsilon}$	Glacial channel width coefficient	(Prasicek et al., 2019)
ϵ	1/2	-	Glacial channel width exponent	(Prasicek et al., 2019)
λ	1 to 3/2	-	Ice constriction exponent	$1/\eta - \epsilon$
ℓ	1 to 2	-	Glacial sliding erosion exponent	(F. Herman et al., 2015)
ρ	3	-	Glen's flow law exponent	(Cuffey & Paterson, 2010)
γ	$1/\rho$ to 1	-	Ice flux approx. exponent	-
ν	$\frac{9}{7}\ell$	-	Slope exponent in SIIM	$\frac{\rho\ell}{1+\gamma(\rho-1)}$
μ	$\frac{4}{7}\ell$	-	Flux exponent in SIIM	$\frac{\gamma(\rho-1)\ell}{1+\gamma(\rho-1)}$
r	1/3 to 3/13	-	Glacial terminus exponent	$\frac{\nu-\lambda\mu}{\nu+\mu}$
θ	4/13	-	-	$\mu/(\nu+\mu)$
a	1/3 to 5/9	-	Ice accumulation exponent	$1-\frac{\lambda\mu}{\nu}$
b	5/9	-	Ice loss exponent	$1-\frac{\mu}{\nu}$
σ	-	-	Glacial steepness index constant	$B(a,b)^\nu;\ x_t \leq L$**
σ	-	-	Glacial steepness index constant	$B_{(L/x_t)}(a,b)^\nu;\ x_t > L$**

*If this text is the reference, equation is shown instead.
†If no established empirical range exists, the most commonly used value is given
‡If there are multiple references, only the most relevant is given
**$B(\cdot,\cdot)$ is the beta function and $B_{(\cdot)}(\cdot,\cdot)$ is the unregularized incomplete beta function

References

Cook, S. J., Swift, D. A., Kirkbride, M. P., Knight, P. G., & Waller, R. I. (2020). The empirical basis for modelling glacial erosion rates. *Nature Communications*, *11*(1), 1–7.

Cuffey, K. M., & Paterson, W. S. B. (2010). *The physics of glaciers.* Elsevier.

Hack, J. T. (1957). *Studies of longitudinal stream profiles in virginia and maryland* (Vol. 294). US Government Printing Office.

Hagen, J. O., Melvold, K., Pinglot, F., & Dowdeswell, J. A. (2003). On the net mass balance of the glaciers and ice caps in svalbard, norwegian arctic. *Arctic, Antarctic, and Alpine Research*, *35*(2), 264–270.

Herman, F., Beyssac, O., Brughelli, M., Lane, S. N., Leprince, S., Adatte, T., ... Cox, S. C. (2015). Erosion by an alpine glacier. *Science*, *350*(6257), 193–195.

Prasicek, G., Hergarten, S., Deal, E., Herman, F., & Robl, J. (2019). A glacial buzzsaw effect generated by efficient erosion of temperate glaciers in a steady state model. *Earth and Planetary Science Letters*.