Poroelastic aspects in geothermics

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What is poroelasticity and why is it important?

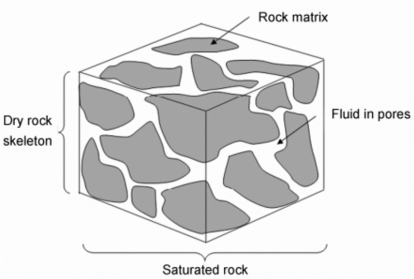


Figure:
http://www.geothermie.de/wissenswelt/lexikon-der-geothermie/p/poroelastizitaet.html

- Poroelasticity is a material research discipline and relates for a material the solids deformation to the fluid flow.
- It describes e.g. in geothermal research the stresses and interactions in an aquifer.
- The mathematical model to describe this behavior is dated back to Biot in the 1930s.

Poroelasticity: Modelling of the coherence between water pressure and stresses in porous or ragged rocks.

What is our aim?

The aim is referring to [Freeden, Blick, 2013] for the scalar case and [Blick, Eberle, 2019] for the tensorial case to

- Construct scaling functions and wavelets that are physically motivated with the application of post-processing
- Introduce low-pass and band-pass filter for the further interpretation of data sets

For this ansatz, we need the fundamental solutions of the corresponding partial differential equations of poroelasticity.

Please note that our intention is less to do an approximation than to do a further decomposition of the data sets.

Quasistatic equations of poroelasticity (QEP)

The quasistatic equations of poroelasticity base on physical laws like the conservation of linear momentum, conservation of mass or linear elasticity. They are partial differential equations and given by

$$\begin{aligned} -\frac{\lambda+\mu}{\mu}\nabla_x(\nabla_x \cdot u) - \nabla_x^2 u + \alpha \nabla_x p &= f, \\ \partial_t(c_0 \mu p + \alpha(\nabla_x \cdot u)) - \nabla_x^2 p &= h, \end{aligned}$$

where λ, μ, α and c_0 are material parameters.
u is the displacement vector and p the pressure.

Unknown: $u(x,t)$ and $p(x,t)$.

With this, we can define the poroelastic differential operator

$$L^{\mathrm{pe}}(\partial) = \begin{pmatrix} -\frac{\lambda+\mu}{\mu}\nabla_x(\nabla_x \cdot u) - \nabla_x^2 u + \alpha \nabla_x p \\ \partial_t(c_0 \mu p + \alpha(\nabla_x \cdot u)) - \nabla_x^2 p \end{pmatrix}$$

For the construction of the scaling functions and wavelets, we will come back to this operator later again.

Fundamental solutions

Another essential component are the fundamental solutions, which can be arranged in a tensor $G(x,t)$ and are given by

$$\boldsymbol{G}(x,t) = \begin{pmatrix} \boldsymbol{u}^{\mathrm{CN}}(x)\delta(t) & p^{\mathrm{St}}(x)\delta(t) \\ u^{\mathrm{Si}}(x,t) & p^{\mathrm{Si}}(x,t) \end{pmatrix},$$

where

$$\boldsymbol{u}_{ki}^{\mathrm{CN}}(x,t) = C_3 \frac{1}{2\pi}\left(-\delta_{ki}\ln(\|x\|) + C_4 \frac{x_i x_k}{\|x\|^2}\right)\delta(t)$$

$$u^{\mathrm{Si}}(x,t) = C_1 \frac{x}{2\pi\|x\|^2}\left(1 - \exp\left(-\frac{\|x\|^2}{4C_2 t}\right)\right)$$

$$p^{\mathrm{Si}}(x,t) = \frac{1}{4\pi t}\exp\left(-\frac{\|x\|^2}{4C_2 t}\right)$$

$$p^{\mathrm{St}}(x,t) = C_1 \frac{x}{2\pi\|x\|^2}\delta(t).$$

Please note that the divergence from the poroelastic operator is meant row-wise.

If we have a look back at the fundamental solutions, we can see that each of them has a singularity at the point $x = (0, 0)$. The main idea for the construction of the scaling functions is, to mollify the fundamental solutions with a Taylor expansion around the critical singularity. We show an example for the component p^{St} and regularize this function in domain with respect to a parameter τ. We write

$$p^{\mathrm{St}} = \frac{C_1}{2\pi} \frac{x}{\|x\|^2} = \frac{1}{2} \nabla_x \left(\ln \|x\|^2 \right).$$

Since we want a linear expansion for p^{St}, we have to do an expansion up to order 2 for $\ln \|x\|^2$:

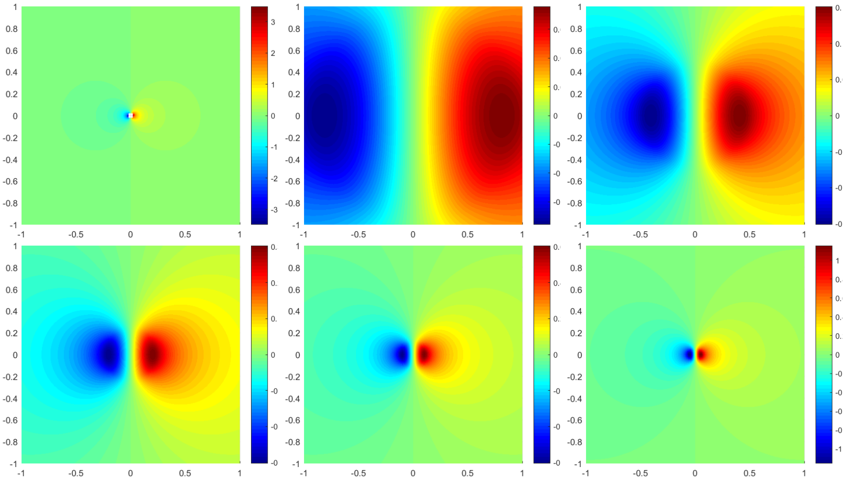
$$\frac{1}{2} \ln \left(\|x\|^2 \right) \approx \frac{1}{2} \left(\ln \tau^2 + \frac{1}{\tau^2}(\|x\|^2 - \tau^2) - \frac{1}{2\tau^4}(\|x\|^2 - \tau^2)^2 \right).$$

The gradient leads us to

$$p_\tau^{\mathrm{St}} = \begin{cases} \frac{C_1}{2\pi} \frac{x}{\|x\|^2}, & \|x\| > \tau, \\ \frac{C_1}{2\pi} \cdot \frac{x}{\tau^2} \cdot \left(2 - \frac{\|x\|^2}{\tau^2} \right), & \|x\| < \tau. \end{cases}$$

In the following, τ will be a sequence of monotonically decreasing values, i.e. $\tau = 2^{-j}$, $j \in \mathbb{N}_0$.
We call these regularized functions scale discrete (potential) scaling functions.

In the figure below, we can see the fundamental solution p^{St} (upper left) and the regularized varieties for $j = 0$ (upper middle), $j = 1$ (upper right), $j = 2$ (lower left), $j = 3$ (lower middle) and $j = 4$ (lower right).



Wavelets

The family $\{\mathcal{W}_{\tau_j}(\Delta;\cdot)\}_{j\in\mathbb{N}}$ defined via

$$\mathcal{W}_{\tau_j}(\Delta;\cdot) = G_{\tau_j}(\Delta;\cdot) - G_{\tau_{j-1}}(\Delta;\cdot)$$

is called a *scale discrete (potential) wavelet function*. We call

$$\Phi_{\tau_j}(\|x-y\|) = L^{\mathrm{pe}}(\partial)G_{\tau_j}(\Delta;\|x-y\|)$$

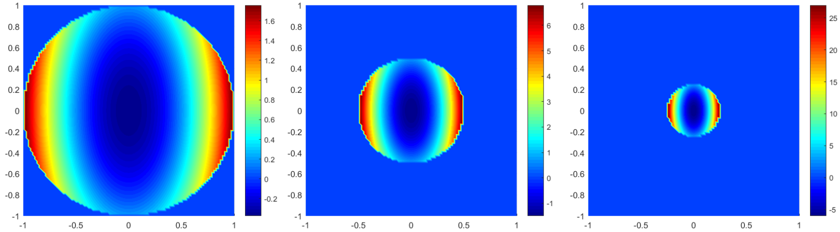
the *scale discrete (source) scaling function* and

$$\Psi_{\tau_j}(\|x-y\|) = \Phi_{\tau_j}(\|x-y\|) - \Phi_{\tau_{j-1}}(\|x-y\|)$$

the *scale discrete (source) wavelet function*.

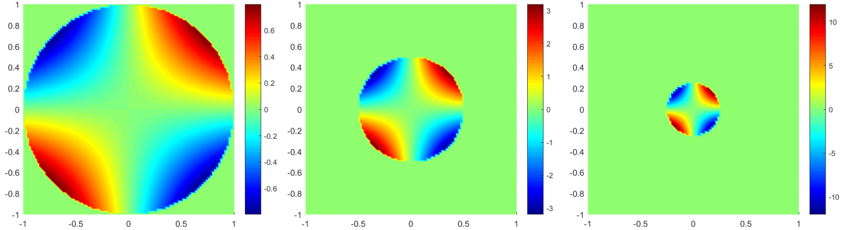
We will have a closer look at the source scaling functions, because these are the ones we are interested in.

As an example, we will show here the source scaling functions $\Phi_{11,\tau}$ and $\Phi_{12,\tau}$ (next slide) for $j = 0$ (first column), $j = 1$ (second column) and $j = 2$ (third column) and introduce the theoretical requirements that are necessary.



Due to the construction, all source scaling functions have compact support and this support gets smaller with increasing j (i.e. decreasing τ).

Here we can see the source scaling function $\Phi_{12,\tau}$ for several j.



Please note that in the case of the time-dependent $\Phi_{31,\tau}$, $\Phi_{32,\tau}$ and $\Phi_{33,\tau}$ (not shown here), we have to restrict the support in time with a variable t_0 depending on τ such that the support also gets smaller in time with increasing j.

We show the necessary theoretical requirements to the source scaling functions i.e. the following approximate identity has to hold true for all $x \in \mathcal{B}$ and $t \in T$

$$\lim_{\substack{\tau \to 0 \\ \tau > 0}} \int_{\mathcal{B}} \int_T \Phi_\tau(x - y, t - \theta) f(y, \theta) \, \mathrm{d}\theta \, \mathrm{d}y = f(x, t).$$

For the proof it is necessary to have

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}} \Phi_\tau(y, \theta) \, \mathrm{d}\theta \, \mathrm{d}y = \mathbb{I},$$

which is fulfilled in our case. Here $\mathbb{I}$ is the identity matrix.

The approximate identity can be shown for the several components of Φ_τ with the help of the mean value theorem and the fact that the integral over the positive parts of the functions can be estimated by a constant (the proof is motivated by techniques used for a different problem in [Blick, Eberle, 2019]).

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