A Scale Invariance Criterion for Geophysical Fluids

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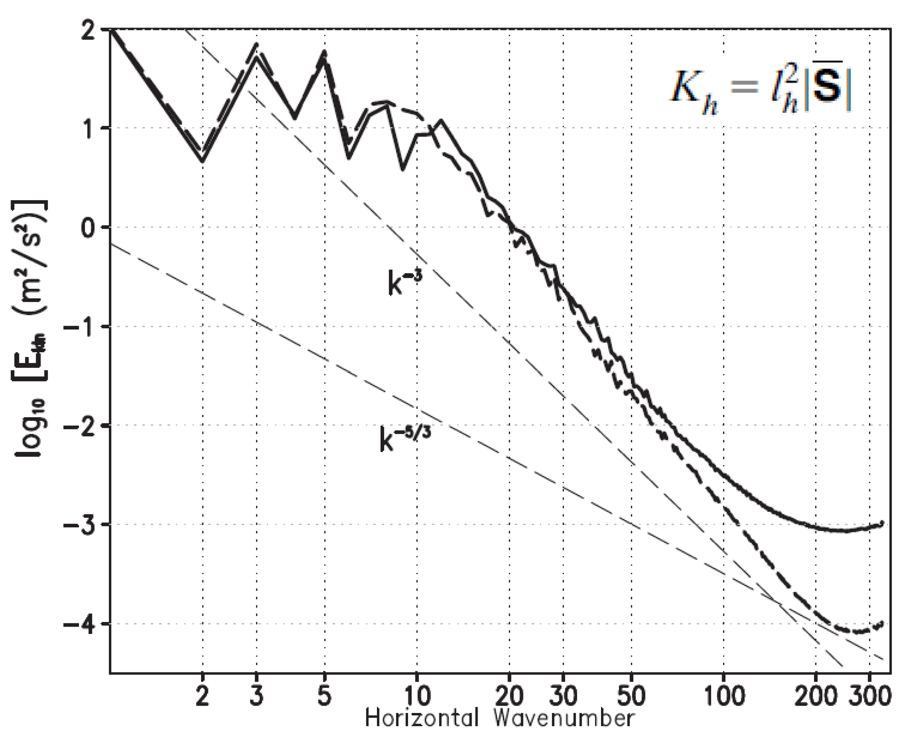
1. Motivation

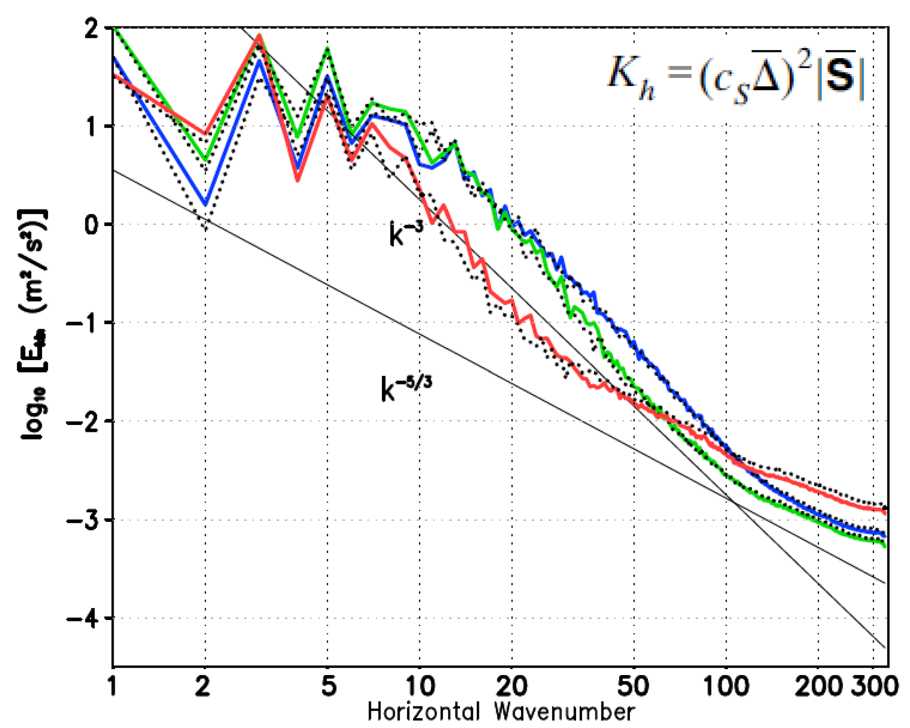
Kinetic energy spectra in our high-resolution (n_{cut}=330) CGM: Classic Smagorinsky model vs. Dynamic Smagorinsky model (DSM)



Left: Classic Smagorinsky model ($l_h = const.$) without higher-order terms exhibits *accumulation of energy*

Right: DSM allows for *continuing -5/3 slope* without higher-order terms

Question: How can we explain the difference between Smagorinsky model and DSM? Can we use the answer to improve parameterizations?



Answer (Oberlack, 1997): Classic Smagorinsky model cannot capture near-wall scaling laws and violates scale invariance

- ➢ Consider mathematical consistencies of parameterizations
  - Represented by mathematical properties (e.g. invariances) for set of equations
  - Examples: Invariances with respect to time, translation, rotation, scaling …

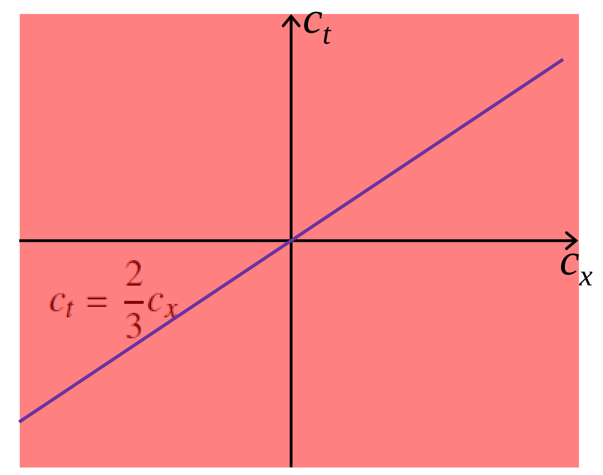
2. Parameter Space of Scaling Factors

Visualization of scale invariance with scaling factors c_a, defined by transformation $a^* = e^{c_a} a.$

- Comparison of Navier-Stokes and Euler equations: Difference due to molecular viscosity

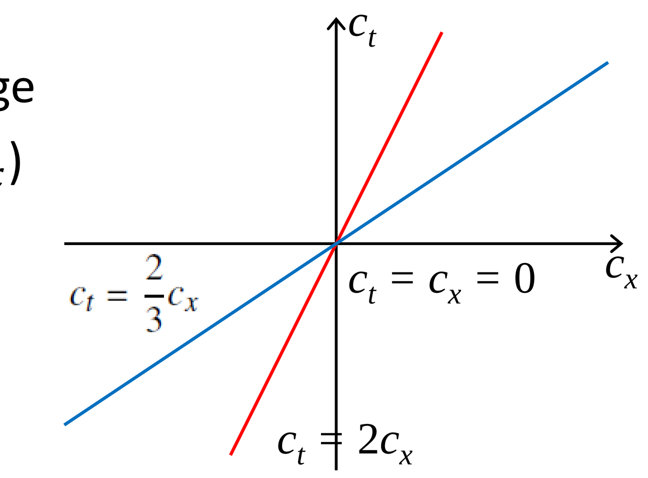
$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{\nabla p}{\rho} + \nu\nabla^2\mathbf{v}$$

- Two independent Euler scaling symmetries (c_x, spatial, and c_t, temporal; red area in diagram below left) ↔ one combined space-time scaling symmetry in Navier-Stokes equations (red line in diagram below right)
- Presence of turbulence yields additional constraint by constant energy transfer rate $\epsilon^* = \epsilon = \text{const.}$
  → $0 = c_\epsilon = 2c_x - 3c_t$ (symmetry breaking, denoted by intersection of blue line with red area/line below)



Left: With turbulence, two Euler symmetries merge to a single space-time scaling symmetry ($c_t = \frac{2}{3}c_x$)

Right: Breaking of Navier-Stokes symmetry due to turbulence leads to $c_t = c_x = 0$
→ no scaling symmetry possible



- Consequence: each parameterization for Euler equations that adds additional constraint on scaling factors would break space-time scaling symmetry

➢ All parameterizations must follow the same Kolmogorov-like scaling symmetry

3. Scale invariance criterion

General formulation (Schaefer-Rolffs *et al.*, 2015)

- For a symmetry transformation of an equation of motion of a scalar a the mathematical structure must be retained under transformation

$$\partial_t a + (\mathbf{v}\cdot\nabla)a = \mathcal{F}_a(t, x_i, a, b_1, b_2, \dots)$$
$$\Leftrightarrow\ \partial_{t^*} a^* + (\mathbf{v}^*\cdot\nabla^*)a^* = \mathcal{F}_a(t^*, x_i^*, a^*, b_l^*)$$

- Application of Kolmogorov-like scaling

$$t^* = e^{\frac{2}{3}c_x}t,\ x_i^* = e^{c_x}x_i,\ v_i^* = e^{\frac{1}{3}c_x}v_i,\ a^* = e^{c_a}a,\ b_l^* = e^{c_{b_l}}b_l,$$

➢ A criterion (in the red box) can be derived as follows

$$\partial_{t^*} a^* + (\mathbf{v}^*\cdot\nabla^*)a^* = \mathcal{F}_a(t^*, x_i^*, a^*, b_l^*)$$
$$e^{c_a - c_t}\partial_t a + e^{c_a - c_t}(\mathbf{v}\cdot\nabla)a = \mathcal{F}_a(e^{c_t}t, e^{c_x}x_i, e^{c_a}a, e^{c_{b_l}}b_l)$$
$$\partial_t a + (\mathbf{v}\cdot\nabla)a = \boxed{e^{\frac{2}{3}c_x - c_a}\mathcal{F}_a(e^{\frac{2}{3}c_x}t, e^{c_x}x_i, e^{c_a}a, e^{c_{b_l}}b_l) \overset{!}{=} \mathcal{F}_a(t, x_i, a, b_l)}$$

- It can be used for each term individually, because scaling is linear

Criterion for quasi-geostrophic regime (Schaefer-Rolffs, 2019)

- Coriolis term significant $\mathbf{f} = f\ \mathbf{e}_z = 2\Omega_E \sin\phi\ \mathbf{e}_z$
- Scaling includes a rotation defined by

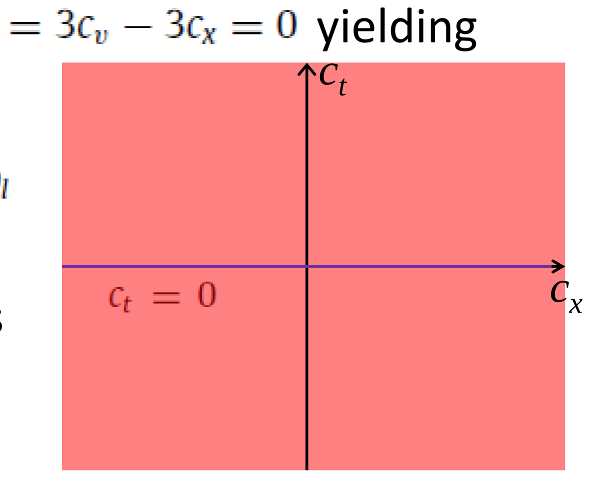
$$A_{ij}[t] = (1 - \cos(\Omega_E t))\, n_i n_j + \cos(\Omega_E t)\, \delta_{ij} + \sin(\Omega_E t)\, \epsilon_{ikj} n_k$$

➢ QG scaling:

$$t^* = e^{c_t}t;\quad x_i^* = e^{c_x}A_{ij}[(1 - e^{c_t})t]x_j;$$
$$v_i^* = e^{c_x - c_t}A_{ij}[(1 - e^{c_t})t]v_j;\quad a^* = e^{c_a}a;\quad b_l^* = e^{c_{b_l}}b_l$$

- Enstrophy cascade with $\eta^* = \eta$, such that $c_\eta = 3c_v - 3c_x = 0$ yielding $c_v = c_x$ and $c_t = 0$ (blue line)

→ $t^* = t;\ x_i^* = e^{c_x}x_i;\ v_i^* = e^{c_x}v_i;\ a^* = e^{c_a}a;\ b_l^* = e^{c_{b_l}}b_l$



➢ The scale invariance criterion for the Euler equations with quasi-geostrophic motion is

$$\boxed{e^{-c_a}\mathcal{F}_a(t, e^{c_x}x_i, e^{c_a}a, e^{c_{b_l}}b_l) = \mathcal{F}_a(t, x_i, a, b_l)}$$

Criterion for anelastic regime (Schaefer-Rolffs, 2019)

- Decomposition of temperature, pressure, and density in primitive equations

$$\begin{aligned}\rho &= \overline{\rho}(z) + \rho'(\mathbf{x}, t),\\ p &= \overline{p}(z) + p'(\mathbf{x}, t),\\ T &= \overline{T}(z) + T'(\mathbf{x}, t),\end{aligned}\quad \rightarrow\quad \begin{aligned} d_z\overline{p} &= -g\overline{\rho} \quad \partial_z\left(\frac{p'}{\overline{\rho}}\right) = -g\frac{\rho'}{\overline{\rho}},\\ D_t\mathbf{u} &= -\frac{\nabla p'}{\overline{\rho}}, \quad \nabla\cdot\mathbf{u} = -\frac{1}{\overline{\rho}}\partial_z(\overline{\rho}w),\end{aligned}\quad D_t T' = -\frac{w\overline{T}}{g}N^2 + \frac{wg}{C_p}\frac{\rho'}{\overline{\rho}}$$

- Assuming constant energy cascade, hence $c_t = \frac{2}{3}c_x$
- Entangled scaling factors, finally leading to

$$\begin{aligned} c_v &= c_x/3,\\ c_z = c_{\overline{p}} - c_{\overline{\rho}} = c_{\overline{T}} = c_{p'} - c_{\overline{\rho}} = c_{T'} &= 2c_x/3 = 2c_v\\ c_w = c_{\rho'} - c_{\overline{\rho}} &= 0,\\ c_N &= -c_x/3.\end{aligned}$$

- Scale invariance criterion remains *as in general case*,

$$\mathcal{G}_a(t, x_i, a, b_l) \equiv \boxed{e^{\frac{2}{3}c_x - c_a}\mathcal{F}_a(e^{\frac{2}{3}c_x}t, e^{c_x}x_i, e^{c_a}a, e^{c_{b_l}}b_l) = \mathcal{F}_a(t, x_i, a, b_l)}$$

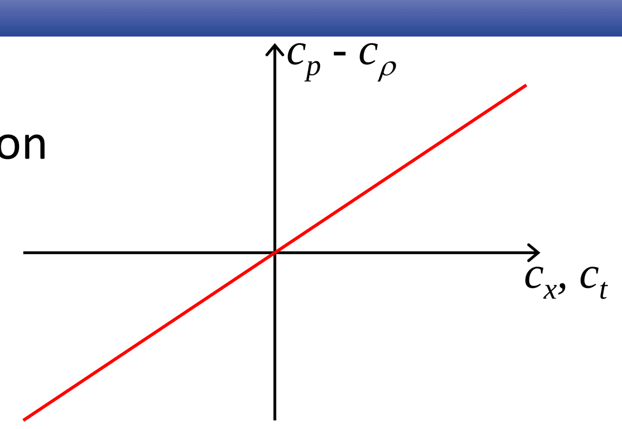
- Further: scaling of z in accordance with aspect ratio of stratified turbulence, ($z \sim x^{1/3}/N$, Lindborg, 2006) $c_z = c_x/3 - c_N = 2c_x/3$

4. Applications

General formulation

Example: Pressure gradient in Euler equation

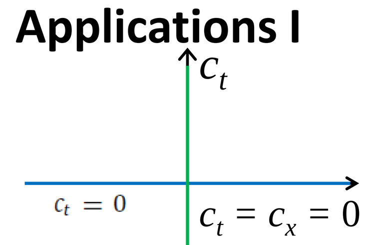
$$\mathcal{G}_\mathbf{v} = -e^{\frac{1}{3}c_x}\frac{\nabla^* p*}{\rho^*} = -e^{-\frac{2}{3}c_x + c_p - c_\rho}\frac{\nabla p}{\rho}$$
$$\rightarrow\quad c_p - c_\rho = \tfrac{2}{3}c_x = 2c_v$$



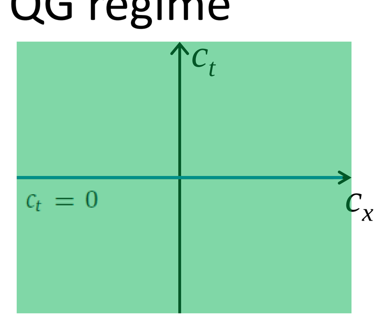
➢ Extension of the parameter space, no constraint in the existing space

Lindborg, 2006: The energy cascade in a strongly stratified fluid. *J. Fluid Mech.*, **550**, 207–242
Oberlack, 1997: Invariant modeling in large-eddy simulation of turbulence. – In: *Annual Research Briefs, Center for Turbulence Research*, Stanford, 3–22.
Schaefer-Rolffs, Knöpfel, Becker, 2015: A scale invariance criterion for LES parametrizations, *Met.Z.* **24**, 3-14, 2015
Schaefer-Rolffs, 2019: The scale invariance criterion for geophysical fluids, *Eur. J. Mech. B Fluids* **74**, 92-98, 2019

Applications I



QG regime

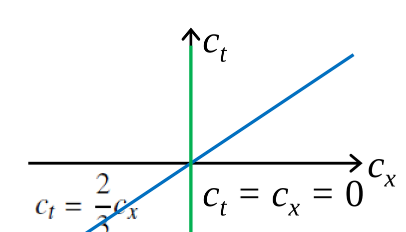


The classic Smagorinsky parameterization adds a new constraint $c_x = 0$ (green line) to the Euler equations

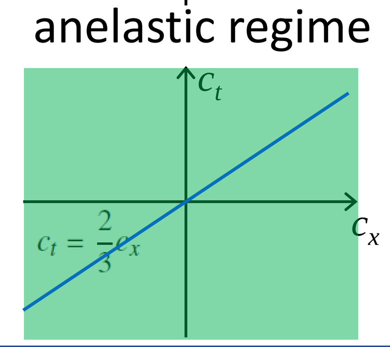
➢ Breaking of scale invariance

The DSM does not add any constraint (denoted by green area) to the scaling

➢ Scale invariance preserved



anelastic regime



Applications II

- Criterion also applicable for a passive tracer equation $\partial_t C + (\mathbf{v}\cdot\nabla)C = \nabla(K_C\nabla C)$
- Application to vertical diffusion in anelastic horizontal momentum equation

$$\mathcal{G}_\mathbf{u} = e^{\frac{1}{3}c_x}\frac{1}{\rho^*}\partial_{z^*}(\rho^* K_z^*\partial_{z^*}\mathbf{u}^*) = e^{\frac{2}{3}c_x - 2c_z + c_{K_z}}\frac{1}{\rho}\partial_z(\rho K_z\partial_z\mathbf{u}) \overset{!}{=} \mathcal{F}_\mathbf{u} \quad\rightarrow\quad c_{K_z} = 2c_z - \frac{2}{3}c_x$$

- From $K_z = l_z^2|\partial_z\mathbf{u}|$ it follows $c_{K_z} = 2c_{l_z} - c_z + c_x/3$ and $c_{l_z} = \frac{3c_z - c_x}{2}$
- Finally, due to $c_z = 2c_x/3$, we have $c_{l_z} = c_x/2$, leading to the conclusions:

➢ a vertical mixing length cannot be constant to ensure scale invariance

➢ one possible realization might be $l_z = \sqrt{l_0\, l_h}$, where l_0 is a constant