1. Motivation

Kinetic energy spectra in our high-resolution ($\nu = 330$) GSM: Classic Smagorinsky model vs. Dynamic Smagorinsky model (DSM)

- Consider mathematical consistencies of parameterizations
  - Represented by mathematical properties (e.g., invariances) for set of equations
  - Examples: Invariances with respect to time, translation, rotation, scaling ...

Answer (Oberlack, 1997): Classic Smagorinsky model cannot capture near-wall scaling laws and violates scale invariance

3. Scale invariance criterion

**General formulation** (Schaefer-Rolffs et al., 2015)
- For a symmetry transformation of an equation of motion of a scalar $a$ the mathematical structure must be retained under transformation
  \[ \partial_t a + (\mathbf{v} \cdot \nabla) a = \mathcal{F}_d(t, x, a, b, \ldots) \]
  \[ \Rightarrow \partial_t a + (\mathbf{v} \cdot \nabla) a = \mathcal{F}_d(t', x', a', b') \]
- Application of Kolmogorov-like scaling
  \[ \nu = \nu(t), x' = \nu x, v' = \nu v, a' = \nu a, b' = \nu b \]
  \[ \Rightarrow a'' = \partial_t a'' + (\mathbf{v}'' \cdot \nabla) a'' = \mathcal{F}_d''(t'', x'', a'', b'') \]
- A criterion (in the red box) can be derived as follows
  \[ \partial_t a'' + (\mathbf{v}'' \cdot \nabla) a'' = \mathcal{F}_d''(t'', x'', a'', b'') \]
  \[ \Rightarrow \mathcal{F}_d''(t'', x'', a'', b'') \]
  \[ \Rightarrow \mathcal{F}_d''(t'', x'', a'', b'') \]
  - It can be used for each term individually, because scaling is linear

**Criterion for quasi-geostrophic regime** (Schaefer-Rolffs, 2019)
- Coriolis term significant $f = f_0 c_t = 2\Omega \sin \phi x$
- Scaling includes a rotation defined by $\mathcal{F}_d(t, x, a, b) = \mathcal{F}_d(t, x, a, b)$
- QG scaling: $x'' = \nu x, y'' = \nu y, z'' = \nu z$
- Enstrophy cascade with $\nu = \nu(x)$ such that $c_t = c_t = 0$ yielding $c_t = c_t = 0$
- Blue line: $x'' = \nu x, y'' = \nu y, z'' = \nu z$
- Euler equations with quasi-geostrophic motion is $x'' = \nu x, y'' = \nu y, z'' = \nu z$
- The scale invariance criterion for the Euler equations with quasi-geostrophic motion is $x'' = \nu x, y'' = \nu y, z'' = \nu z$

**Criterion for anelastic regime** (Schaefer-Rolffs, 2019)
- Decomposition of temperature, pressure, and density in primitive equations
  \[ \mathbf{u}(t, x) \pm \mathbf{u}(t, x) \rightarrow \mathbf{u}(t, x) \pm \mathbf{u}(t, x) \]
- Assumption constant energy cascade, hence $c_t = c_t$
  - Entangled scaling factors, finally leading to
  - Scale invariance criterion remains as in general case, $G_d(t, x, a, b) = \frac{1}{2} x'' + y'' + z''$
  - Further: Scaling of $\nu$ in accordance with aspect ratio of stratified turbulence, $L_z \sim x^{1/2}/\nu$, Lindborg, 2006

4. Applications

**Applications I**
- Example: Pressure gradient in Euler equation
  \[ \mathbf{u} = \mathbf{u}(t, x) \pm \mathbf{u}(t, x) \rightarrow \mathbf{u}(t, x) \pm \mathbf{u}(t, x) \]
  - Extension of the parameter space, no constraint in the existing space

**Applications II**
- Criterion also applicable for a passive tracer equation $\partial_t C + (\mathbf{v} \cdot \nabla) C = \nabla \cdot (K_C \nabla C)$
- Application to vertical diffusion in anelastic horizontal momentum equation $\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \nabla \cdot (\nu_k \nabla \mathbf{u})$
  - Anelastic regime
  - From $K_C = \frac{1}{2} k_L \mathbf{u}$ it follows $c_t = c_t = c_t$ and $c_t = c_t$
  - Finally, due to $c_t = c_t$, we have $c_t = c_t$, leading to the conclusions:
  - A vertical mixing length cannot be constant to ensure scale invariance
  - One possible realization might be $\ell_z = \sqrt{\ell_y \ell_5}$, where $\ell_z$ is a constant

**Visualization of scale invariance with scaling factors $c_{\nu}$ defined by transformation $\nu'' = \nu(c_{\nu})$**
- Comparison of Navier-Stokes and Euler equations: Difference due to molecular viscosity
  \[ \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = - \nabla p + \nu \nabla^2 \mathbf{v} \]
- Two independent Euler scaling symmetries ($c_{\nu}$ spatial, and $c_{\nu}$ temporal; red area in diagram below left) $\rightarrow$ one combined space-time scaling symmetry in Navier-Stokes equations (red line in diagram below right)
- Presence of turbulence yields additional constraint by constant energy transfer rate $\epsilon'' = \epsilon = \text{cons}$
  - $0 = c_{\nu} = 2c_x - 3c_t$, leading to the conclusions:
  - No scaling symmetry possible

- Conclusion: each parameterization for Euler equations that adds additional constraint on scaling factors would break space-time scaling symmetry
- All parameterizations must follow the same Kolmogorov-like scaling symmetry