A Parametrization for Triad Interactions of Internal Gravity Waves in Varying Background Flows for WKBJ Ray-Tracing Methods

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A Non-Hydrostatic, Non-Rotating, Weakly Non-Linear Bousinesq Theory

$$D_t \boldsymbol{v} = -\nabla p + \boldsymbol{e}_z b$$
$$D_t b = -N^2 w$$
$$0 = \nabla \cdot \boldsymbol{v}$$

$$\boldsymbol{u}(x,t) = \sum_{k=0}^{\infty} \epsilon^k \boldsymbol{U}_0^{(k)} + \Re \sum_{\beta} \sum_{n=1}^{\infty} \epsilon^n e^{i\phi_\beta/\epsilon^2} \boldsymbol{U}_\beta^{(n)}$$

- $L_\mathrm{w} \sim H_\mathrm{w}$ (non-hydrostatic)
- large scale mean flow:

  $\epsilon^2 = L_\mathrm{w} \,/\, L_\mathrm{mf}$
- no rotation $f = 0$
- constant stratification

- superposition of waves
- weak wave amplitudes
- wave oscillation on $(T_0, \boldsymbol{X}_0)$
- interactions on $(T_1, \boldsymbol{X}_1)$
- modulation on $(T_2, \boldsymbol{X}_2)$

cf. Achatz et al. [2010, 2017]; Wilhelm et al. [2018]

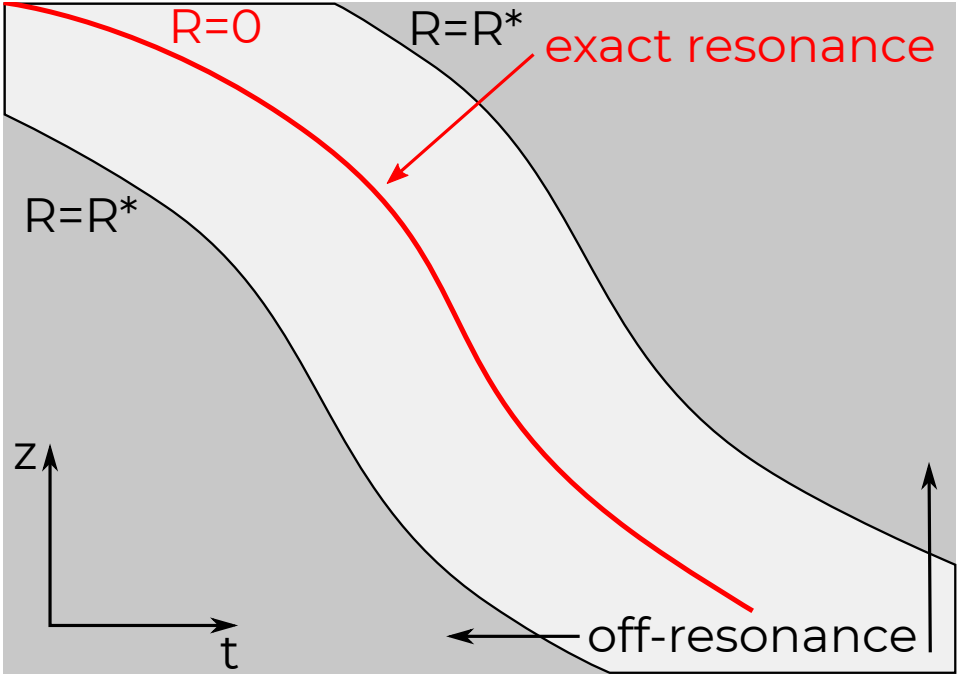
WKBJ Expansion: Non-Linearities

$$\partial_T Y_\beta \sim Y_\gamma Y_\delta e^{i(\phi_\gamma+\phi_\delta-\phi_\beta)/\epsilon^2} + Y_\gamma{}^* Y_\delta e^{i(-\phi_\gamma+\phi_\delta-\phi_\beta)/\epsilon^2} + Y_\gamma{}^* Y_\delta{}^* e^{i(-\phi_\gamma-\phi_\delta-\phi_\beta)/\epsilon^2}$$

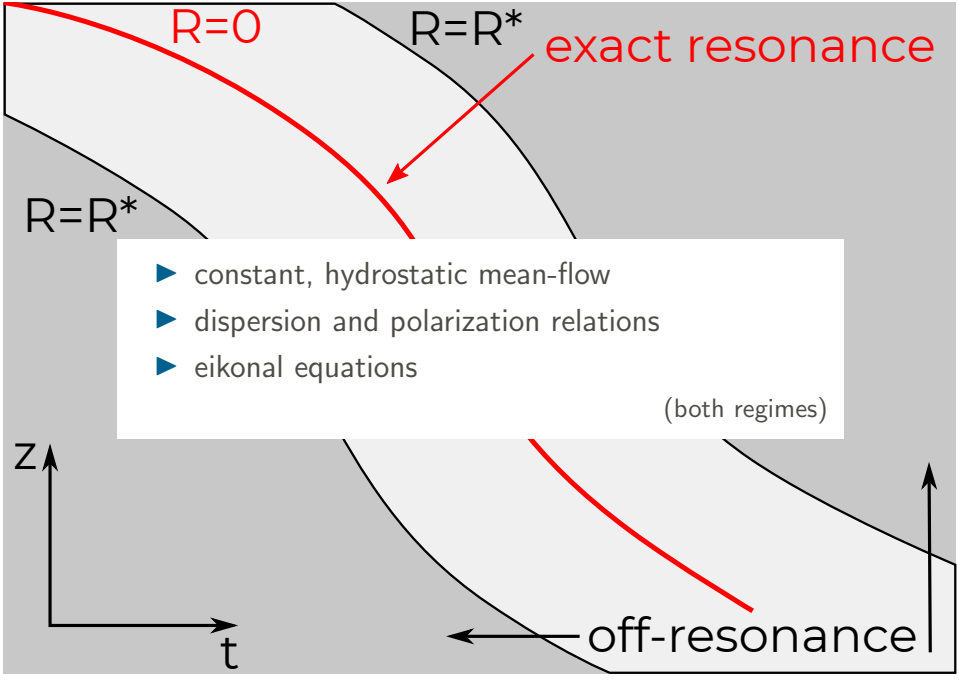
$$\frac{\Delta\phi}{\epsilon^2} = \left(\frac{\Delta\phi}{\epsilon^2}\right)^* + \underbrace{(\partial_{T_2}\Delta\phi)^*}_{=-(\Delta\omega)^*}(T_0 - T_0^*) + \frac{1}{2}(\partial_{T_2}^2\Delta\phi)^*(T_1 - T_1^*)^2$$

- defines resonance conditions: $\omega_1 = \omega_2 + \omega_3$, $\boldsymbol{k}_1 = \boldsymbol{k}_2 + \boldsymbol{k}_3$
- non-linearities persist only near resonance, i.e. where $\Delta\omega \in \mathcal{O}(\epsilon)$, $\Delta\boldsymbol{k} \in \mathcal{O}(\epsilon)$
- define: $0 = R(\boldsymbol{x}, t) = \dfrac{|\omega_2 + \omega_3 - \omega_1|}{\omega_2 + \omega_3}$ with $\omega_1 = \omega(\boldsymbol{k}_2 + \boldsymbol{k}_3)$

WKBJ Expansion: Two Regimes



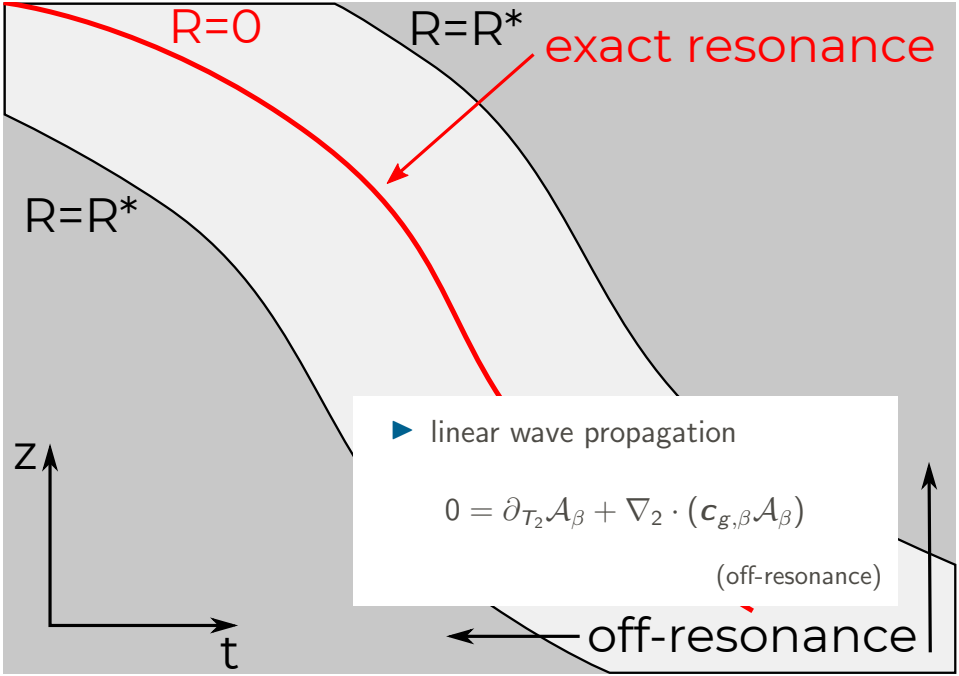
WKBJ Expansion: Two Regimes



- constant, hydrostatic mean-flow
- dispersion and polarization relations
- eikonal equations

(both regimes)

WKBJ Expansion: Two Regimes

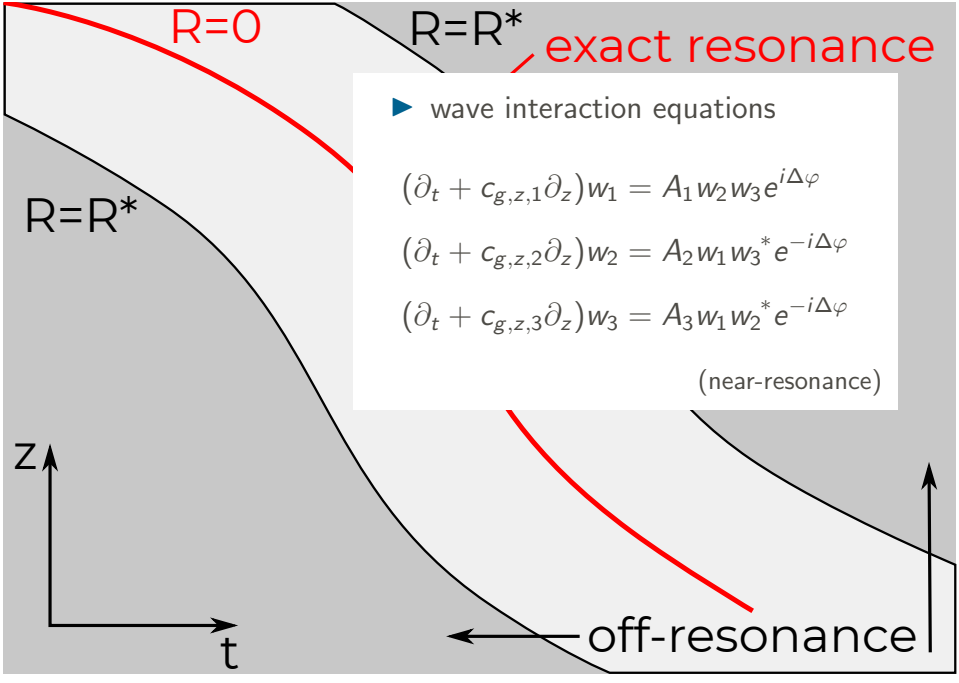


- linear wave propagation

$$0 = \partial_{T_2}\mathcal{A}_\beta + \nabla_2 \cdot (\boldsymbol{c}_{g,\beta}\mathcal{A}_\beta)$$

(off-resonance)

WKBJ Expansion: Two Regimes



- wave interaction equations

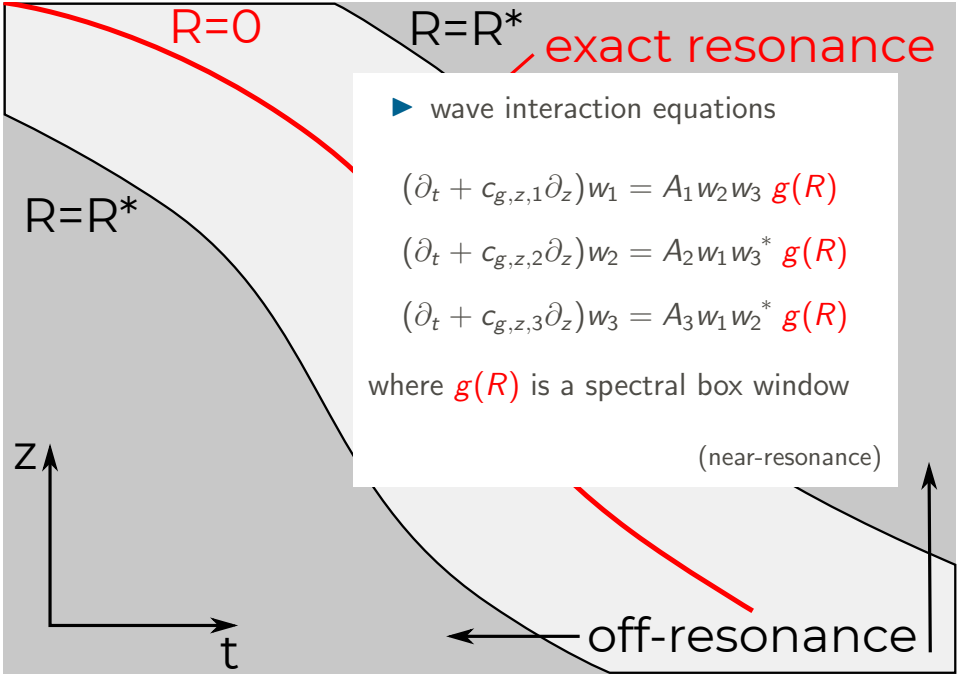
$$(\partial_t + c_{g,z,1}\partial_z)w_1 = A_1 w_2 w_3 e^{i\Delta\varphi}$$

$$(\partial_t + c_{g,z,2}\partial_z)w_2 = A_2 w_1 w_3^* e^{-i\Delta\varphi}$$

$$(\partial_t + c_{g,z,3}\partial_z)w_3 = A_3 w_1 w_2^* e^{-i\Delta\varphi}$$

(near-resonance)

WKBJ Expansion: Two Regimes



- wave interaction equations

$$(\partial_t + c_{g,z,1}\partial_z)w_1 = A_1 w_2 w_3 \; g(R)$$

$$(\partial_t + c_{g,z,2}\partial_z)w_2 = A_2 w_1 w_3^* \; g(R)$$

$$(\partial_t + c_{g,z,3}\partial_z)w_3 = A_3 w_1 w_2^* \; g(R)$$

where $g(R)$ is a spectral box window

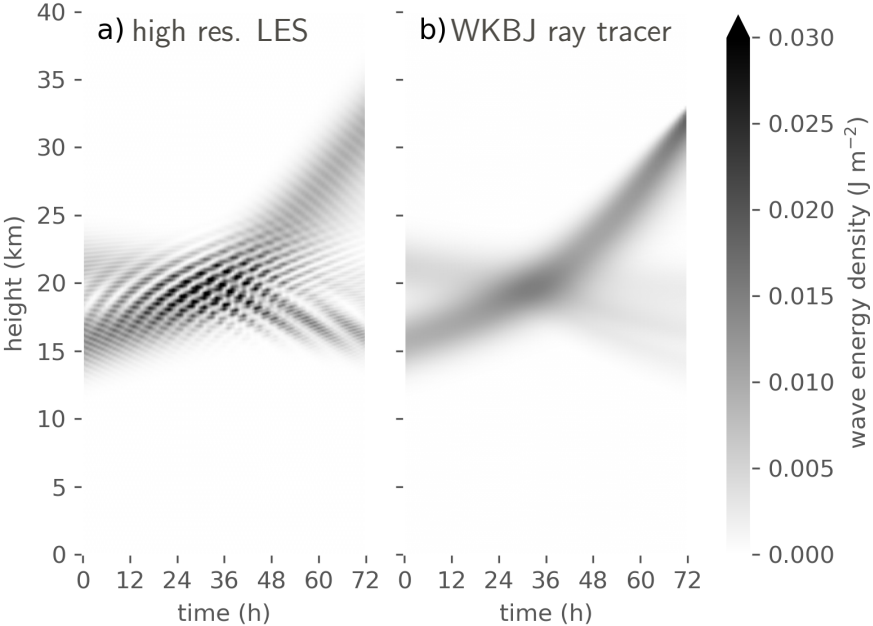
(near-resonance)

LES - WKBJ Model Comparisons: Total Wave Energy Densities

$\hat{k}_{2,3} = 0.1$

$\hat{m}_2 = 10$

$u_0 = 1\,m\,s^{-1}$



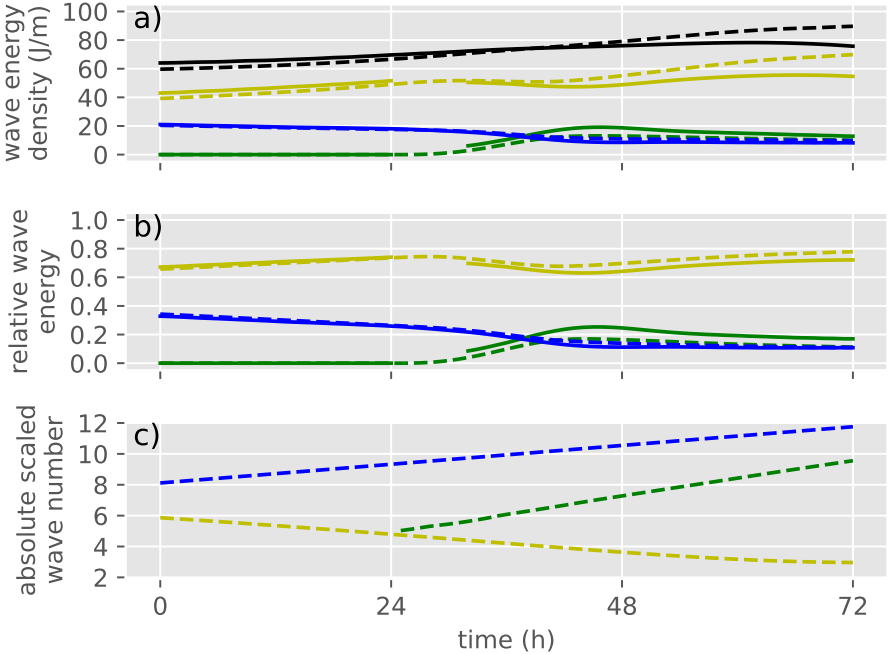
not shown: sinusoidal background shear

LES - WKBJ Model Comparisons: Properties of Individual Wave Trains

$\hat{k}_{2,3} = 0.1$

$\hat{m}_2 = 10$

$u_0 = 1\,m\,s^{-1}$



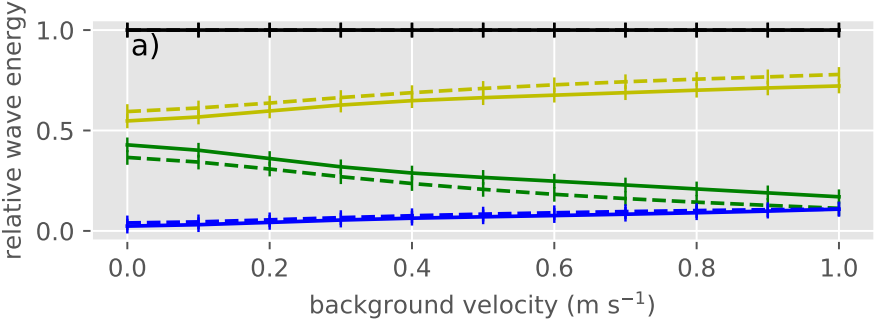
- solid: LES
- dashed: ray tracer

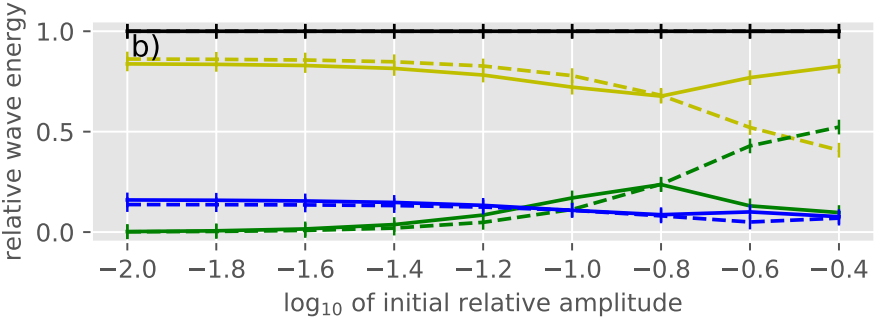
LES - WKBJ Model Comparisons: Varying Wave and Mean-Flow Amplitude

relative energy after interaction

$\hat{k}_{2,3} = 0.1$

$\hat{m}_2 = 10$





- solid: LES
- dashed: ray tracer

Summary

- weakly non-linear, Boussinesq, WKBJ theory
  - three scales
  - two scaling regimes: off-resonance / near-resonance solution
  - spectral passage through resonance
- simplification: parametrize dephasing as spectral window
- comparison WKBJ ray tracer and LES
  - varying $\partial_z u_0$, α, and m_β
- we find generally a good agreement for small α
- strong shear suppresses triad interactions
- recently submitted to QJRMS
- work in progress: wave mean-flow interaction

Achatz, U., R. Klein, and F. Senf, 2010: Gravity waves, scale asymptotics and the pseudo-incompressible equations. *Journal of Fluid Mechanics*, **663**, 120–147, doi:10.1017/S0022112010003411, URL http://www.journals.cambridge.org/abstract{_}S0022112010003411.

Achatz, U., B. Ribstein, F. Senf, and R. Klein, 2017: The interaction between synoptic-scale balanced flow and a finite-amplitude mesoscale wave field throughout all atmospheric layers: weak and moderately strong stratification. *Quarterly Journal of the Royal Meteorological Society*, **143 (702)**, 342–361, doi:10.1002/qj.2926.

Wilhelm, J., and Coauthors, 2018: Interactions between Meso- and Sub-Mesoscale GravityWaves and their Efficient Representation in Mesoscale-Resolving Models. *Journal of the Atmospheric Sciences*, JAS–D–17–0289.1, doi:10.1175/JAS-D-17-0289.1, URL http://journals.ametsoc.org/doi/10.1175/JAS-D-17-0289.1.