Synchronization of traveling waves in coupled dispersive systems

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Abstract

Bifurcations of periodic traveling wave solutions to the nonlinear system of weakly coupled KdV-type equations are studied. Solutions close to cnoidal and harmonic waves are considered. Lyapunov – Schmidt procedure, allowing one to reduce the origin problem to the system of bifurcation equations, is used. The dimension reduction of the bifurcation equations system involves different techniques in both cases. These techniques are based on symmetry and cosymmetry properties of the origin KdV-type equations. Sufficient conditions for the solutions orbits branching in terms of Poincare – Pontryagin functional are formulated.

1. Motivation

System of two coupled KdV equations arises in describing strong interaction of internal waves in stratified fluid **(Gear & Grimshaw 1983, Grimshaw 2013)**. It means there are two different modes with near coincided phase speeds c_p and $c_p + a^2\Delta$ **(Eckart 1961)**. Here $a << 1$ and Δ is detuning parameter. In this situation particle vertical displacement is given by

$$\hat{\zeta}(z,s,\tau) = a^2\big(A_1(\tau,s)\hat{\varphi}_1(z) + A_2(\tau,\hat{\xi})\hat{\varphi}_2(z)\big) + \dots,$$

where $\hat{\xi} = s + \Delta\tau$. At leading order in a modal functions $\hat{\varphi}_{1,2}$ satisfy the following spectral problem

$$\Big\{\rho_0(u_0 - c_p)^2\hat{\varphi}_{iz}\Big\}_z + \rho_0 N^2\hat{\varphi}_i = 0, \quad (-h < z < 0), \quad \hat{\varphi}_i = 0, \quad (z = -h), \qquad (u_0 - c_p)^2\hat{\varphi}_{iz} = g\hat{\varphi}_i, \quad (z = 0).$$

Here $N^2 = -g\rho_{0z}/\rho_0$ is Brunt – Vaisala frequency. Finally, the amplitude functions $A_{1,2}$ satisfy the following system

$$\hat{\alpha}_1 A_{1\tau} + \hat{\gamma}_{11} A_1 A_{1s} + \hat{\delta}_{11} A_{1sss} + \hat{\nu}_{211}\big\{A_1 A_2\big\}_s + \hat{\gamma}_{21} A_2 A_{2s} + \hat{\delta}_{21} A_{2sss} = 0, \tag{1}$$
$$\hat{\alpha}_2 A_{2\tau} + \hat{\gamma}_{22} A_2 A_{2s} + \hat{\delta}_{22} A_{2sss} + \hat{\nu}_{122}\big\{A_1 A_2\big\}_s + \hat{\gamma}_{12} A_1 A_{1s} + \hat{\delta}_{12} A_{1sss} + \hat{\alpha}_2\Delta A_{2s} = 0.$$

3. Lyapunov – Schmidt method

Let $\mathscr{E}$ and $\mathscr{F}$ be real Banach spaces and $\mathscr{U} \subset \mathscr{E}$ be an open set. Suppose $\mathbb{F} : \mathscr{U} \times (-\varepsilon_0, \varepsilon_0) \to \mathscr{F}$ is a smooth mapping with $\varepsilon_0 \in \mathscr{R}$. We are looking for a solution to the operator equation

$$\mathbb{F}(w;\varepsilon) = 0. \tag{2}$$

(Operator formulation) For a known w_0, s. t. $\mathbb{F}(w_0;0) = 0$ one can look for w as a perturbation $w = w_0 + \vartheta$, where ϑ satisfies the equation

$$\mathbb{A}\vartheta = \mathbb{R}(\vartheta;\varepsilon) \tag{3}$$

with $\mathbb{R}(\vartheta;\varepsilon) = \mathbb{A}\vartheta - \mathbb{F}(w_0 + \vartheta;\varepsilon)$.
(Fredholm property) Frechet derivative $\mathbb{A} = \mathbb{F}'_w(w_0;0)$ is supposed to be a Fredholm operator and $\dim\operatorname{Ker}\mathbb{A} = \operatorname{codim}\operatorname{Im}\mathbb{A} = n \geq 1$.
(Projectors) One can define projectors $\mathbb{P} : \mathscr{E} \to \operatorname{Ker}\mathbb{A}$ and $\mathbb{Q} : \mathscr{F} \to \mathscr{Y}$ generating the following decompositions of spaces $\mathscr{E}$ and $\mathscr{F}$

$$\mathscr{E} = \operatorname{Ker}\mathbb{A} \oplus \mathscr{X}, \qquad \mathscr{F} = \operatorname{Im}\mathbb{A} \oplus \mathscr{Y}.$$

Let $\{e_j\}_{j=1}^n$ be a basis in $\operatorname{Ker}\mathbb{A}$. The function ϑ is sought in the form $\vartheta = \sum_{i=1}^{n} \xi_i e_i + \sigma$ where $\xi_1, \dots, \xi_n \in \mathscr{R}$ and σ is defined imlpicitly by

$$\sigma = \widetilde{\mathbb{A}}^{-1}(\mathbb{I} - \mathbb{Q})\mathbb{R}\Big(\sum_{i=1}^{n} \xi_i e_i + \sigma;\varepsilon\Big) = 0. \tag{4}$$

Here $\widetilde{\mathbb{A}} : \mathscr{X} \to \operatorname{Im}\mathbb{A}$ is a restriction of $\mathbb{A}$ onto $\mathscr{X}$. Thus, equation (2) is equivalent to the following n-dimensional system of functional equations on the coefficients $\xi = (\xi_1, \dots, \xi_n)$

$$\mathbb{Q}\mathbb{R}\left(\sum_{i=1}^{n} \xi_i e_i + \sigma;\varepsilon\right) = 0, \tag{5}$$

called the **system of bifurcation equations**.

4. Cnoidal-wave type solution

In this case δ is finite and hence omitted below. Thus, we are looking for $T(\varepsilon)$-periodic solution where $T(\varepsilon) = T_0/\omega(\varepsilon)$ with $\omega(0) = 1$. Operator $\mathbb{A} : \mathscr{E} \to \mathscr{F}$ defined by $\mathbb{A}w = (u'' + (3u_0 - 1)u, v'' + (3v_0 - 1)v)$ **is a Fredholm linear operator**. It's kernel is two-dimensional and spanned by vectors e_1 and e_2. The **system of bifurcation equations** is given by

$$\int_0^{T_0(\delta)} R_1(u_1, v_1; \dots)u_0'\,ds = 0, \qquad \int_0^{T_0(\delta)} R_2(u_1, v_1; \dots)v_0'\,ds = 0 \tag{8}$$

with $u_1 = \xi_1 u_0' + \sigma_1(\xi_1, \xi_2; \dots)$ and $v_1 = \xi_2 v_0' + \sigma_2(\xi_1, \xi_2; \dots)$. The origin system admits a potential formulation with potential

$$l(w;\varepsilon) = \int_0^{T(\varepsilon)} \left\{\frac{u'^2}{2} + \frac{v'^2}{2} + H(u,v;\varepsilon)\right\} ds, \quad T_g l(w;\varepsilon) = l(T_g w;\varepsilon),$$
$$\Longrightarrow$$
$$0 = \langle \nabla l(w;\varepsilon),\, Xw\rangle_\varepsilon = -\langle \mathbb{Q}\mathbb{R}(w_1;\omega,c,\varepsilon), Xw\rangle_0, \tag{9}$$

(Makarenko 1996) where $g \in \mathscr{R}$, $w = w_0 + \varepsilon w_1$, $w_1 = \xi_1 e_1 + \xi_2 e_2 + \sigma(\xi_1, \xi_2, \omega, c, \varepsilon)$, $\langle\cdot,\cdot\rangle_\varepsilon$ is an inner product in the space $\mathscr{L}_2[0, T(\varepsilon)] \times \mathscr{L}_2[0, T(\varepsilon)]$, ∇l is a Frechet derivative of l. The infinitesimal operator of the time translation group $X = \partial_\zeta$ **plays role of cosymmetry** here **(Yudovich 1991)** . Due to (9) one of the equations in (8) can be expressed via another. Thus, at leading order in ε one has

$$\Psi(c) \overset{def}{=} \int_0^{T_0} \Phi_u(u_0(\tau), u_0(\tau + c), 0)\, u_0'(\tau)d\tau = 0. \tag{10}$$

Here $\Psi(c)$ is a Poincare – Pontryagin function **(Poincare 1890, Pontryagin 1934)**. If $c = c_0$ is a root of $\Psi(c)$, then the mapping σ is defined at $\varepsilon = 0$. Finally, the first equation from (8) is of the form

$$\Psi'(c)(\xi_1 - \xi_2) + \Gamma(\omega, c) + \varepsilon\Pi(\xi_1, \xi_2; \omega, c, \varepsilon) = 0. \tag{11}$$

Here an explicit form of smooth functions Γ and Π is inessential for analysis. Thus, one can apply an implicit function theorem to express ξ_1 as a function of all other parameters. The coefficient ξ_2 stays free here. Finally, we get the following statement
If the phase shift c is a simple root of the function $\Psi(c)$, then for sufficiently small ε, system (6) has a $T(\varepsilon)$-periodic solution with $\omega(\varepsilon) \to 1$ as $\varepsilon \to 0$

2. Statement of the problem

Looking for traveling wave solutions and integrating once (integration constants are neglected), one get a system of coupled autonomous second order ODEs. We will work with the following system

$$u'' = H_u(u,v;\varepsilon), \quad v'' = H_v(u,v;\varepsilon), \quad H = (u^2 + v^2 - u^3 - v^3)/2 + \varepsilon\Phi(u,v;\varepsilon), \quad \varepsilon << 1 \tag{6}$$

with $\Phi(0,0;\varepsilon) = \Phi_u(0,0;\varepsilon) = \Phi_v(0,0;\varepsilon) = 0$. Such a type system appears in an appropriate choosing of coefficients in (1). When $\varepsilon = 0$ decoupled system has a cnoidal-wave solution

$$u_0(t) = \alpha_2 + \delta\, cn^2(rt;m), \quad v_0(t;c) = u_0(t + c), \quad r = \sqrt{\delta + \lambda}/2, \quad m^2 = \delta/(\delta + \lambda),$$

where $\delta = \alpha_3 - \alpha_2$, $\lambda = \alpha_2 - \alpha_1$ and α_i are roots of polynomial $-u^3 + u^2 + 2h = 0$ with a given constant h. Since system (6) is invariant wrt time translations, phase shift c is arbitrary at leading order in ε. **The problem is to find the value of c, providing $T(\varepsilon,\delta)$-periodic solution branching when $\varepsilon \neq 0$.** Let $k \geq 1$ be an integer. Define the space $\mathscr{H}_\delta^k$ as a Sobolev space $\mathscr{W}_2^k[0, T_0(\delta)]$ of real periodic functions. For a pair $w = (u,v)$ we denote $\mathscr{E}_\delta = \mathscr{H}_\delta^{k+2} \times \mathscr{H}_\delta^{k+2}$ and $\mathscr{F}_\delta = \mathscr{H}_\delta^k \times \mathscr{H}_\delta^k$. The **operator formulation** is following. We seek the solution in the form $w(t;\omega,\varepsilon,\delta,c) = w_0(t;\omega(0,\delta),\delta,c) + \varepsilon w_1(t;\omega,\varepsilon,\delta,c)$. After introducing new independent variable $\zeta = \omega(\varepsilon,\delta)t$, the solution period $T_0(\delta) = T(0,\delta)$ becomes a fixed one. Thus, one can define w_1 from

$$\mathbb{A}w_0 + \varepsilon\mathbb{A}w_1 = 3w_0^2/2 + \varepsilon\mathbb{R}(w_1;\varepsilon,\omega,c), \quad \mathbb{A}w_1 = \big(u_1'' + (3u_0 - 1)u_1,\, v_1'' + (3v_0 - 1)v_1\big), \quad w_0^2 = (u_0^2, v_0^2), \quad (\cdot)' = d/d\zeta. \tag{7}$$

Here $\mathbb{A} : \mathscr{E}_\delta \to \mathscr{F}_\delta$ and nonlinear operator $\mathbb{R} = (R_1, R_2)$ components are following

$$R_1(u_1, v_1;\varepsilon,\delta,\omega,c) = \varepsilon^{-1}(1-\omega^2)(u_0''+\varepsilon u_1'') - \frac{3}{2}\varepsilon u_1^2 + \Phi_u(u_0+\varepsilon u_1, v_0+\varepsilon v_1;\varepsilon), \quad R_2(u_1, v_1;\varepsilon,\delta,\omega,c) = \varepsilon^{-1}(1-\omega^2)(v_0''+\varepsilon v_1'') - \frac{3}{2}\varepsilon v_1^2 + \Phi_v(u_0+\varepsilon u_1, v_0+\varepsilon v_1;\varepsilon).$$

The linear system $\mathbb{A}w = 0$ has a solution space spanned by the following vectors

$$e_1 = (u_0', 0), \quad e_2 = (0, v_0'), \quad e_3 = (u_*, 0), \quad e_4 = (0, v_*), \quad u_*(\zeta) = u_0'(\zeta)\int_{\zeta_0 \neq 0}^{\zeta} \frac{ds}{u_0'^2(s)}, \quad v_* = u_*(\zeta + c).$$

The elements e_3 and e_4 are non-periodic functions in a general case, but it become periodic in the case when the cnoidal-wave solution transforms to a harmonic wave packet. Note that **soliton limit** was considered in **(Makarenko 1996, Wright & Scheel 2007)**.

5. Small-amplitude harmonic waves

Consider the case when $\delta \to 0$, then $T_0(\delta) \to 2\pi$ and $T(\varepsilon,\delta) = 2\pi/\omega(\varepsilon,\delta)$ where $\omega^2 = \mu^2(\delta) - \varepsilon\omega_*(\varepsilon,\delta)$ with an analytic functions μ, s. t. $\mu(0) = 1$ and ω_*. The Vieta's theorem leads to

$$\lambda = -\frac{\delta}{2} + \left(1 - \frac{3\delta^2}{8} - \frac{9\delta^4}{128}\right) + \dots, \quad \alpha_2 = \frac{1}{3} - \frac{\delta}{2} + \left(\frac{1}{3} - \frac{\delta^2}{8} - \frac{3\delta^4}{128}\right) + \dots.$$

Thus, an asymptotic formula for solution $w_0 = (u_0, v_0)$ when $\varepsilon = 0$ takes the form $u_0(t;\delta) = 2/3 + \delta\varphi(t;\delta)$, $v_0(t;\delta,c) = u_0(t + c;\delta)$ where $\varphi = \varphi_0 + \delta\varphi_1$ satisfies the equation

$$A_0\varphi_1 = R_0(\varphi_1;\varrho,\eta,\delta), \quad A_0\varphi_1 = \varphi_1'' + \varphi_1, \quad (\cdot)' = d/d\zeta,$$
$$R_0 = \eta(\varphi_0'' + \delta\varphi_1'') - 3(\varphi_0 + \delta\varphi_1)^2/2, \quad \eta = \delta^{-1}(1 - \mu^2),$$

where $\varphi_0 = \varrho\cos\zeta$ and $\zeta = \mu(\delta)t$. We apply LS method again. The null space of the linear operator A_0 is invariant wrt translations of a time variable. They generate the representation of a compact Lie group SO(2) in the space of parameters $\kappa = (\kappa_1, \kappa_2) \in \mathscr{R}^2$, where $\varphi_1 = \kappa_1\cos\zeta + \kappa_2\sin\zeta + \sigma_0$. Thus, we can use the reduction theorem **(Loginov & Trenogin 1971)**, which gives an invariant form of the solution φ_1:

$$\varphi_1 = T_g\big\{|\kappa|\cos\zeta + \sigma_0(\zeta;\varrho,|\kappa|,\eta,\delta)\big\}, \quad g \in [0, 2\pi]$$

Without loss of generality one can set $g = 0$. In addition, the operator R_0 is also invariant wrt the scaling group:

$$L_\gamma R_0(|\kappa|\cos\zeta + \sigma_0;\varrho,\eta,\delta) = R_0(L_\gamma\{|\kappa|\cos\zeta + \sigma_0\}; L_{\frac{\gamma}{2}}\varrho, L_{\frac{\gamma}{2}}\eta, L_{-\frac{\gamma}{2}}\delta)$$

with $L_{\gamma/2}\varphi = e^{\gamma/2}\varphi$. The invariants of this group can be taken in the form

$$\zeta = \zeta, \quad \eta_* = \delta\eta, \quad \varrho_* = \delta\varrho, \quad \kappa_* = \delta^2|\kappa|, \quad \hat{\sigma}_0 = \sigma_0(\varrho + \delta|\kappa|)^{-2}.$$

Hance $\sigma_0 = (\varrho + \delta|\kappa|)^2\hat{\sigma}_0(\zeta;\eta_*,\varkappa)$ with $\varkappa = \varrho_* + \kappa_*$. Here $\hat{\sigma_0}$ should satisfy the factor-equation

$$\hat{\sigma}_0 = \tilde{A}^{-1}(I - Q)\big\langle\beta(-\cos\zeta + \varkappa\hat{\sigma}_0'') - 3(\cos\zeta + \varkappa\hat{\sigma}_0)^2/2\big\rangle, \quad \beta = \eta_*\varkappa^{-1},$$

which does not change under the transformation $\zeta \to -\zeta$. So the function $\hat{\sigma}_0$ should also be even wrt ζ. Finally, the **system of bifurcation equations** reduces to a one scalar equation

$$\beta - 15\varkappa/8 + 19\varkappa^2\beta/32 - 1755\varkappa^3/512 + \dots = 0,$$

giving an explicit form for η. Thus, one obtains the following asymptotic formula

$$u_0(\zeta;\varkappa) = 2/3 + \varkappa\varphi, \quad \varphi(\zeta;\varkappa) = \cos\zeta + \varkappa(\cos(2\zeta)/4 - 3/4) + \dots$$

Now we consider bimodal equation (7), taking into account that $w_0 = 2/3 + \varkappa\theta$, where $\theta = (\varphi, \psi)$ with $\psi = \varphi(\zeta + c)$:

$$\varkappa\mathbb{A}_0\theta + \varepsilon\mathbb{A}_0 w_1 = -\varkappa^2\theta^2 - 3\varkappa\varepsilon\theta w_1 + \varepsilon\mathbb{R}(w_1;\omega,\varepsilon,\delta,c).$$

Here is denoted $\theta^2 = (\varphi^2, \psi^2)$, $\theta w_1 = (\varphi u_1, \psi v_1)$ and $\mathbb{A}_0 : \mathscr{E}_0 \to \mathscr{F}_0$ defined by $\mathbb{A}_0 = (A_0, A_0)$. As written above θ satisfy the equation $\mathbb{A}_0\theta = (1 - \mu^2)\theta'' - 3\varkappa\theta^2/2$. Thus, using the reasoning as above, concerning group theoretical reduction, one obtains the following **system of bifurcation equations** for $u_1 = \varrho_1\cos\zeta + \Phi_u^0$, $v_1 = \varrho_2\cos(\zeta + c) + \Phi_v^0$ with $\Phi_{u,v}^0 = \Phi_{u,v}(2/3, 2/3; 0)$:

$$-\omega_*\varrho_1 + a_1\varrho_1 + \varrho_2\Phi_{uv}^0\cos c + \varepsilon\chi_1(\varrho_1, \varrho_2, \omega_*;\varepsilon) = 0,$$
$$-\omega_*\varrho_2 - \varrho_1\Phi_{uv}^0\cos c + a_2\varrho_2 + \varepsilon\chi_2(\varrho_1, \varrho_2, \omega_*;\varepsilon) = 0,$$
$$\sin c\,\big(\varrho_1\Phi_{uv}^0 + \varepsilon\chi_3(\varrho_1, \varrho_2, \omega_*;\varepsilon)\big) = 0, \quad \sin c\,\big(-\varrho_2\Phi_{uv}^0 + \varepsilon\chi_4(\varrho_1, \varrho_2, \omega_*;\varepsilon)\big) = 0.$$

Here one of the equations can be eliminated due to (9) and explicit form of constants a_i and functions χ_i is inessential for analysis. **In this case the Poincare – Pontryagin function is degenerate and has the following asymptotics:**

$$\Psi(c;\varkappa) = -\varkappa^2\Phi_{uv}^0\sin c + \dots$$

Even in this degenerate case, simple roots $c = \pm\pi k$, $k = 0, 1, \dots$ provide an existence of phase-locked modes here.

References

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