Quasi-three-dimensional simulation of crescent-shaped waves

A. Dosaev, Yu. Troitskaya

dosaev@appl.sci-nnov.ru

Institute of Applied Physics of the Russian Academy of Sciences

Modeling potential deep water waves

- To integrate the governing equations

$$\begin{aligned} \eta_t + \eta_x\varphi_x + \eta_q\varphi_q - \varphi_y &= 0, \\ \varphi_t + g\eta + \frac{1}{2}|\nabla\varphi|^2 &= 0 \quad (\text{at } y = \eta), \\ \Delta\varphi &= 0, \\ |\nabla\varphi| \to 0 \quad &\text{as} \quad y \to -\infty. \end{aligned}$$

  we need to compute $\nabla\varphi$ on the free surface from a known distribution of φ.
- For 2D flows ($\partial/\partial q = 0$) powerful techniques based on conformal mapping were developed. The free surface is mapped onto a straight line via a conformal transform $x + iy = Z(u + iv, t)$, and derivatives of the "complex potential" $\Psi(u) = \varphi(u) + i\psi(u) = (1 + i\hat{H})\varphi(u)$ are then easily computed.
- If one extends this conformal mapping approach to 3D flows, i.e.

$$x + iy = Z(u + iv, \boxed{q,} t),$$

  while keeping only first order 3D corrections to the exact 2D governing equations (in a small parameter $\epsilon = (l_x/l_q)^2 \ll 1$, changes along the q-axis are assumed to be slow)...
  ...**What 3D effects would such a model retain?**

Quasi-three-dimensional wave model

- Equations of motion can be derived from the principle of least action with the Hamiltonian

$$\mathcal{H} = \frac{g}{2}\int \eta^2 \; dx \; dq + \mathcal{K}.$$

However, in a 3D problem a compact expression for the kinetic energy functional $\mathcal{K}$ in terms of Z and Ψ is not known.

- Exact 2D + first order 3D corrections (*Ruban, 2005*):

$$\mathcal{K} = \frac{1}{2}\int dx \; dq \int_{-\infty}^{\eta} (\varphi_x^2 + \varphi_q^2 + \varphi_y^2) \; dy = -\frac{1}{2}\int \varphi \hat{H} \varphi_u \; du \; dq + \mathcal{F}$$

where

$$\mathcal{F} = \frac{i}{8}\int (Z_u\Psi_q - Z_q\Psi_u)\hat{G}\overline{(Z_u\Psi_q - Z_q\Psi_u)} \; du \; dq \; +$$

$$+ \frac{i}{16}\int \left\{ [(Z_u\Psi_q - Z_q\Psi_u)^2/Z_u] \; \hat{E}\overline{(Z-u)} - (Z-u) \; \hat{E}\overline{[(Z_u\Psi_q - Z_q\Psi_u)^2/Z_u]} \right\} \; du \; dq.$$

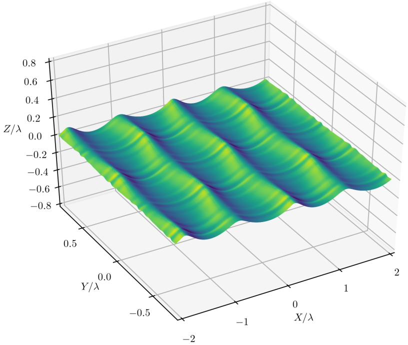
- Regularization (*Ruban, 2010*) restores correct linear dispersion relation on a whole wavenumber plane:

$$G(k,m) = \frac{-2i}{\sqrt{m^2+k^2}+|k|}, \qquad E(k,m) = \frac{2|k|}{\sqrt{m^2+k^2}+|k|},$$

where k, m are wavenumbers in the x and q directions.

Crescent-shaped waves

- Formation of crescent-shaped waves is an essentially 3D effect (*Su, 1982*) originating from an instability of Stokes wave to three-dimensional perturbations (*McLean, 1982*).
- If we actually run a simulation, we also observe a fast growth of another instability giving rise to perpendicular ripples:



That spurious instability only manifests itself for the regularized model and is caused by cubic terms in $\mathcal{H}$:

$$\frac{i}{16}\int\left\{\Psi_q^2\hat{E}\overline{(Z-u)}-(Z-u)\hat{E}\overline{\Psi_q^2}\right\}\,du\,dq.$$

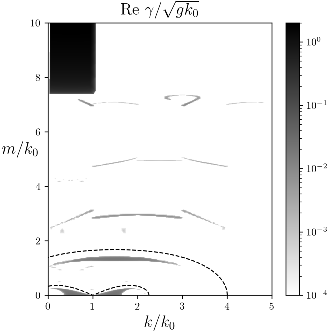
Zones of instability of Stokes wave

The spurious instability can be mitigated by masking the problematic terms in the neighbourhood of the q-axis:

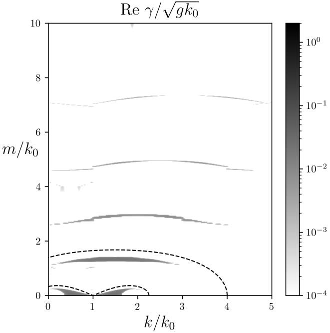
$$
\begin{aligned}
[(Z_u\Psi_q - Z_q\Psi_u)^2/Z_u]\ \hat{E}\overline{(Z-u)} \quad &\rightarrow \quad [(\hat{M}(Z_u\Psi_q - Z_q\Psi_u))^2/Z_u]\ \hat{E}\overline{(Z-u)} \\
(Z-u)\ \hat{E}\overline{[(Z_u\Psi_q - Z_q\Psi_u)^2/Z_u]} \quad &\rightarrow \quad (Z-u)\ \hat{E}\overline{[(\hat{M}(Z_u\Psi_q - Z_q\Psi_u))^2/Z_u]}
\end{aligned}
$$

where

$$M(k,m) = 1 - \left|\frac{m}{\sqrt{k^2+m^2}}\right|^p.$$



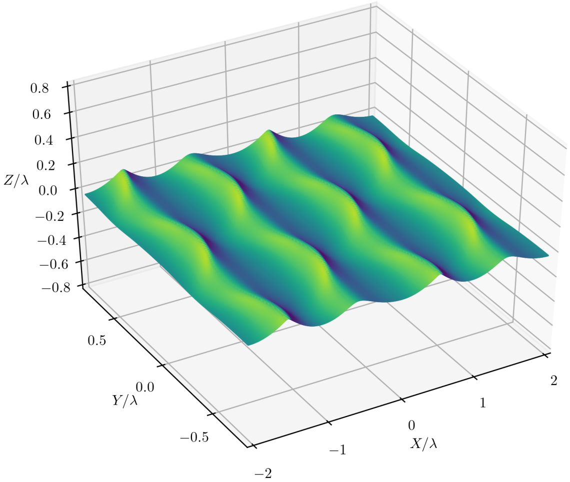
original model



modified

Crescent-shaped waves

The modified quasi-three-dimensional model exhibits a plausible dynamics leading to formation of crescent-shaped waves:



Increment of instability

Maximum increments of instability for the zone correspoding to 5-wave interactions (solid line) are reasonably close to the values from an exact 3D analysis (white markers, *McLean, 1982*):