Physical measures and tipping points in a changing climate

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Parameter-dependent “bistable” systems

As subsystems of the climate system respond to changes in forcing, tipping points may appear. Nonlinear feedbacks can cause the “unperturbed" subsystem to have multiple stable states, and external inputs can then cause the system to change between these states.

AIM: To develop an understanding of the predictability of tipping between nontrivial attractors. In particular:

- We present a notion of “physical measure” for nonautonomous systems.
- We use this to define a “tipping probability” between two attractors.

Parameter-dependent "bistable" systems

Consider a model "unperturbed" ODE for $\mathbf{x}$ on some manifold M:

$$\dot{\mathbf{x}} = f(\mathbf{x};\lambda)$$

where λ is a parameter from some interval $[\lambda_-,\lambda_+]$, with

- "desirable state", represented by an attactor $A_\lambda^{\mathrm{des}} \subset M$
- $U_\lambda^{\mathrm{des}} :=$ interior of basin of attraction of A_λ^{des}
- "undesirable state", denoted by an attactor $A_\lambda^{\mathrm{undes}} \subset M$
- $U_\lambda^{\mathrm{undes}} :=$ interior of basin of attraction of $A_\lambda^{\mathrm{undes}}$
- the "boundary" $M \setminus (U_\lambda^{\mathrm{des}} \cup U_\lambda^{\mathrm{undes}})$ a null set.

Parameter shift and physical measures

Now consider the “forced” nonautonomous ODE

$$\dot{\mathbf{x}} = f(\mathbf{x}; \lambda(t))$$

with smooth $\lambda\colon \mathbb{R} \to [\lambda_-, \lambda_+]$ where $\lambda(t) \to \lambda_\pm$ as $t \to \pm\infty$.

Definition. A physical measure is a time-dependent probability measure μ_t ($t \in \mathbb{R}$) on M such that for every continuous $g\colon M \to \mathbb{R}$ and every probability measure ρ on $U^{\mathrm{des}}_{\lambda_-}$ with smooth density, for all $t \in \mathbb{R}$,

$$\int_M g(\mathbf{x}(t))\, \rho(d\mathbf{x}(s)) \to \int_M g\, d\mu_t \quad \text{as } s \to -\infty.$$

Remark. Cf. (a) time-dependent invariant measures (Checkroun et al. 2011), (b) parameter-shift driven systems (Drótos et al. 2015).

Tipping probability

Definition. We will say that *the probability of tipping is well-defined* if

- the physical measure $(\mu_t)_{t\in\mathbb{R}}$ exists;
- $\mu_t(U_{\lambda_+}^{\mathrm{des}} \cup U_{\lambda_+}^{\mathrm{undes}}) = 1$ for all (sufficiently large) t;
- $\lim_{t\to\infty} \mu_t(U_{\lambda_+}^{\mathrm{undes}})$ exists.

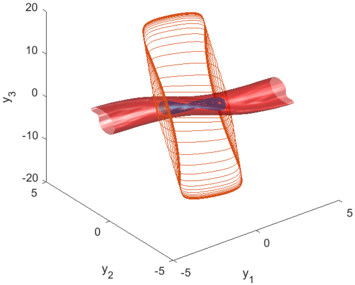
In this case, the probability of tipping is

$$p = \lim_{t\to\infty} \mu_t(U_{\lambda_+}^{\mathrm{undes}}).$$

Remark. If $\lambda(t)$ converges sufficiently fast to λ_- as $t \to -\infty$, then the question of existence of the physical measure $(\mu_t)_{t\in\mathbb{R}}$ should be reducible to whether the autonomous system $\dot{\mathbf{x}} = f(\mathbf{x}; \lambda_-)$ has a physical measure supported on $A_{\lambda_-}^{\mathrm{des}}$ (details in preparation).

Example: shifted double scroll

A prototype example is the double scroll system: a 3D system of coupled ODEs (Chua et al. 1986) with bistability between a chaotic "double scroll" attractor and a large amplitude limit cycle:



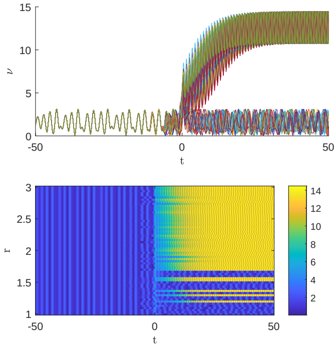
$$\begin{aligned}
\dot{x}_1 &= F_1(x_1, x_2, x_3) := a(x_2 - \phi(x_1)) \\
\dot{x}_2 &= F_2(x_1, x_2, x_3) := x_1 - x_2 + x_3 \\
\dot{x}_3 &= F_3(x_1, x_2, x_3) := -bx_2
\end{aligned}$$

for $(x_1, x_2, x_3) \in \mathbb{R}^3$, where

$$\phi(x) = x^3/16 - x/6.$$

Red tube shows basin boundary between attractors.

Example: shifted double scroll



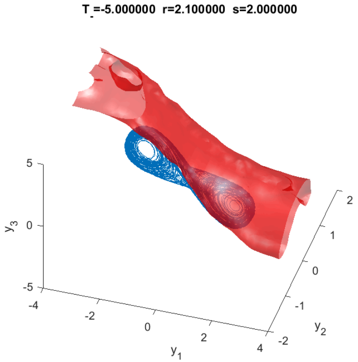
Shifted double scroll – time-dependent translation of origin to $(1,1,0)\Lambda(rt)$ where

$$\Lambda(s) = 1 + \tanh(s).$$

Tipping from chaos to periodic seen as the sharp growth of $\nu(t) = \sqrt{9x_2^2 + x_3^2}$ around $t = 0$.

Top: ensemble of runs varying r.
Bottom: same shown colored by ν.

Example: shifted double scroll

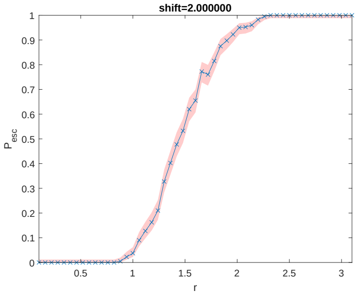


Blue – approximation of physical measure at time $T = -5$ for shifted double scroll.

Red – basin boundary at $T = -5$ for chaos in future system.

Tipping probability is mass of physical measure that is exterior to red tube (i.e. in basin of future periodic attractor).

Example: shifted double scroll



Probability of tipping p for shifted double scroll system on varying rate r.

Note region of partial tipping

$$0 < p < 1$$

for $0.85 < r < 2.2$.

Further questions: random systems

Now consider a stationary white-noise-driven system on M, e.g.

$$d\mathbf{x}_t = f(\mathbf{x}_t)\,dt + \sum_{i=1}^{n} \sigma_i(\mathbf{x}_t) \circ dW_t^i$$

Theorem (Arnold 1998, Sec. 1.7)

Given any stationary probability measure ρ on M, for almost every sample path ω of the noise there is a time-dependent probability measure μ_t^ω ($t \in \mathbb{R}$) on M such that for every continuous $g\colon M \to \mathbb{R}$, for all $t \in \mathbb{R}$,

$$\int_M g(\mathbf{x}_t)\,\rho(d\mathbf{x}_s) \to \int_M g\,d\mu_t^\omega \quad \textit{as } s \to -\infty.$$

Similar results also hold for Markovian-coloured-noise-driven systems (Crauel 1991, Newman 2020).

Further questions: random systems

- Under what conditions will we have that

$$\int_M g(\mathbf{x}_t)\,\tilde{\rho}(d\mathbf{x}_s) \to \int_M g\,d\mu_t \ \text{ as } s \to -\infty$$

  for every probability measure $\tilde{\rho}$ with smooth density? (I.e. "when can we regard $(\mu_t^\omega)_{t\in\mathbb{R}}$ as a physical measure?")
- Can we use this to give useful predictions in parameter-drifting random dynamical systems?
- Can we apply these results to better quantify probabilities of tipping in climate models? (cf. Ashwin & von der Heydt 2019)

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Thank you.

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