Revealing a temporal symmetry/asymmetry dichotomy in a Markovian setting, and a parametrization based on fractional low-order joint moments

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Abstract

We show that “an arrow of time”, which is reflected by the joint distributions of successive variables in a stochastic process, may exist (or not) solely on grounds of marginal probability distributions, without affecting stationarity or involving the structural dependencies within the process. The temporal symmetry/asymmetry dichotomy thus revealed, is exemplified for the simplest case of stably-distributed Markovian recursions, where the lack of Gaussianity, even when the increments of the process are independent and identically distributed with symmetric marginal, is generating a break of temporal symmetry. We devise a statistical tool to evidence this striking result, based on fractional low-order joint moments, whose existence is guaranteed even for the case of “fat-tailed” strictly-stable distributions, and is thereby suited for parameterizing structural dependencies within such a process.

Temporal reversibility of stochastic processes has different facets and interpretations from the points of view of Statistics and Mathematical Physics, (see e.g. Osawa, 1988; Georgiou and Lindquist, 2014). In stochastics, the strictest interpretation requires all marginal and finite-dimensional joint distributions of the process variables to be identical between the cases when the time indices are being run either way, and we shall refer to stochastic processes that fulfill this strict requirement as time-symmetric. Under this setting, we show how a stationary process, with hence time-invariant marginals, can have its symmetry broken, or not, at the level of its joint distributions, depending only on the particular stationary marginal distribution of the process.

Let us note that the implications of such a probability-based symmetry/asymmetry dichotomy have a deep meaning in terms of the distinction that physical laws make between past and future, a distinction that is taken to be depicted, essentially, by the evolution of entropy, an intrinsically statistical notion (see e.g. Boltzmann, 1877; Shannon, 1948; Prigogine and Géhéniau, 1986; Christakos, 1990; Arneodo et al., 1995; and more recently, Koutsoyiannis, 2017; Christakos, 2017), but has been connected only loosely with phenomenology (see e.g. Zwanzig, 1961).

We shall analyze herein the simplest case, that of a Markovian process, with a single-variable, linear iteration function, as in:

$$X_{t+1} = rX_t + \epsilon_t, \ \forall t \in \mathbb{N}_0, \tag{1}$$

where $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$, $\{\epsilon_t\}_{t\in\{1,\ldots,n\}}$ independent $\forall n \in \mathbb{N}_0$, and (ϵ_s, X_t) independent $\forall t \le s \in \mathbb{N}_0$.

The stationary AR(1) process (Box and Jenkins, 1976) $\{X_t\}_{t\in\mathbf{N_0}}$, with $|r| < 1$ has joint lag-1 density given by:

$$\begin{aligned} f_{X_t,X_{t+1}}(x,y) &= f_{X_{t+1}|X_t=x}(y) f_{X_t}(x) = \\ &= f_{rX_t+\epsilon_t|X_t=x}(y) f_{X_t}(x) = \\ &= f_{\epsilon_t+rx}(y) f_{X_t}(x) = \\ &= f_{\epsilon_t}(y-rx) f_{X_t}(x). \end{aligned} \tag{2}$$

In an AR(1) process, strictly stable (Lévy, 1937) probability distributions ($S\alpha S_0$ in the notation of Samorodnitsky and Taqqu, 1994) of the independent and identically distributed (i.i.d.) variables ϵ allow for stationary marginals of the process variables X of the same distribution family (see Carsteanu and Langousis, 2020). An interesting detail should be mentioned at this point: In Rao (1966), Lemma 5 establishes a relationship that implies Gaussianity of the marginals, in a context of non-degenerate random variables. This latter condition has been overlooked when the main theorem of the aforementioned paper was used; e.g. to establish a connection between Gaussianity and time-reversibility of linear stochastic processes (Weiss, 1975). Since other stable distributions fulfill the conditions of the cited Lemma 5, they may (or may not) have properties derived from that lemma.

We present herein two examples illustrating a shift from a symmetric to an asymmetric lag-1 joint distribution, based solely on the kind of $S\alpha S_0$ marginal distribution function of the increments.

- Example 1 (case of Gaussian i.i.d. increments):

Increments ϵ_t are distributed according to a Gauss-Moivre ("normal") probability distribution function (Gauss, 1809), i.e. $S\alpha S_0$ with $\alpha = 2$, with standard deviation σ_ϵ. Then, the process variables X have standard deviation $\sigma = \sigma_\epsilon/\sqrt{1-r^2}$, and

$$f_{X_t,X_{t+1}}(x,y) = f_\epsilon(y-rx)f_X(x) = \frac{\exp\left(-\frac{x^2-2rxy+y^2}{2\sigma^2(1-r^2)}\right)}{2\pi\sigma^2\sqrt{1-r^2}}, \qquad (3)$$

a joint density that is bivariate normal with autocorrelation r, and obviously symmetric in $x \leftrightarrow y$ (so there is no "arrow of time" embedded in a series generated by such a process). Also, one can easily check that indeed the marginal distribution of the innovations is time-invariant and symmetric:

$$\int_{-\infty}^{\infty} f_{X_t,X_{t+1}}(x,y)\mathrm{d}y = \int_{-\infty}^{\infty} f_{X_t,X_{t+1}}(y,x)\mathrm{d}y = \frac{\exp\left(-\frac{x^2}{2\sigma^2}\right)}{\sigma\sqrt{2\pi}}. \qquad (4)$$

- Example 2 (case of Cauchy i.i.d. increments):

The increments ϵ_t are distributed according to a Cauchy probability distribution function (Cauchy, 1853), i.e. $\alpha = 1$, with scale parameter γ_ϵ. Then, the process variables X have scale parameter $\gamma = \gamma_\epsilon/(1-|r|)$ (Carsteanu and Langousis, 2020), and

$$f_{X_t,X_{t+1}}(x,y) = f_\epsilon(y-rx)f_X(x) = \frac{\gamma^2(1-|r|)}{\pi^2[(y-rx)^2+\gamma^2(1-|r|)^2](x^2+\gamma^2)}, \tag{5}$$

a joint density that is an asymmetric bivariate Cauchy (Ferguson, 1962) (the denominator being a non-degenerate polynomial of degree 4 in x for $r \neq 0$, and degree 2 in y), so there is an "arrow of time" embedded in a series generated by such a process, an arrow that is being determined by the asymmetry of the joint distribution of successive process variables. The aforementioned property is generated by the asymmetry of the joint density of consecutive process variables, even in the case when their marginal distribution is time-invariant and symmetric:

$$\int_{-\infty}^{\infty} f_{X_t,X_{t+1}}(x,y)\mathrm{d}y = \int_{-\infty}^{\infty} f_{X_t,X_{t+1}}(y,x)\mathrm{d}y = \frac{\gamma}{\pi(x^2+\gamma^2)}. \tag{6}$$

Let us now define the p^{th}-order signed Fractional low-order joint moment (FLOJM) of $S\alpha S_0$-distributed X and Y as:

$$E\left\{|XY|^p \mathrm{sign}(XY)\right\}, \ \forall p < \alpha/2, \tag{7}$$

and hereby, the p^{th}-order nonlinear correlation as:

$$\rho_p\{X,Y\} = \frac{E\left\{|XY|^p \mathrm{sign}(XY)\right\}}{\sqrt{E\left\{(X^2)^p\right\} E\left\{(Y^2)^p\right\}}}, \ \forall p < \alpha/2. \tag{8}$$

.

As $p \to 0$ in (8), we obtain the sign correlation, which exists for all stable distributions, and can hence be used herein for comparison purposes. For $\alpha = 2$ (bivariate Gauss-Moivre) we obtain:

$$\rho_0\{X_t, X_{t+1}\} = \frac{2}{\pi}\arcsin(r), \tag{9}$$

whereas for the $\alpha = 1$ (bivariate Cauchy in equation (5)), we obtain:

$$\rho_0\{X_t, X_{t+1}\} = \frac{r}{1-|r|}\frac{\Phi\left(\left(\frac{r}{1-|r|}\right)^2, 2, \frac{1}{2}\right)}{\pi^2} - \frac{2\ln(1-2|r|)}{\pi^2}\ln\left(\frac{1+\mathrm{sign}(r)-2r}{1-\mathrm{sign}(r)+2r}\right), \tag{10}$$

where Φ is Lerch's transcendent (Lerch, 1887) of parameters $(2, 1/2)$ evaluated at $r^2/(1-|r|)^2$: $\Phi\left(x, 2, 1/2\right) = \sum_{n=0}^{\infty} \frac{x^n}{(n+1/2)^2}$, and $\ln(\cdot)$ is the natural logarithm. The estimator $\widehat{r}$ in such a process is obtained by numerically inverting equation (10).

Figures 1 and 2 show the theoretical $\rho_0\{X_t, X_{t+1}\}$ as a function of r, as well as estimations thereof, for $r = (-0.9, 0.8, \ldots, 0, \ldots, 0.8, 0.9)$, from a time series of 10000 samples of process variables. The sign correlation (since it exists for all $S\alpha S_0$-distributed stationary processes), can be used to detect temporal dependencies, and also, provide a first-order parametrization of their dependence structure in a linear setting (i.e., through the proportionality factor r).

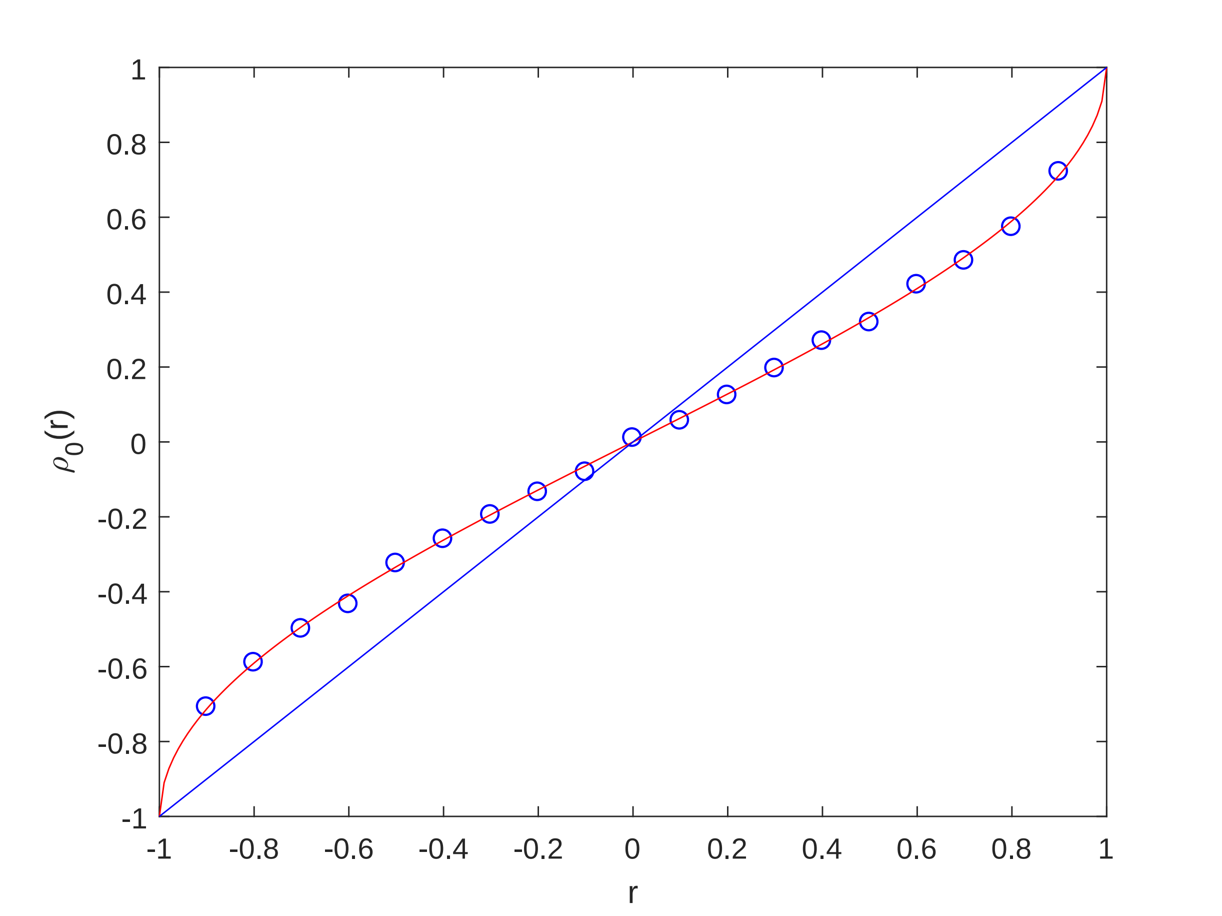

Figure 1: The theoretical $\rho_0\{X_t, X_{t+1}\}$ (red line) as a function of r, for a normally-distributed AR(1) process, and estimations of the same (blue circles), for $r = (-0.9, 0.8, \ldots, 0, \ldots, 0.8, 0.9)$. The main diagonal is shown (in blue) as a visual reference.

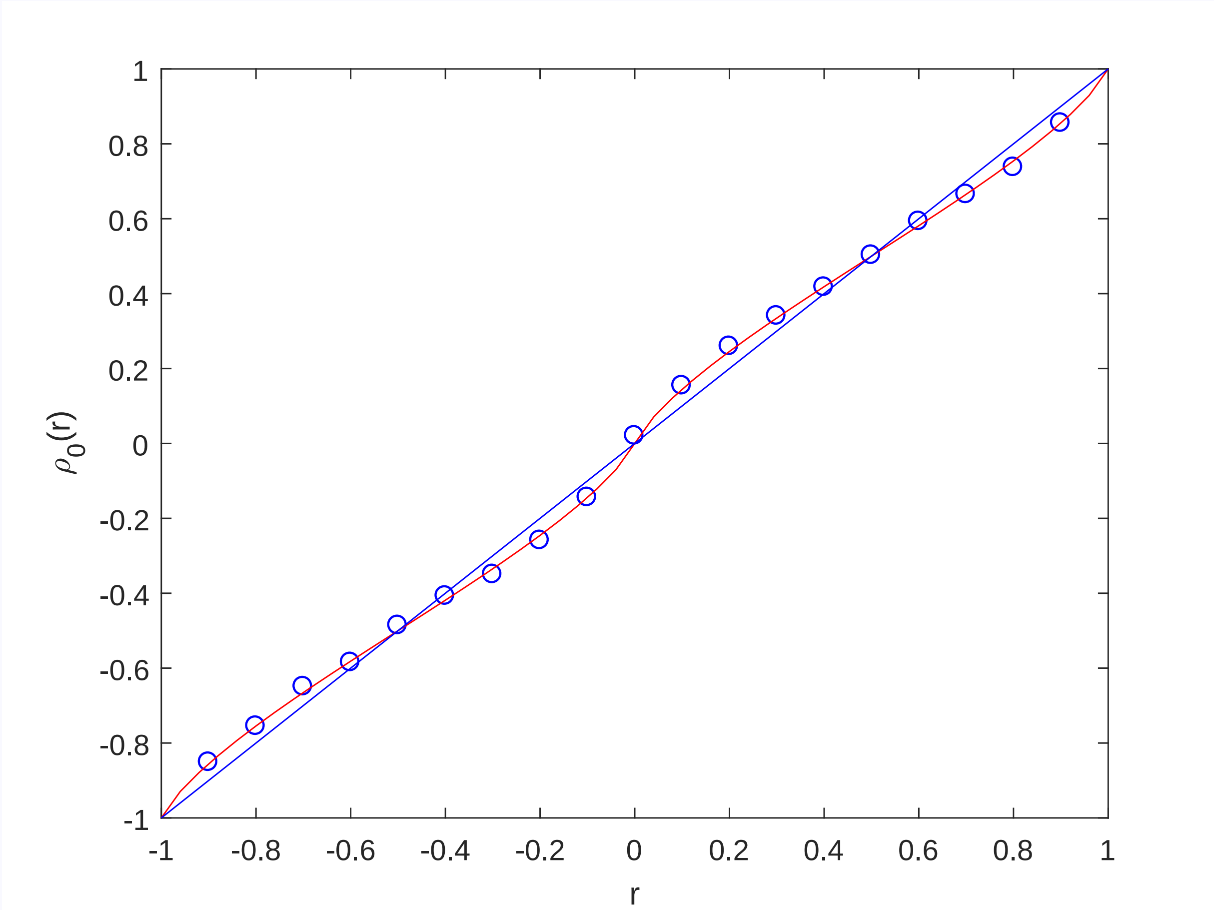

Figure 2: The theoretical $\rho_0\{X_t, X_{t+1}\}$ (red line) as a function of r, for a Cauchy-distributed AR(1) process, and estimations of the same (blue circles), for $r = (-0.9, 0.8, \ldots, 0, \ldots, 0.8, 0.9)$. The main diagonal is shown (in blue) as a visual reference.

Conclusions and future work

One of the most striking results that arise from the analysis of discrete-time, linear Markovian stochastic processes with symmetric, strictly stable ($S\alpha S_0$) marginals of the process variables and their independent increments, is the fact that the temporal symmetry of the process is preserved or broken as a result of its temporal dependence structure and the marginal probability distribution function of the increments, without any changes in the dynamic recursion equation of the process. In order to characterize temporal dependencies in a stationary setting, even in the stable case where moments of order lower than 2 may diverge, we evaluated the sign correlation for AR(1) processes (i.e., an estimator based on fractional low-order joint moments, which exists for all $S\alpha S_0$-distributed stationary processes), and used it to parameterize their dependence structure (that is, through the proportionality factor r).

Future research endeavors could focus on: (i) investigating possible implications of irreversibility of stochastic processes in time series analysis, statistical inference and forecasting; (ii) widen the understanding regarding which marginal distributions generate, or alternatively do not generate, an "arrow of time" within the joint distributions of a given stationary process; and (iii) generalizing the expressions of FLOJMs for the parameterization of different processes, particularly stable ones, which are ubiquitous in nature due to their attractive property under aggregation.

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