The strange instability of the equatorial Kelvin wave

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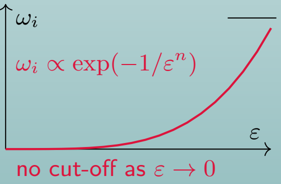
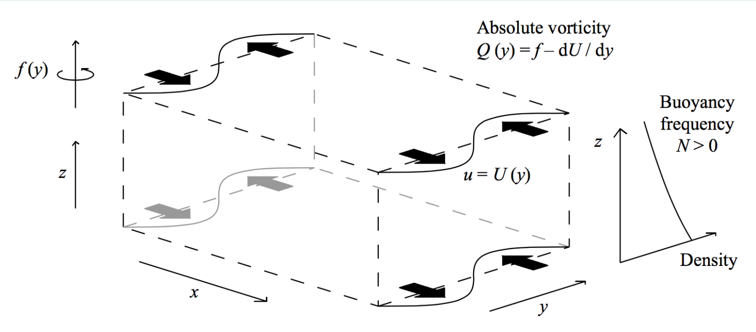
Overview

- Linear instability of shear flows (inviscid; rotating, stratified).
- Consider a (simple) basic state $\boldsymbol{u} = U(y)\hat{\boldsymbol{x}}$; seek small disturbances $\propto \exp i(kx - \omega t)$, and find $\omega = \omega(k, \text{flow parameters})$.
- Typically, instability appears at some critical non-zero parameter.

- Place an **equatorial Kelvin wave** in a shear flow, the strength of which is measured by a nondimensional parameter ε. There is strange behaviour when $\varepsilon \ll 1$, discovered by John Boyd. We are motivated by (and to some extent follow) a series of his papers:
  - Boyd (1981): Sturm–Liouville eigenproblems with an interior pole (J. Math. Phys.)
  - Boyd and Christidis (1982): Low wavenumber instability on the equatorial beta-plane (J. Atmos. Sci.)
  - Boyd & Natarov (1998): A Sturm–Liouville eigenproblem of the fourth kind: a critical latitude with equatorial trapping (Studies in App. Math.)
  - Natarov & Boyd (2001): Beyond-all-orders instability in the equatorial Kelvin wave (Dyn. Atmos. Oceans)



(Cartesian) Equations of motion



With $\boldsymbol{u} = (u, v, w)$ in (x, y, z) and $D/Dt = \partial/\partial t + \boldsymbol{u}\cdot\nabla$,

$$\frac{Du}{Dt} - fv = -\frac{1}{\bar{\rho}}\frac{\partial p}{\partial x}, \quad \frac{Dv}{Dt} + fu = -\frac{1}{\bar{\rho}}\frac{\partial p}{\partial y},$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} - g\rho, \quad \frac{D\rho}{Dt} = 0, \quad \nabla\cdot\boldsymbol{u} = 0.$$

Boussinesq
Hydrostatic

At rest, density $\rho = \bar{\rho} + \rho_0(z)$, and $N^2 = -g\rho_0'/\bar{\rho}$ (constant).

Linear disturbances for uniform shear

- Take $U(y) = \Lambda y$, $f = \beta y$, and disturbances $\propto \mathrm{e}^{ik(x-ct)+imz}$.
- Nondimensionalise lengths by $L_d = \sqrt{c_{\mathrm{gw}}/\beta}$, time by L_d/c_{gw}, (u, v) by c_{gw} and $p/\overline{\rho}$ by c_{gw}^2, where $c_{\mathrm{gw}} = \frac{N}{m}$ (gravity-wave speed):

$$(\varepsilon y - c)u + (\varepsilon - y)v + p = 0, \quad \text{(1a)}$$
$$\cancel{-k^2(\varepsilon y - c)}v + yu + p' = 0, \quad \text{(1b)}$$
$$(\varepsilon y - c)p + u + v' = 0, \quad \text{(1c)}$$

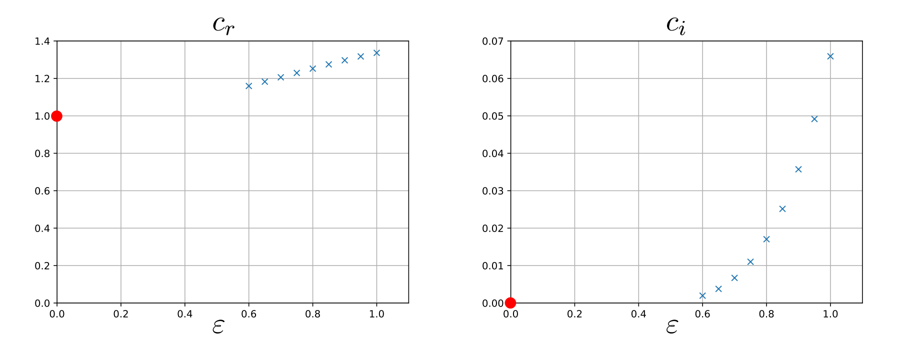
where $\varepsilon = \left(\frac{\Lambda^2}{\beta c_{\mathrm{gw}}}\right)^{1/2} \lesssim 1,$

$k = L_d k_{\mathrm{dim}} \lesssim 1.$

- When $k \ll 1$ (cf. Gill, 1980s), there is only one parameter, ε.
- In atmosphere, typically $\Lambda \approx 10\,\mathrm{m\,s^{-1}}/1000\,\mathrm{km} \approx 10^{-5}\,\mathrm{s}^{-1}$. With $c_{\mathrm{gw}} = 30\,\mathrm{m\,s^{-1}}$, find $\varepsilon \approx 0.4$ (but smaller for deeper modes).
- The $\varepsilon = 0$ Kelvin wave has $v = 0$, $u = p \propto \mathrm{e}^{-y^2/2}$, and $c = 1$.
- Seek solutions with $c \to 1$ as $\varepsilon \to 0$, and $c_i > 0$ (unstable: $k \in \mathbb{R}^+$).

Numerical solutions: standard shooting

Solve on $[-10, 10]$, with $v(\pm 10) = 0$. Shoot to equator with 4th-order Runge Kutta, and $\Delta y = 0.005$. Secant iteration.



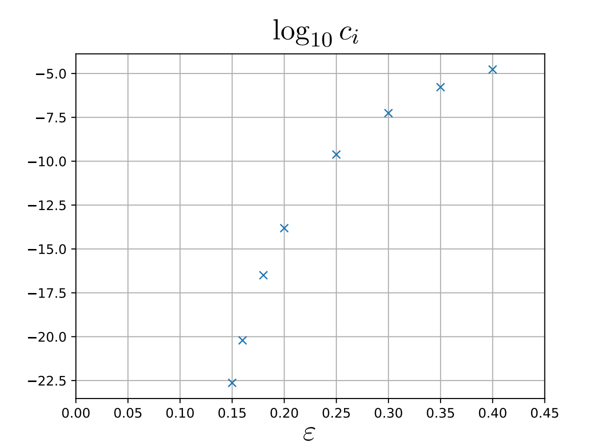
Here ● denotes the $\varepsilon = 0$ Kelvin wave.

But the scheme breaks down as ε decreases. Since $\mathrm{Im}(c) = c_i \ll 1$, there is a near singularity in the ODEs (at $y = c/\varepsilon \approx c_r/\varepsilon \approx 1/\varepsilon$).

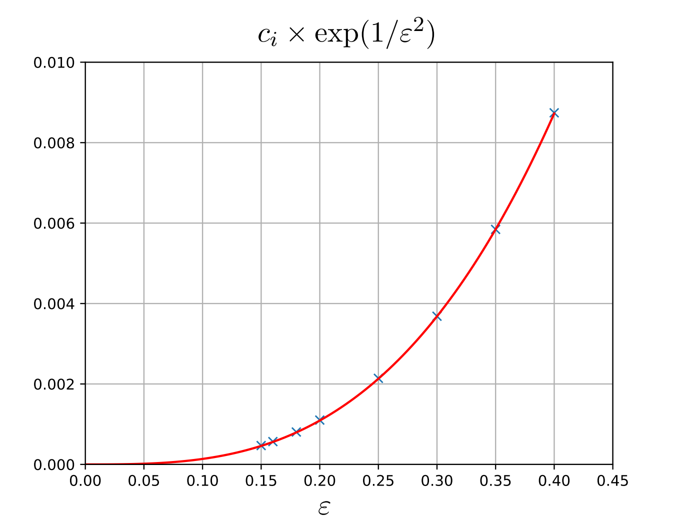
Note: already $\mathrm{Im}(c) = c_i = 0.00196$ at $\varepsilon = 0.6$.

Numerical solutions: high-precision shooting

- Shoot in complex plane, detouring below singularity at $y \approx 1/\varepsilon$ (cf. Boyd).
- Use arithmetic accurate to 25 decimal places (mpmath), and other tricks.
- Secant method converges after 5 iterations, in less than 1 minute (on a laptop).



Scaling of c_i as $\varepsilon \to 0$



The red line is $0.14\varepsilon^3$, so $c_i \approx 0.14\,\varepsilon^3 \exp(-1/\varepsilon^2)$ as $\varepsilon \to 0$. Natarov and Boyd suggested $c_i \propto \text{constant} \times \exp(-1/\varepsilon^2)$ as $\varepsilon \to 0$, but the growth is yet weaker!

Outlook

- How can this instablity be explained?
- Possible to make progress using asymptotics, but hard to extract c_i, which is exponentially small as $\varepsilon \to 0$ (and no insight!).
- Boyd showed importance of critical latitude for instability to exist.
- Mechanism probably involves two-way interaction between equatorial wave and critical layer:
    - Kelvin wave in shear has weak latitudinal flow v. This induces $v \propto \varepsilon^2 \exp(-\varepsilon^2/2)$ in critical layer.
    - This induces highly localised vorticity perturbations (approximately quasi-geostrophic, with very short deformation radius of $O(\varepsilon)$).
    - These induce flows that feedback upon (remote) equatorial Kelvin wave, reducing effect by another factor of $\exp(-\varepsilon^2/2)$ (?).
- Possible similarities to wind-wave instability mechanism proposed by Carpenter et al. (2017, J. Phys. Oceanogr.), which involves a surface water wave interacting with a critical layer in the air above.