Non-linear resonant instability of short surface waves as the first stage “bag-breakup” process at the air-sea interface at high winds

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Introduction
The recent experimental study [1, 2] identify “bag breakup” as the dominant mechanism by which spume droplets are generated at hurricane wind speeds. These droplets can significantly affect the exchanging processes in the air-ocean boundary layer. In order to estimate spray-mediated heat, momentum and mass fluxes we need not only reliable experimental data, but a theoretical model of this process.

The “bag-breakup” fragmentation is a strongly nonlinear process, and we focus only on its first stage which includes the small-scale elevation of the water surface.

Our model of the bag’s initiation is based on a weak nonlinear interaction of a longitudinal surface wave and two oblique waves propagating at equal and opposite angles to the flow as it was done in [3, 4].

Model
Let’s consider the piecewise velocity profile as
\[ \mathbf{u}(z) = \begin{cases} \mathbf{u}_1 & -1 < z \leq 0 \\ \mathbf{u}_2 & -\infty < z \leq -1 \end{cases} \]
which was along OX axis. All quantities are made dimensionless relative to the stream velocity \( \mathbf{V} \), density \( \rho \) of the fluid and the thickness \( \delta \) of the boundary layer. We assume that the interaction occurs between three waves: two oblique, which propagate at the same angle to the wind flow and one longitudinal wave.

Assuming the fluid incompressible and inviscid, the components of the velocity perturbation should satisfy the nonlinear equations of motion:
\[ \partial_t \mathbf{v}_x + \mathbf{u}_0 \partial_x \mathbf{v}_x + v_x \partial_x \mathbf{u}_0 + \frac{1}{\rho} \partial_x p = -\nabla \mathbf{\phi} \]
\[ \partial_t \mathbf{v}_y + \mathbf{u}_0 \partial_y \mathbf{v}_y + v_y \partial_y \mathbf{u}_0 + \frac{1}{\rho} \partial_y p = -\nabla \mathbf{\phi} \]
with non-linear boundary conditions:
1) Dynamic boundary conditions (taking into account capillary effect and gravity)
\[ p|_{z=0} = (\partial_0 + \mathbf{\Sigma} \cdot \eta) = 0 \]
\[ p|_{z=\pm \eta} = \pm p \]
2) Kinematic boundary conditions at \( z = 0 \) \( \pm \eta \)
\[ \partial_t \mathbf{\phi} + (\partial_0 + \mathbf{\Sigma} \cdot \eta) \mathbf{\phi} = v_x \frac{\partial}{\partial x}, v_y \frac{\partial}{\partial y} \]
3) Decay at infinity
\[ \mathbf{v}, \mathbf{v}' \rightarrow 0 (z \rightarrow -\infty) \]

First order
We seek a uniform expansion by using the method of multiple scales in the form
\[ \psi(\vec{r}, t) = \psi_0(\vec{r}, t, \epsilon \eta) + \epsilon^2 \psi_2(\vec{r}, t, \epsilon \eta) + \epsilon^4 \psi_4(\vec{r}, t, \epsilon \eta) + \ldots \]
For the first order of the multiple scales we give that all waves propagate independently, and we obtain a homogeneous system of algebraic equations for each wave in the following form
\[ -i\omega \psi_0''(\vec{r}, t, \epsilon \eta) - \Phi_0'(\vec{r}, t, \epsilon \eta) = 0 \]
\[ -i(\omega + k \partial_x \cos \alpha \mathbf{u}_0) \psi_0''(\vec{r}, t, \epsilon \eta) - \Phi_0'(\vec{r}, t, \epsilon \eta) = 0 \]
\[ -i(\omega + k \partial_y \cos \alpha \mathbf{u}_0) \psi_0''(\vec{r}, t, \epsilon \eta) - \Phi_0'(\vec{r}, t, \epsilon \eta) = 0 \]
where \( \alpha = 65^\circ \).

Dispersive relations
We assume that the amplitudes of oblique waves are the same. For effective interaction of these waves, the synchronism conditions \( k_3 = 2k_2 \cos \alpha, Re \omega_3(k_3) = Re \omega_2(k_3) \) must be satisfied. Analysis of the dispersion relations shows that these conditions can be fulfilled at certain values of velocity and angle.

Second order
To derive the three-wave interaction system, it is necessary to take into account second order terms, but the term \( \psi_0''(\vec{r}, t, \epsilon \eta) \), like \( \psi_2(\vec{r}, t, \epsilon \eta) \), contains not only an irrotational part, but a rotational one. As a result, the stream function can be written in the following way
\[ \psi(\vec{r}, t) = \epsilon \lambda_3(\xi) \psi_2(\vec{r}, t, \epsilon \eta, \lambda_2 \xi) + \epsilon^2 \psi_4(\vec{r}, t, \epsilon \eta, \lambda_2 \xi) + \frac{W(\epsilon \eta)}{\lambda_2} \exp[i(k \xi - \omega t)] \]

Considering the nonlinearity of the boundary conditions, we get an inhomogeneous system of equations for each of the waves in the form:
\[ L_2(\psi_0, \psi_2, \psi_4) = \mathbf{b} \]
Their homogeneous parts have a nontrivial solution if their right-hand sides are orthogonal to the solution of the adjoint homogeneous problem.

The irrotational term \( W(z) \) is determined from the equations of motion of the second order in the inviscid limit. However, in this case, the integral will have a high-order pole in the critical layer at \( z = z_0 = \{ -1 \leq z_0 \leq 0 \} \). The solution of the viscous problem for a purely shear flow is described in detail in [4] and all mathematical calculations for present study almost coincide with [4]. The evaluation of the last term \( \mathbf{b}(\omega, \mathbf{u}_0) \) is quite straightforward. That resulted in the equations describing the modulational of amplitudes \( \mathbf{A}_1, \mathbf{A}_2 \) as follow:
\[ \frac{d \mathbf{A}_i}{d \tau} = \lambda_i \mathbf{A}_i \]
Numerical simulation of the receiving system demonstrates explosive growth within finite time \( \tau_{max} \).

The dynamics of the simulated structure is similar to the initial stage of the “bag-breakup” phenomenon. To compare the experimental data and our model, we define the transversal scale of the structure formed by these surface waves as \( L_0 = \frac{2 \pi d}{\omega_{max}} \) where we come back to dimensional variables.

Numerical calculations showed the transverse scale of the most unstable triads \( L_0 \) and its time of growth \( \tau_{max} \) have following dependencies on the friction velocity
\[ L_0 \propto u_{z}^{-1.23} \]
\[ \tau_{max} \propto u_{z}^{-1.89} \]
As it was reported in [1] the experimental data give us the following dependencies for size of bags and their lifetime
\[ R_{1} > \tau > u_{z}^{-2} \]

Outlook
In the future research, we will continue develop this model and will focus on more thorough comparison with the “bag-breakup” instability.

References