Bifurcations of periodic traveling wave solutions to the nonlinear system of weakly coupled KdV-type equations are studied. Solutions close to cnoidal and harmonic waves are considered. Lyapunov–Schmidt procedure, allowing one to reduce the original problem to the system of bifurcation equations, is used. The dimension reduction of the bifurcation equations system involves different techniques in both cases. These techniques are based on symmetry and cosymmetry properties of the origin KdV-type equations. Sufficient conditions for the solutions orbits branching in terms of Painlevé–Pontryagin functional are formulated.

1. MOTIVATION

System of two coupled KdV equations arises in describing strong interaction of internal waves in stratified fluid (Gear & Grimshaw 1983, Grimshaw 2013). It means there are two different modes with near coincided phase speeds based on symmetry and cosymmetry properties of the origin KdV-type equations. Sufficient conditions for the solutions orbits branching in terms of Painlevé–Pontryagin functional are formulated.

2. STATEMENT OF THE PROBLEM

Looking for traveling wave solutions and integrating once (integration constants are neglected), one gets a system of coupled autonomous second order ODEs. We will work with the following system

\[ u'' = R(u,v), \quad v'' = R(u,v), \]

with \( R(u,v) = \phi(u,v) + \psi(u,v) \), \( \phi(u,v) = u^3 - 3uv^2 + \epsilon \omega \) and \( \psi(u,v) = \beta u^2 + \gamma v^2 \). Such a type system appears in an appropriate choosing of coefficients in (1). When \( \epsilon = 0 \) decoupled system has a cnoidal-wave solution

\[ u(0) = A_0 + \delta \cos \eta, \quad v(0) = B_0 + \delta \cos \eta, \quad \beta = \epsilon \omega \]

which does not change under the transformation \( \zeta = \zeta - \Delta \). So the function \( A_0 \) should also be even \( \zeta \). Finally, the system of bifurcation equations reduces to a one scalar equation

\[ \lambda = - \frac{1}{\sqrt{n}} + \frac{3\beta^2}{2} \lambda - \frac{3\beta^2}{2} \lambda^2 + \frac{1}{2} \lambda^3. \]

giving an explicit form for \( \lambda \). Thus, one obtains the following asymptotic form

\[ u(\zeta) = c_1 + 2c_2 \cosh \left( \frac{x}{q_0} \right) + \cosh \left( \frac{x}{q_0} \right), \]

with \( c_1, c_2, q_0 \) being determined by the boundary conditions.

3. LYAPUNOV–SCHMIDT METHOD

In this case \( \delta \) is finite and hence omitted below. Thus, we are looking for \((T, \phi)\)-periodic solution where \( T = T(u,v) \) with \( \lambda(\delta, \epsilon) = 1 \). Operator \( A : \mathcal{D} \to \mathcal{D} \) defined by \( A = \phi(u,v) + \psi(u,v) \) is a Fredholm linear operator. It’s kernel is two-dimensional and spanned by \( \nu \) and \( \tau \). The system of bifurcation equations is formulated:

\[ \begin{align*}
\lambda & = \frac{1}{\sqrt{n}} + \frac{3\beta^2}{2} \lambda - \frac{3\beta^2}{2} \lambda^2 + \frac{1}{2} \lambda^3, \\
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with \( c_1, c_2, q_0 \) being determined by the boundary conditions.

5. SMALL-AMPLITUDE HARMONIC WAVES

Consider the case when \( \lambda = 0 \), then \( T(u,v) = 2\tau \) and \( T(u,v) = 2\sigma \), with an analytic functions \( \mu, \lambda \). Let \( \mu(0) = 1 \) and \( \lambda(0) \). The Vieta’s theorem leads to

\[ \lambda = \frac{1}{\sqrt{n}} + \frac{3\beta^2}{2} \lambda - \frac{3\beta^2}{2} \lambda^2 + \frac{1}{2} \lambda^3. \]

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with \( c_1, c_2, q_0 \) being determined by the boundary conditions.

Now we consider bimodal equation (7), taking into account that \( u(0) = 2 \cosh \sigma \) and \( v(0) = 2 \cosh \sigma \), giving an explicit form for \( \lambda \). Thus, one obtains the following asymptotic form

\[ u(\zeta) = c_1 + 2c_2 \cosh \left( \frac{x}{q_0} \right) + \cosh \left( \frac{x}{q_0} \right), \]

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