Log-log linearity of the asymptotic distribution – a valid indicator of multi-fractality?

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Introduction

It has been claimed (Cheng et al. 1994, Cheng 1999, etc.), that the presence of anomalies of spatial fields (typical: geochemical) resulting from its multi-fractal nature, is indicated by

$$area(Z > z) \sim z^{-\beta}$$

for high threshold z, i.e. a power-law behaviour.

Call A(z) := area(Z>z) the excursion set (Adler 1981)

statement equivalent to $\lim\limits_{z \to \infty} \frac{\ln A_z}{\ln z} = -\beta$

<u>Question:</u> Does multifractality imply the power-law behaviour?

Multifractal

- reminder: locally, $|Y(x+h)-Y(x)| \sim h^{\beta(x)}$

  ß – local singularity or Hölder exponent; (Hausdorff-)fractal dimension of set of x with same ß(x): fractal spectrum. (In environmental applications: box-dim.)

- Notation:
  $V_i(s)$ = i-the cell of size s, covering domain D; their areas A(s)
  $m_i(s)$= mass in cell i of size s = A(s) EZ (Z $\in V_i(s)$)
  $p_i(s)$ = probability measure = $m_i(s)/\Sigma m_i(s)$ = E_iZ / ΣE_iZ.
  S(s):= $\Sigma m_i(s)$
  $\mu_i(q,s) := p_i(s)^q/\Sigma p_i(s)^q = m_i(s)^q / S_q(s)$, $S_q(s) := \Sigma m_i(s)^q$.

- According Chhabra & Jensen (1989),

$$\alpha(q) = \lim_{s\to 0} \frac{\sum_i \mu_i(q,s) \log\ p_i(s)}{\log s} \qquad f(\alpha(q)) = \lim_{s\to 0} \frac{\sum_i \mu_i(q,s) \log \mu_i(q,s)}{\log s}$$

- $a(q,s) := \Sigma\mu_i(s) \ln p_i(s) = \ldots = (1/S_q(s))\, \Sigma m_i(s)^q - \ln S(s)$
  $f(q,s) := \Sigma\mu_i(s) \ln \mu_i(s) = \ldots = (q/S_q(s))\, \Sigma m_i(s)^q - \ln S_q(s)$

- $\alpha(q)$ and $f(\alpha(q))$ estimated from slopes of graphs a(q,s) and f(q,s) vs. ln s.

Generation of multifractals by multiplicative cascade

Ansatz: multiplicative binomial cascade

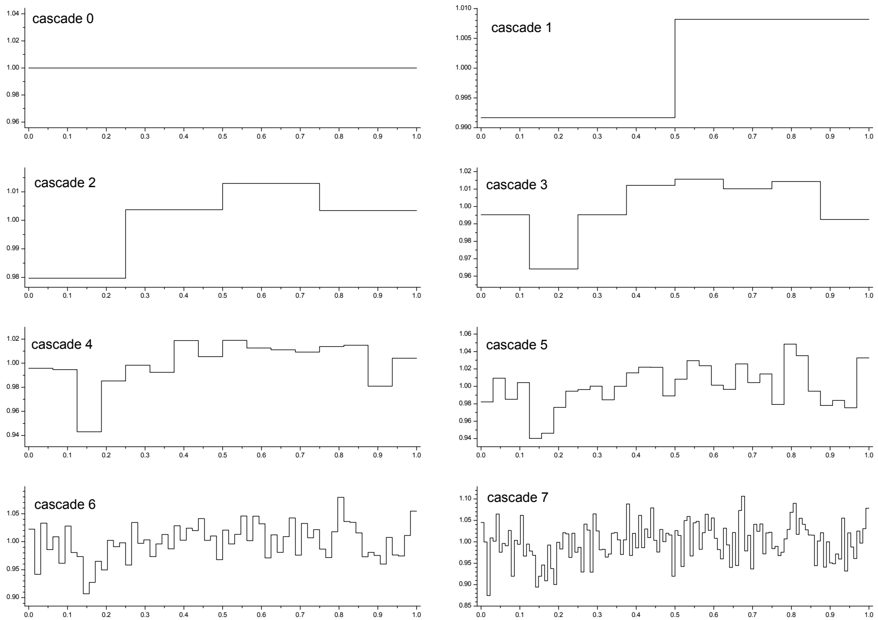
1-dim example (next slide):
1) Domain, length=1. Contains n_0 events.
2) Divided into 2 equal halves. Left half: n_1 events, right half: n_0-n_1.
$E[n_1]=n_0/2=E[n_0-n_1]$, $n_1 \sim f(n_1) = \mathrm{Bin}(n_0,1/2)$.
➔Sample n_1 from f.
3)…i) 2^i bins in i-th cascade, $n_{i+1} \sim \mathrm{Bin}(n_i, 1/2)$.

Technicalities:
a) De Moivre (1733), CLT: $\mathrm{Bin}(n,p) \rightarrow \mathrm{N}(np,\sqrt{(np(1-p))})$
b) $n_{i+1} \equiv (n_i/2)(1-\gamma_i)$, $n_i-n_{i+1} \equiv (n_i/2)(1+\gamma_i)$: **$\gamma_I$ = splitting factor**.
Construction ensures mass conservation.
c) Transition: events n → intensity $z_i = n_i/2^{-i}$, 2^{-i}=length of bin i-th cascade.
d) Traditional random **p-model**: $|\gamma_i| \equiv \gamma$, random $+\gamma$ or $-\gamma$.

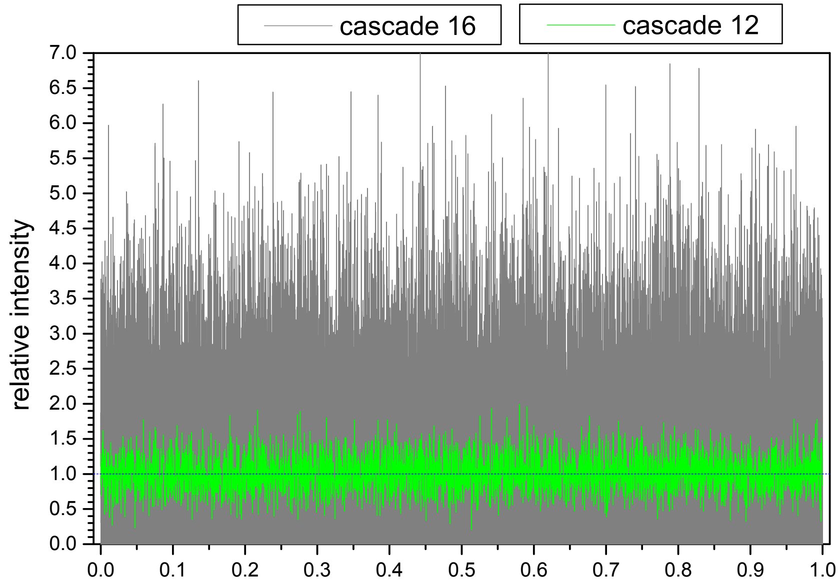
History: Kolmogorov (1940s), De Wijs (1950s), Mandelbrot (1960-70s), Schertzer & Lovejoy (1980s), Agterberg & Chen (1990s-)

[from Bossew et al. 2008]

Example:
1-dim, splitting factor γ=0.6



First 8 cascades



Cascades 12 and 16 (2^{12}=4096 and 2^{16}=65536 bins)

Properties of cascades

De Wijs 2-dim cascade (p model) is asymptotically LN.
After k generations, σ=ln(GSD):

$$\sigma^2 = \frac{k}{4}\left(\ln\left(\frac{1+\gamma}{1-\gamma}\right)\right)^2$$

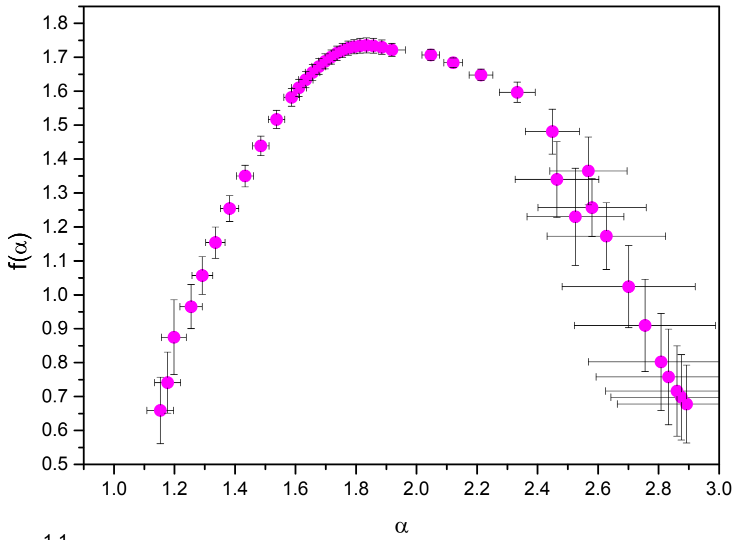
Minimum singularity exponent:

$$\alpha_{\min} = \log_2\left(\frac{2}{1+\gamma}\right)^2 = 2\left(1-\frac{\ln(1+\gamma)}{\ln 2}\right)$$

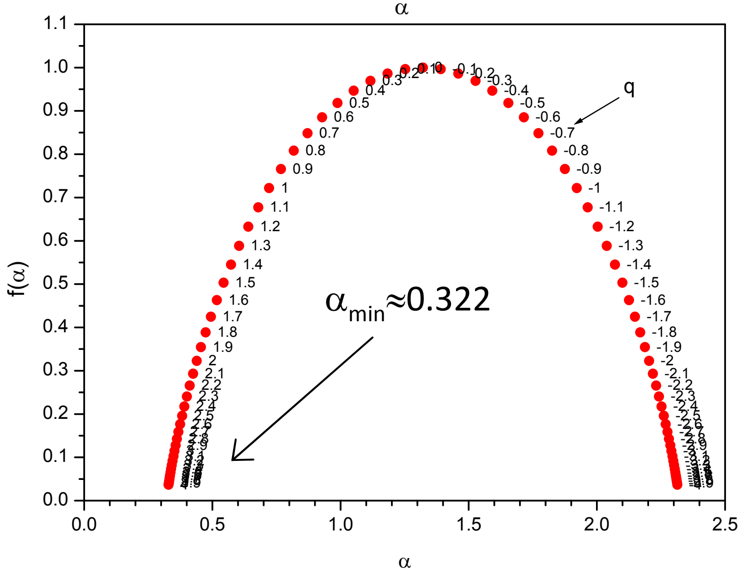
1-dim: $$\alpha_{\min} = 1-\frac{\ln(1+\gamma)}{\ln 2}$$ for γ=0.6: $\alpha_{min} \approx 0.322$

De Wijs 1951; Agterberg 2007a,b; and other.

examples



Friedmann-Radon potential (Austria) (indoor Rn concentration normalized for anthropogenic factors)



1-dim multiplicative cascade, splitting factor γ=0.6

Maximum of the curve: f(α)= box dimension

Cheng's derivation

Idea:

- Mass in cell i (size s) scale as $\mu_i(s) \sim s^{\alpha i}$; therefore density or concentration, $z_i = \mu_i / s^2$ in 2-dim case:
  $z_i(s) \sim s^{\alpha i-2}$.
- Number of cells with $\alpha_i = \alpha$: $N(s) \sim s^{-f(\alpha)}$, therefore area(cells with α)$\sim s^{2-f(\alpha)}$.
- Calculate area(cells from α_{min} to $\alpha_{min}+\delta$), δ=small, i.e. the area of high anomalies (hot spots), defined by α near α_{min}:

$$area(\alpha_{\min} \leq \alpha \leq \alpha_{\min} + \delta) \sim \int_{\alpha_{\min}}^{\alpha_{\min}+\delta} s^{2-f(\alpha)} d\alpha$$

- This can be approximated according to Cheng (1994); annex A, leading to

  log area(cells from α_{min} to $\alpha_{min}+\delta$) ≈ const. - β log z ,

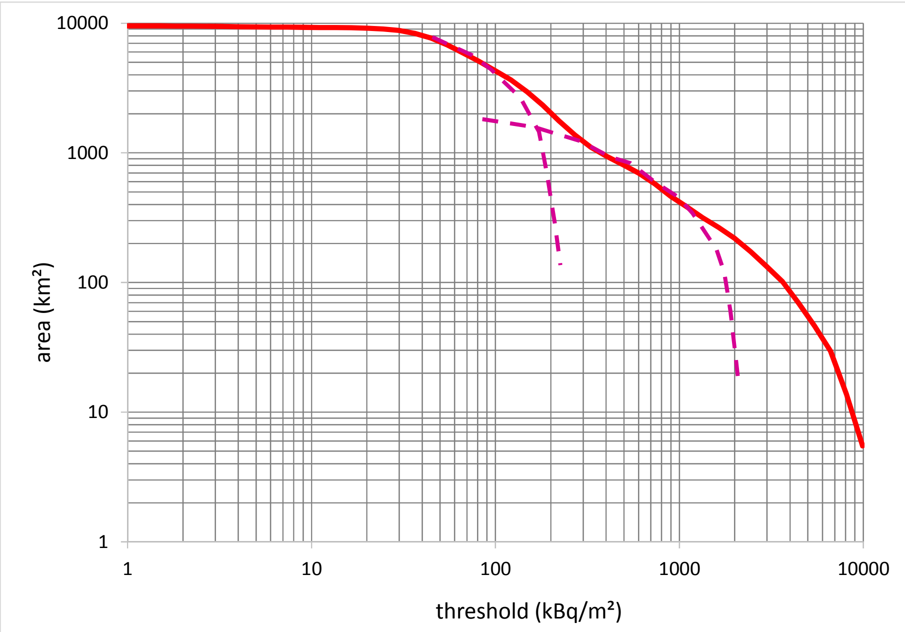
  i.e. a power-law behaviour (const. and ß defined appropriately). However, the derivation (eq. (13) in the paper) is difficult to follow.

LN distribution

- CDF of LN distribution ~ area(Z<z): $CDF = \frac{1}{2} erfc\left[-\frac{\ln z - \mu}{\sigma\sqrt{2}}\right]$,
- p_z := prob(Z>z)=1-CDF=1/2 erfc(+)
- Asymptotic expansion: $erfc(x) = \frac{e^{-x^2}}{x\sqrt{\pi}} \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!!}{(2x^2)^k}$ Abramowitz & Stegun 7.1.23
- with $erfc(x) - \frac{e^{-x^2}}{x\sqrt{\pi}} \sum_{k=0}^{n-1} (\ldots) = O\left(x^{1-2n}\, e^{-x^2}\right)$
- n=1: $erfc(x) \sim \frac{e^{-x^2}}{x\sqrt{\pi}} + O\left(\frac{e^{-x^2}}{x}\right)$
- ln: $\ln \frac{e^{-x^2}}{x\sqrt{\pi}} = -x^2 - \ln x - const$

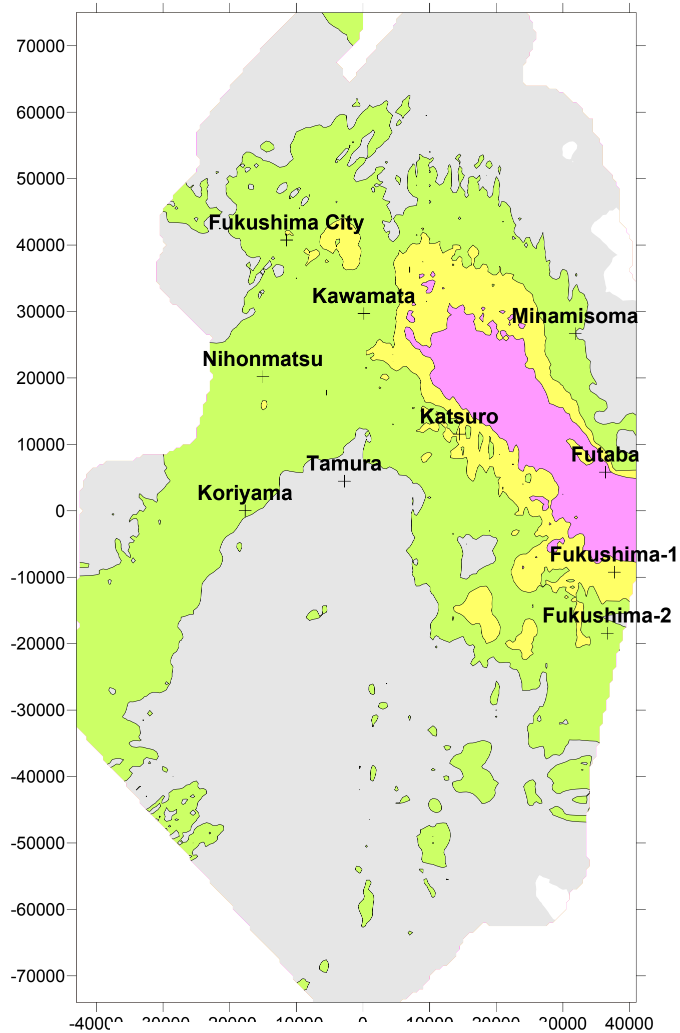
  which is dominated by x^2 for high x. Therefore, the log survival function is asymptotically shaped quadratic. – not linear!

Mixture of populations; Example: Fukushima



“log-survival plot”:
log(area in which Z>z) against log(z)
e.g. area (^{137}Cs > 1 MBq/m²) ≈ 400 km²

Lognormal: would be falling parabola; here mixture of several (3-4) parabolas;
“Breakpoints” → separation of populations



breakpoints:
green > 100 kBq/m²,
yellow >330, pink >900

[Bossew & Debayle, NARE-9, 2014]

Number and size of maxima

- Number M_z of maxima above Z=z for Gaussian fields, Adler 1981 (p.133, Theorem 6.3.1), 2-dim case:

$$EM_z = \frac{area(D)\,(\det\Lambda)^{1/2}\,z}{(2\pi)^{3/2}\,\sigma^3}\exp\left(-\frac{z^2}{2\sigma^2}\right)\left(1+O(1/z)\right)$$

  Λ=matrix of 2nd order spectral moments E(∇_iZ ∇_jZ)
  ($\approx$ inv FT of covariance, e.g. Hristopulos 2020, sec. 5.3.2)

- Expected total excursion area Σarea(z) = area(D) p_z.

- Mean area of maximal region > z: $\langle$area(B)(z)$\rangle$=Σarea(z) / $M_z \sim 1/z^2$ or

  log $\langle$area(B)(z)$\rangle \sim$ -log z. (Bossew 2010)

  LN fields: log z → z

- So, the log-survival function ln(1-CDF) total area decreases by $\sim z^2$, while the mean size of individual maxima, by $\sim$ -z.

- Mixture of populations: more complicated.

Questions 1

- Multifractal generated by multiplicative cascades is asymptotically (with generation) LN;
- (1) Total excursion area(Z>z) behaves as falling parabola in log-log plot for high z;
- (2) But according to analysis of Cheng: linear (=power law).
- Mean size of individual anomalous patches indeed falling linear.
- **Solution of discrepancy between (1) and (2)?**

Questions 2

- Shall certain local maxima be understood (1) as realizations of separate processes, anomaly process on top of BG process? –
- Or (2) as “rare” realization of one process?
- (1) mixture of individual (LN) processes
  (2) extreme outcomes of cascades
- Possibly philosophical question, dependent on scale on which the field is viewed? Because sometimes anomalies appear only at “closer look”.

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Thank you!

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