# Turbulent cascade in the solar wind on ion scales

## Prelude: Compressible Kármán-Howart-Monin law

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EGU2020-8738, EGU 2020

# Turbulence in the solar wind

Solar wind

- large-amplitude turbulent fluctuations
- energy cascade
- transition on ion scales
  - onset of Hall cascade (Hellinger et al., 2018; Bandyopadhyay et al., 2020)?
  - ion energization (Yang et al., 2019)?
  - increase of compressibility (Andrés et al., 2019)?

$\Rightarrow$ compressible Kármán-Howart-Monin (KHM) approach

The approach of Galtier and Banerjee (2011), . . . , Andrés et al. (2018) includes the isothermal internal energy. This is in many respects problematic.

Here we investigate KHM and coarse-graining approaches in compressible HD for the kinetic energy.

We also show preliminary results for the KHM equation in compressible Hall MHD turbulence.

# Statistical Approach – Structure Functions

Statistically homogeneous, incompressible turbulence (de Karman and Howarth, 1938; Kolmogorov, 1941; Monin and Yaglom, 1975; Frisch, 1995)

$$\underbrace{\frac{\partial S^{(i)}}{\partial t}}_{\text{decay}} + \underbrace{\nabla_{\boldsymbol{l}} \cdot \boldsymbol{Y}^{(i)}}_{\text{cascade}} = \underbrace{2\nu \Delta_{\boldsymbol{l}} S^{(i)}}_{\text{dissipation}} - \underbrace{4\epsilon}_{\text{dissip. rate}} \quad (1)$$

$$\delta \boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x} + \boldsymbol{l}) - \boldsymbol{u}(\boldsymbol{x}) \quad S^{(i)} = \langle |\delta \boldsymbol{u}|^2 \rangle \quad \boldsymbol{Y}^{(i)} = \left\langle \delta \boldsymbol{u} |\delta \boldsymbol{u}|^2 \right\rangle \quad \epsilon = \nu \langle \nabla \boldsymbol{u} : \nabla \boldsymbol{u} \rangle$$

Inertial range: $\nabla_{\boldsymbol{l}} \cdot \boldsymbol{Y}^{(i)} = -4\epsilon$, i.e., cascade rate = dissipation rate

“Exact law” (Re$\rightarrow \infty$, isotropic medium)

$$\left\langle \delta u_l |\delta \boldsymbol{u}|^2 \right\rangle = -\frac{4}{3}\epsilon l$$

# Compressible HD

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0, \tag{2}$$

$$\frac{\partial (\rho \boldsymbol{u})}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}\boldsymbol{u}) = -\nabla p + \boldsymbol{\nabla} \cdot \boldsymbol{\tau}, \tag{3}$$

$$\frac{\partial T}{\partial t} + (\boldsymbol{u} \cdot \nabla) T = \alpha \Delta T + (\gamma - 1) \frac{1}{\rho} (-p\theta + \boldsymbol{\Sigma} : \boldsymbol{\tau}) \tag{4}$$

viscous stress tensor: $\boldsymbol{\tau} = \mu \left( \boldsymbol{\Sigma} + \boldsymbol{\Sigma}^t - \frac{2}{3} \boldsymbol{I} \theta \right)$

dilatation: $\theta = \boldsymbol{\nabla} \cdot \boldsymbol{u}$

stress tensor: $\boldsymbol{\Sigma} = \nabla \boldsymbol{u}$

# Compressible KHM for the kinetic energy

Structure-function energy conservation equation (Hellinger et al., 2020):

$$\underbrace{\frac{\partial S}{\partial t}}_{\text{decay}} + \underbrace{\nabla_{\boldsymbol{l}} \cdot \boldsymbol{Y} + R}_{\text{cascade}} = \underbrace{C_p - C_\tau}_{\text{corrections}} + \underbrace{2\left\langle \delta p \delta \theta \right\rangle}_{\text{pressure dilatation}} \underbrace{- 2\left\langle \delta \boldsymbol{\tau} : \delta \boldsymbol{\Sigma} \right\rangle}_{\text{dissipation}}, \qquad (5)$$

$$S = \left\langle \delta \boldsymbol{u} \cdot \delta\left(\rho \boldsymbol{u}\right) \right\rangle \quad \boldsymbol{Y} = \left\langle \delta \boldsymbol{u} \left[\delta\left(\rho \boldsymbol{u}\right) \cdot \delta \boldsymbol{u}\right] \right\rangle \quad R = \left\langle \delta \boldsymbol{u} \cdot \left(\theta' \rho \boldsymbol{u} - \theta \rho' \boldsymbol{u}'\right) \right\rangle$$

$S$ and $\boldsymbol{Y}$ are compressible equivalents of $S^{(i)}$ and $\boldsymbol{Y}^{(i)}$.

$R$ represents an additional, compressible energy transfer.

$$C_p = \mathcal{C}\left[\boldsymbol{u}, \nabla p\right] \quad C_\tau = \mathcal{C}\left[\boldsymbol{u}, \nabla \cdot \boldsymbol{\tau}\right] \quad \mathcal{C}\left[\boldsymbol{a}, \boldsymbol{b}\right] = \left(\frac{\rho'}{\rho} - 1\right) \boldsymbol{a}' \cdot \boldsymbol{b} + \left(\frac{\rho}{\rho'} - 1\right) \boldsymbol{a} \cdot \boldsymbol{b}'$$

# Coarse-graining approach

Filtered energy conservation in compressible HD (Eyink and Aluie, 2009; Aluie, 2011, 2013)

$$\underbrace{\frac{\partial\langle\mathcal{E}_\ell\rangle}{\partial t}}_{\text{decay}} + \underbrace{\langle\Pi_\ell + \Lambda_\ell\rangle}_{\text{cascade}} - \underbrace{\langle\overline{p}_\ell\nabla\cdot\overline{\boldsymbol{u}}_\ell\rangle}_{\text{pressure dilatation}} + \underbrace{\langle D_\ell\rangle}_{\text{dissipation}} = 0 \tag{6}$$

$$\mathcal{E}_\ell = \frac{1}{2}\overline{\rho}_\ell|\tilde{\boldsymbol{u}}_\ell|^2 \quad \Pi_\ell = -\overline{\rho}_\ell\nabla\tilde{\boldsymbol{u}}_\ell : (\widetilde{\boldsymbol{u}\boldsymbol{u}}_\ell - \tilde{\boldsymbol{u}}_\ell\tilde{\boldsymbol{u}}_\ell) \quad \Lambda_\ell = (\tilde{\boldsymbol{u}}_\ell - \overline{\boldsymbol{u}}_\ell)\cdot\nabla\overline{p}_\ell$$

$$D_\ell = \nabla\tilde{\boldsymbol{u}}_\ell : \overline{\boldsymbol{\tau}}_\ell \qquad \overline{a}_\ell(\boldsymbol{x}) = \int_V G_\ell(\boldsymbol{r})a(\boldsymbol{x}+\boldsymbol{r})\mathrm{d}^3\boldsymbol{r} \quad \tilde{a}_\ell(\boldsymbol{x}) = \frac{\overline{\rho a}_\ell(\boldsymbol{x})}{\overline{\rho}_\ell(\boldsymbol{x})}$$

# Coarse-graining approach

Incompressible version of Eq. (6) (cf., Eyink and Aluie, 2009)

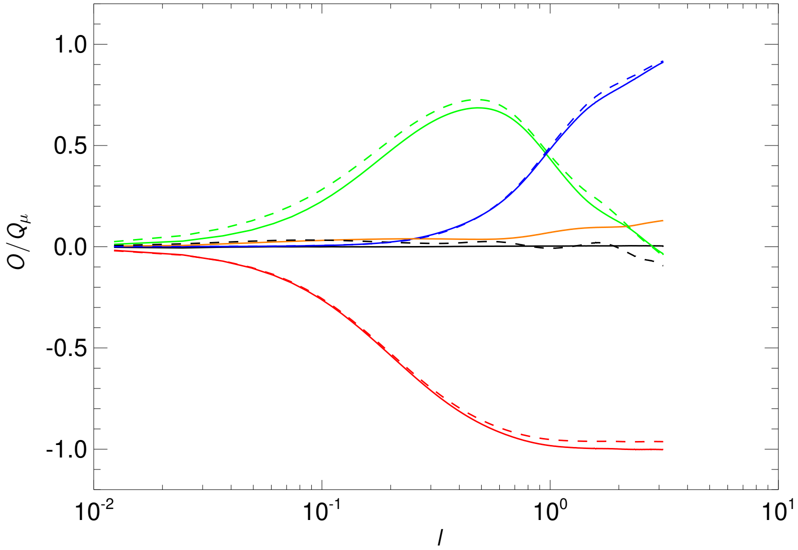
$$\underbrace{\frac{\partial \langle \mathcal{E}_\ell^{(i)} \rangle}{\partial t}}_{\text{decay}} + \underbrace{\langle \Pi_\ell^{(i)} \rangle}_{\text{cascade}} + \underbrace{\langle D_\ell^{(i)} \rangle}_{\text{dissipation}} = 0 \tag{7}$$

$$\mathcal{E}_\ell^{(i)} = \frac{1}{2}\rho_0 |\overline{\boldsymbol{u}}_\ell|^2 \qquad \Pi_\ell^{(i)} = -\rho_0 \nabla \overline{\boldsymbol{u}}_\ell : (\overline{\boldsymbol{u}\boldsymbol{u}}_\ell - \overline{\boldsymbol{u}}_\ell \overline{\boldsymbol{u}}_\ell) \qquad D_\ell^{(i)} = \mu \nabla \overline{\boldsymbol{u}}_\ell : \nabla \overline{\boldsymbol{u}}_\ell$$

# 3D compressible decaying HD simulation

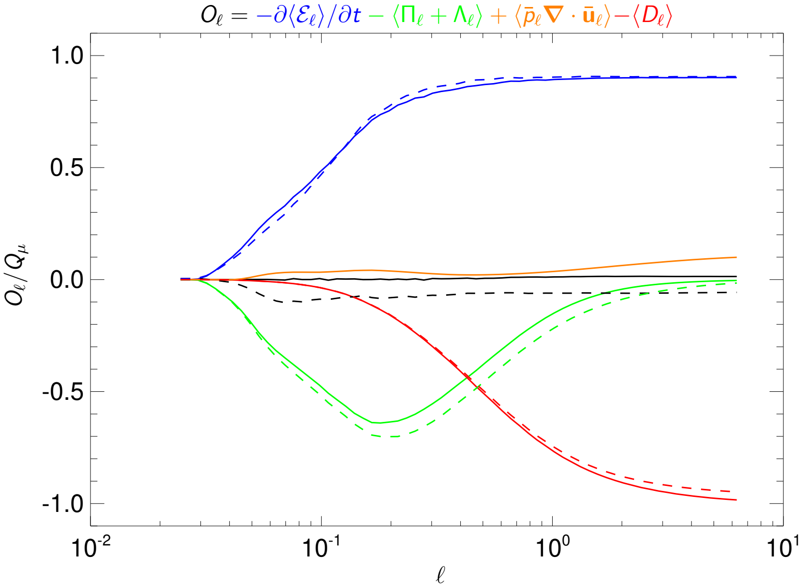
- grid $1024^3$
- size $(2\pi)^3$
- solenoidal init.
- $M = 1$
- $\alpha = \mu = 2.8\ 10^{-3}$
- $\gamma = 5/3$
- $Q_\mu = \langle \boldsymbol{\Sigma} : \boldsymbol{\tau} \rangle$
- Dashed lines: the incompressible equivalents

$$O = -\frac{1}{4}(2\langle \delta\boldsymbol{\tau} : \delta\boldsymbol{\Sigma}\rangle + C_\tau) + \frac{1}{4}(2\langle \delta p \delta\theta \rangle + C_p) - \frac{1}{4}(\boldsymbol{\nabla}\cdot\mathbf{Y} + R) - \frac{1}{4}\frac{\partial S}{\partial t}$$



# 3D compressible HD simulation – Coarse graining

- grid $1024^3$
- size $(2\pi)^3$
- solenoidal init.
- $M = 1$
- $\alpha = \mu = 2.8\ 10^{-3}$
- $\gamma = 5/3$
- $Q_\mu = \langle \boldsymbol{\Sigma} : \boldsymbol{\tau} \rangle$
- Dashed lines: the incompressible equivalents



# Compressible KHM for the internal energy

Following Galtier and Banerjee (2011) one can take $\langle \delta\rho\delta e \rangle$ to represent the internal energy. From Eq. temperature one gets (Hellinger et al., 2020)

$$\frac{\partial S_e}{\partial t} + \nabla_{\boldsymbol{l}} \cdot \boldsymbol{Y}_e + R_e = \alpha \left\langle \delta\rho\delta\left(\Delta e\right)\right\rangle - \mathcal{D}\left(p\theta\right) + \mathcal{D}\left(\boldsymbol{\tau} : \boldsymbol{\Sigma}\right), \qquad (8)$$

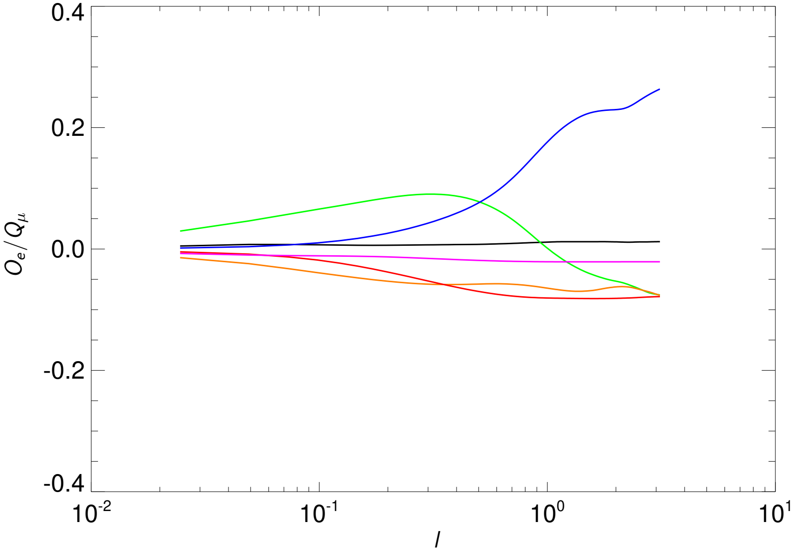
where

$$e = T/(\gamma - 1) \quad S_e = \langle \delta\rho\delta e \rangle \quad \boldsymbol{Y}_e = \langle \delta\boldsymbol{u}\delta\rho\delta e \rangle \quad R_e = \langle \delta e \rho \nabla' \cdot \boldsymbol{u}' - \rho' \delta e \nabla \cdot \boldsymbol{u} \rangle$$

# 3D compressible HD simulation – internal energy

- grid $1024^3$
- size $(2\pi)^3$
- solenoidal init.
- $M = 1$
- $\alpha = \mu = 2.8\ 10^{-3}$
- $\gamma = 5/3$
- $Q_\mu = \langle \boldsymbol{\Sigma} : \boldsymbol{\tau} \rangle$

$$O_e = \frac{1}{2}\mathcal{D}\left(\boldsymbol{\tau} : \boldsymbol{\Sigma}\right) - \frac{1}{2}\left(\boldsymbol{\nabla} \cdot \mathbf{Y}_e + R_e\right) + \frac{1}{2}\alpha \left\langle \delta\rho\delta\left(\boldsymbol{\Delta} e\right)\right\rangle - \frac{1}{2}\mathcal{D}\left(p\theta\right) - \frac{1}{2}\frac{\partial S_e}{\partial t}$$



# Solar wind turbulence

Incompressible Hall-MHD (Politano and Pouquet, 1998; Galtier, 2008; Hellinger et al., 2018; Ferrand et al., 2019)

$$\underbrace{\frac{\partial S^{(i)}}{\partial t}}_{\text{decay}} + \underbrace{\nabla \cdot \boldsymbol{Y}^{(i)}}_{\text{MHD cascade}} + \underbrace{\nabla \cdot \boldsymbol{H}^{(i)}}_{\text{Hall cascade}} = \underbrace{D^{(i)}}_{\text{dissipation}}$$

$$S^{(i)} = S_b^{(i)} + S_u^{(i)} \quad S_u^{(i)} = \langle |\delta \boldsymbol{u}|^2 \rangle \quad S_b = \langle |\delta \boldsymbol{b}|^2 \rangle$$

$$\boldsymbol{Y}^{(i)} = \left\langle \delta \boldsymbol{u} \, |\delta \boldsymbol{u}|^2 + \delta \boldsymbol{u} \, |\delta \boldsymbol{b}|^2 - 2\delta \boldsymbol{b} \, (\delta \boldsymbol{u} \cdot \delta \boldsymbol{b}) \right\rangle$$

$$\boldsymbol{H}^{(i)} = \left\langle \delta \boldsymbol{b} \, (\delta \boldsymbol{b} \cdot \delta \boldsymbol{j}) - \frac{1}{2} \delta \boldsymbol{j} \, |\delta \boldsymbol{b}|^2 \right\rangle$$

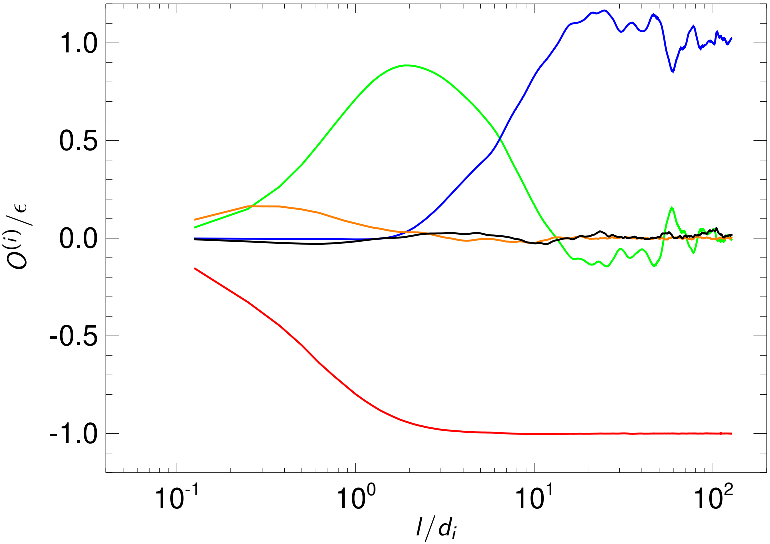
$$D^{(i)} = 2\nu \Delta S_u^{(i)} + 2\eta \Delta S_b^{(i)} - 4\epsilon$$

$$\epsilon = \nu \langle \nabla \boldsymbol{u} : \nabla \boldsymbol{u} \rangle + \eta \langle \nabla \boldsymbol{b} : \nabla \boldsymbol{b} \rangle \quad \boldsymbol{b} = \boldsymbol{B} / \sqrt{\mu_0 \rho} \quad \boldsymbol{j} = \boldsymbol{u}_p - \boldsymbol{u}_e$$

# 2D Hall-MHD simulation vs incompressible KHM equation

- $\boldsymbol{B}_0$ along $z$
- $x$-$y$ grid $2048^2$
- $\Delta x = \Delta y = d_i/8$
- size $256^2 d_i^2$
- Alfvénic fluctuations $\delta B_{rms}/B_0 = 0.17$ for $kd_i < 0.2$
- zero cross-helicity
- $\beta = 0.5$
- $\eta = \nu = 10^{-3}$

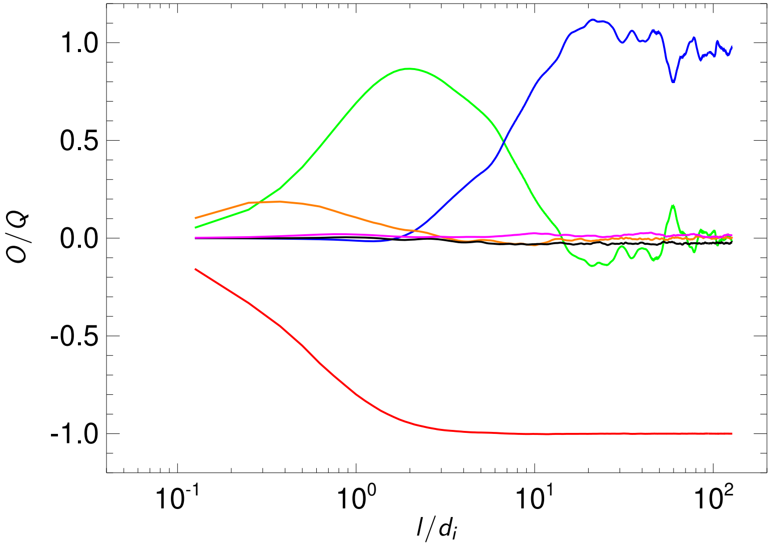
$$O^{(i)} = \frac{1}{4}\left(D^{(i)} - \nabla \cdot \mathbf{H}^{(i)} - \nabla \cdot \mathbf{Y}^{(i)} - \partial S^{(i)}/\partial t\right)$$



# 2D Hall-MHD simulation vs compressible KHM equation

- $\boldsymbol{B}_0$ along $z$
- $x$-$y$ grid $2048^2$
- $\Delta x = \Delta y = d_i/8$
- size $256^2 d_i^2$
- Alfvénic fluctuations $\delta B_{rms}/B_0 = 0.17$ for $kd_i < 0.2$
- zero cross-helicity
- $\beta = 0.5$
- $\eta = \nu = 10^{-3}$

$$O = \frac{1}{4}\left(D + 2\langle \delta p \delta \theta \rangle - K_{\mathrm{Hall}} - K_{\mathrm{MHD}} - \partial S/\partial t\right)$$



# Summary

Compressible HD:

- Energy conservation Eqs. (5,6) well satisfied in the simulation
- Possible inertial range "exact laws"

$$\nabla_{\boldsymbol{l}} \cdot \boldsymbol{Y} + R = -4Q_{\mu} \quad \text{or} \quad \langle \Pi_{\ell} + \Lambda_{\ell} \rangle = -\frac{1}{2}\frac{\partial \langle \rho |\boldsymbol{u}|^2 \rangle}{\partial t}$$

$\Rightarrow$ estimation of heating/cascade rate

- $S_e$ decreases wheareas $\rho e$ increases in the simulation $\Rightarrow$ $S_e$ does not represent well the internal energy. Inclusion of $S_e$ ($\boldsymbol{Y}_e$) to the KHM equation is questionable.

Compressible Hall MHD:

- Compressible Hall MHD KHM equation is better conserved (compared to the incompressible one) in a weakly compressible simulation.

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