Instability of surface quasigeostrophic spatially periodic flows

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INTRODUCTION

Baroclinic instability is the instability of flows of stratified rotating fluid with vertical and horizontal velocity shear. The formation of large-scale atmospheric eddies at mid-latitudes (cyclones, anticyclones) is associated with the realization of baroclinic instability. Awareness of this fact is one of the greatest achievements of 20th century meteorology.

This contribution considers the problem of baroclinic instability of spatially periodic surface geostrophic flows. The geophysical prototype for such flows are the periodic zonal flows in the atmosphere of Jupiter and other giant planets, as well as recently discovered multiple jet systems in the Southern Ocean.

Main governing equation and boundary condition

A semi-infinite atmosphere (z> 0) with a constant buoyancy frequency *N* and an inertial frequency *f* is considered. The atmospheric motion with a characteristic velocity *U* and a horizontal scale *D* is described by the equation

$$q_t + [\psi, q] = 0 \qquad q = \Delta\psi = \psi_{xx} + \psi_{yy} + \psi_{zz} \qquad [\psi, q] = \psi_x q_y - \psi_y q_x \qquad (1)$$

Velocity components and buoyancy disturbance are: $u = -\psi_y$ $\quad$ $\mathrm{v} = \psi_x$ $\quad$ $\sigma = \psi_z$

Here, *D* and *H* = *Df* / *N* as the horizontal and vertical scales, respectively, and *T* = *D* / *U* and *UD* are the time scale, and the stream function scale, respectively.
A fundamentally important boundary condition is attached to equation (1):

$$z = 0: \qquad \psi_{zt} + [\psi, \psi_z] = 0$$

In advective form it reads: $z = 0: \qquad \sigma_t + u\sigma_x + \mathrm{v}\sigma_y = 0$

A more general dimensional form of equation (1) is (Charney, 1948; Обухов 1949; Charney & Stern,1962)

$$q_t + [\psi, q] = 0 \qquad q = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{f^2}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{N^2}\frac{\partial\psi}{\partial z}\right)$$

SQG model; exact solution and its stability problem

Equation (1) is satisfied for flows with q=0. The dynamics of such flows is described by solutions of Laplace equation with non-linear boundary condition at $z = 0$

$$\psi_{xx} + \psi_{yy} + \psi_{zz} = 0, \quad z > 0.$$

$$z = 0: \qquad \psi_{zt} + [\psi, \psi_z] = 0$$

Held et al. (1995).

For harmonic functions at the boundary $\psi = F(\psi_z)$ F is the integral Hilbert operator

Conservation laws: $E_t = 0 \quad V_t = 0$

$$E = \frac{1}{2}\int_0^\infty \overline{\left(\psi_x^2 + \psi_y^2 + \psi_z^2\right)} dz$$

$$V = (1/2)\overline{\psi_z^2}$$

,

Exact stationary solution:

$$\overline{\psi} = -e^{-z}\cos y \qquad \overline{u} = e^{-z}\sin y \qquad \overline{\sigma} = e^{-z}\cos y$$

,

Dimensional velocity profile:

$$\overline{u} = U_0 e^{-z/H}\sin(y/D)$$

,

Stability problem formulation:

$$\psi = \overline{\psi} + \psi'$$

$$\psi_{xx} + \psi_{yy} + \psi_{zz} = 0, \quad z > 0$$

$$z = 0: \qquad \psi_{zt} + \sin y(\psi_{xz} + \psi_x) + [\psi, \psi_z] = 0\,.$$

Linear stability problem; continued fractions

The linear theory is similar to the theory of Meshalkin and Sinai (1961), developed for the Kolmogorov flow. Spatially periodic solutions are sought in the form of a series:

$$\psi = e^{\lambda t} e^{ikx} \sum_{n=-\infty}^{n=+\infty} e^{-k_n z} e^{iny} \varphi_n$$

$$k_n = \sqrt{k^2 + n^2}$$

Substitution into the boundary condition yields:

$$a_n d_n + d_{n-1} - d_{n+1} = 0$$

$$d_n = (1 - k_n)\varphi_n$$

$$a_n = 2\lambda k_n / k(1 - k_n)$$

From the regularity condition follows the equation for finding the increment λ

$$-(a_0/2) = \langle 0, a_1, a_2, \ldots \rangle$$

$$\langle 0, a_1, a_2, \ldots \rangle = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}$$

By truncating the continued fraction, one can obtain successive approximations for increment λ

Dependence of the instability increment on the wavenumber

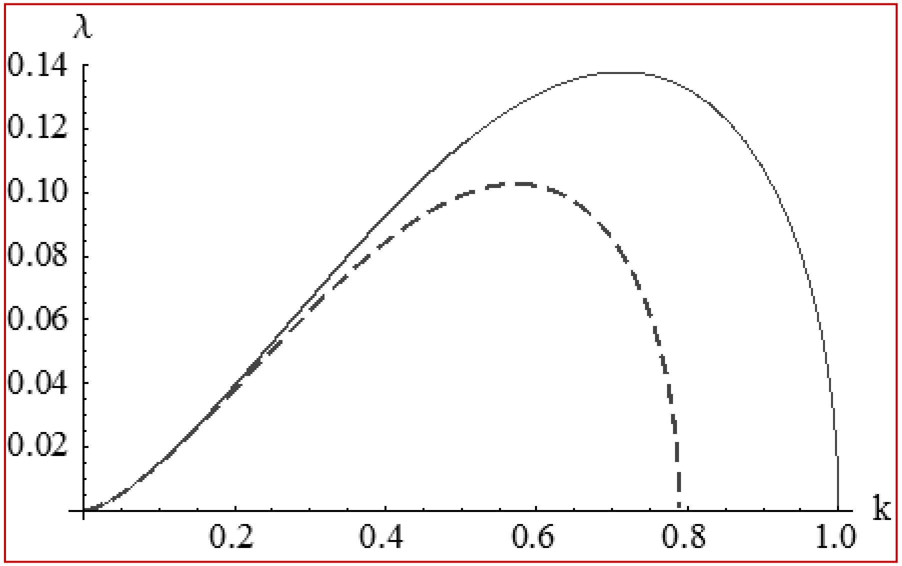
First approximation: $-(a_0/2) = 1/a_1$

$$\lambda^2 = \frac{k(1-k)(k_1-1)}{2k_1}$$

$$k_1 = \sqrt{k^2+1}$$

Second approximation: $-(a_0/2) = \langle 0, a_1, a_2 \rangle$

$$\lambda^2 = \frac{k\left(2k_2-(3k_2-1)k\right)}{4k_1k_2}$$



Longwave instability with a preferred horizontal scale on the order of the main flow wavelength.

Linear stability theory; Galerkin method

A solution is sought in the form of an expansion into functions $f_1 = \sin y \quad f_2 = \cos y \quad f_3 = 1$

$$\psi = A(x,z,t)\sin y + B(x,z,t)\cos y + C(x,z,t), \quad (1)$$

From the Laplace equation it follows:

$$A_{xx} + A_{zz} - A = 0 \qquad B_{xx} + B_{zz} - B = 0 \qquad C_{xx} + C_{zz} = 0, \quad (2)$$

Substituting (1) in the boundary condition and using the orthogonality conditions gives

$$z = 0: \quad A_{zt} + C_{xz} + C_x = 0 \qquad C_{zt} + (1/2)\left(A_z + A\right)_x = 0 \qquad B_{zt} = 0, \quad (3)$$

Solution of (2) is sought as: $A = a(t)e^{-k_1 z}\sin(kx)$ $\quad C = c(t)e^{-kz}\cos(kx)$ $\quad B = 0$

From the boundary condition equations (3), the ODE system follows:

$$a_t + \alpha c = 0 \qquad c_t + \gamma a = 0 \qquad \alpha = k(1-k)/k_1 \qquad \gamma = (k_1 - 1)/2$$

The system has solutions with the increment: $\lambda^2 = \dfrac{k(1-k)(k_1-1)}{2k_1}$

The Galerkin method with three basis functions gives the result of the first approximation in the theory of continued fractions.

A second approximation to the increment can be obtained for the solution:

$$\psi = A^{(1)}(x,z,t)\sin y + B^{(1)}(x,z,t)\cos y + A^{(2)}(x,z,t)\sin 2y + B^{(2)}(x,z,t)\cos 2y + C(x,z,t)$$

Galerkin method and nonlinear perturbation dynamics

$$\psi = A(x,z,t)\sin y + B(x,z,t)\cos y + C(x,z,t)$$

$$A = a(t)e^{-\kappa z}\sin(kx) \qquad C = c(t)e^{-kz}\cos(kx) \qquad B = b(t)e^{-z}$$

Substitution in the nonlinear version of the boundary conditions and the use of orthogonality conditions leads to a system:

$$a_t + \alpha\tilde{b}c = 0 \qquad c_t + \gamma\tilde{b}a = 0 \qquad \tilde{b}_t - \beta ac = 0 \qquad \tilde{b} = b + 1$$

$$\alpha = k(1-k)/k_1 \qquad \gamma = (k_1 - 1)/2 \qquad \beta = k(k_1 - k)/2$$

The system is similar to the system describing the motion of a symmetric top in classical mechanics (or the motion of a fluid in an ellipsoidal cavity).

Conservation laws:

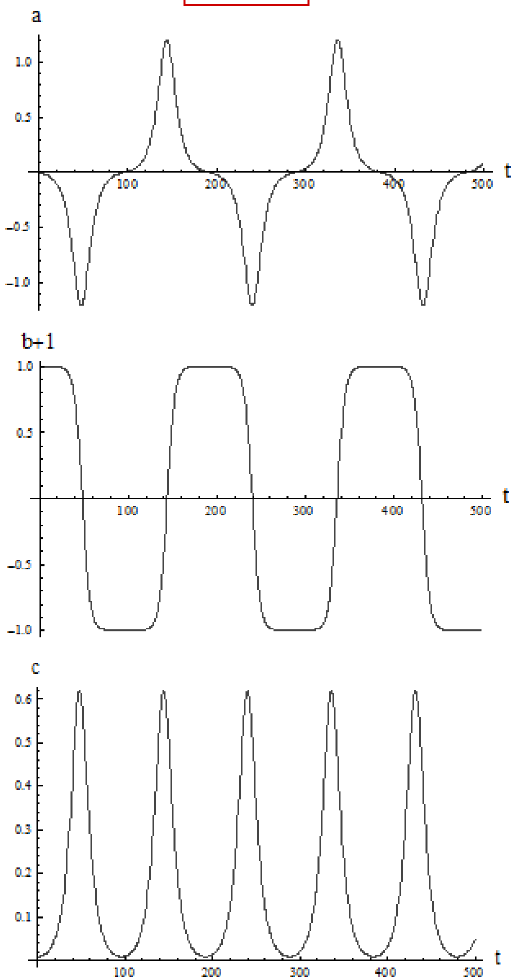
$$\gamma a^2 - \alpha c^2 = I_1 = \text{const} \qquad \beta a^2 + \alpha\tilde{b}^2 = I_2 = \text{const}$$

From the second conservation law it follows that all solutions are limited. The linear stage of exponential growth is replaced by the stage of nonlinear oscillations.

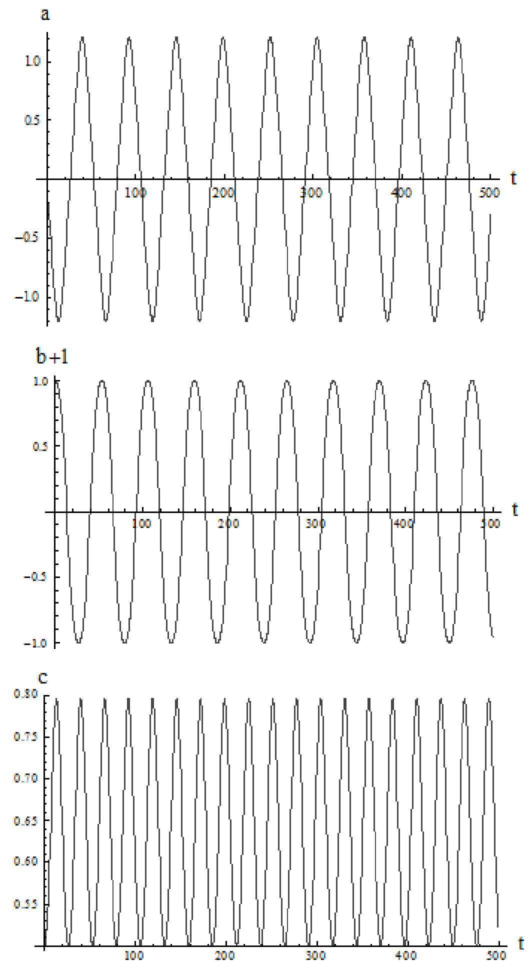
Examples of nonlinear oscillations

$t = 0: \; a = b = 0, \; c = c_0$

$c_0 = 0.01$



$c_0 = 0.5$



Formula for the oscillation period

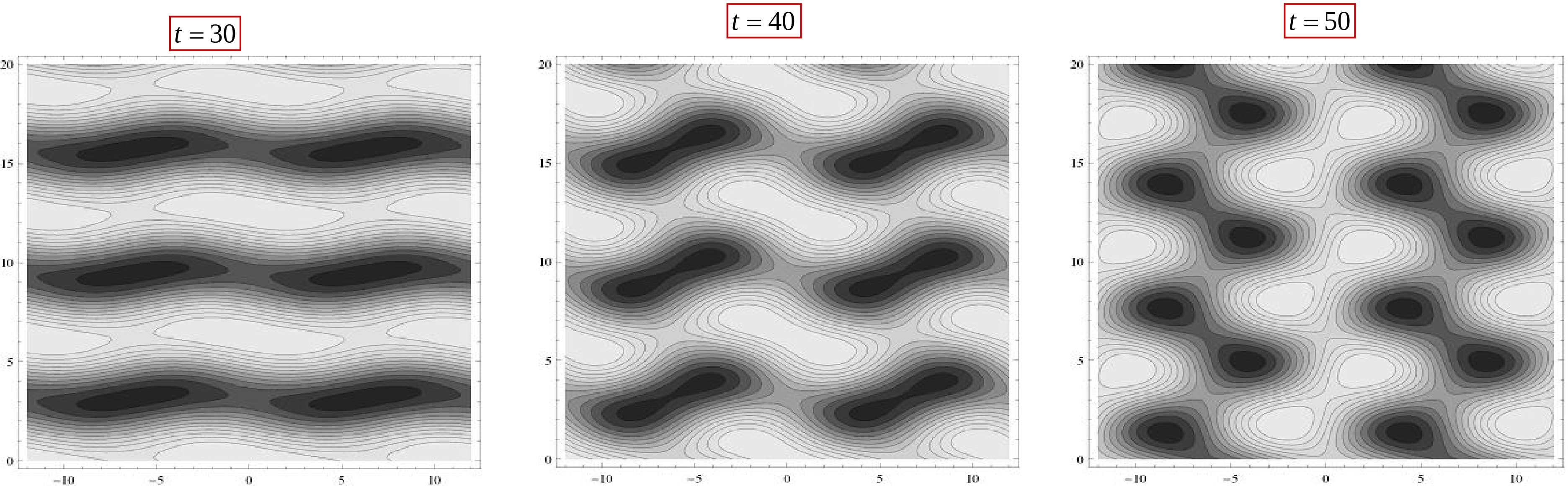
$$T = (4/n)K(m)$$

$$n^2 = \alpha(\gamma + \beta c_0^2)$$

$$m^2 = \left(1 + (\beta/\gamma)c_0^2\right)^{-1}$$

$$K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m^2 \sin^2(\theta)}}$$

The periodic process of birth and death of vortices



An approximate solution for the full stream function

$$\psi_* = \left(b(t)+1\right)e^{-z}\cos y + a(t)e^{-k_1 z}\sin(kx)\sin y + c(t)e^{-kz}\cos(kx)$$

Very important: for the solution, the conservation laws of the SQG model are satisfied.

$$E = (1/2)\int_0^{\infty}\overline{\left(\psi_x^2+\psi_y^2+\psi_z^2\right)}dz = \left(I_2 - kI_1\right)/2\alpha$$

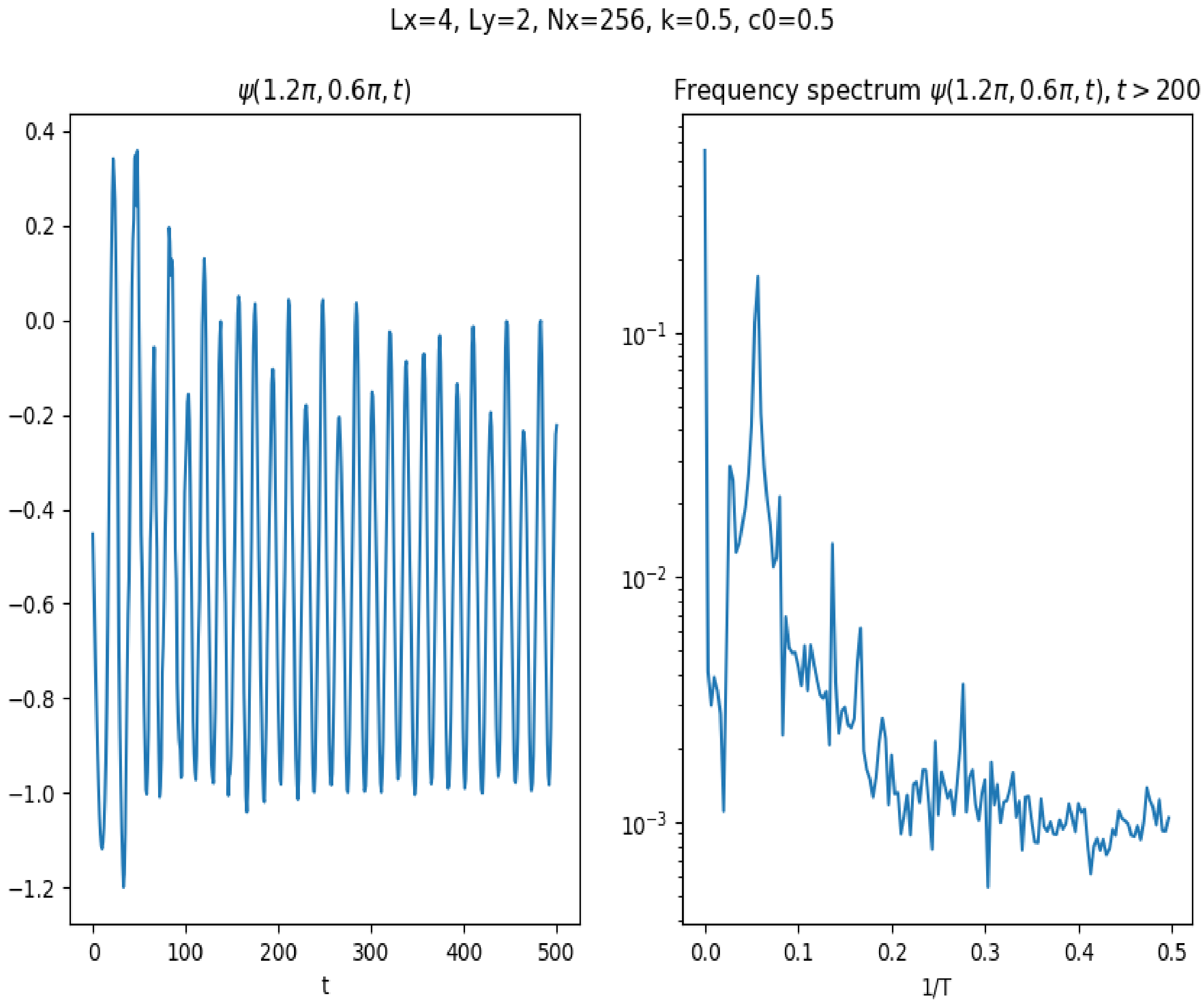
$$V = (1/2)\overline{\psi_z^2} = \left(I_2 - k^2 I_1\right)/2\alpha$$

Numerical integration of SQG model equations

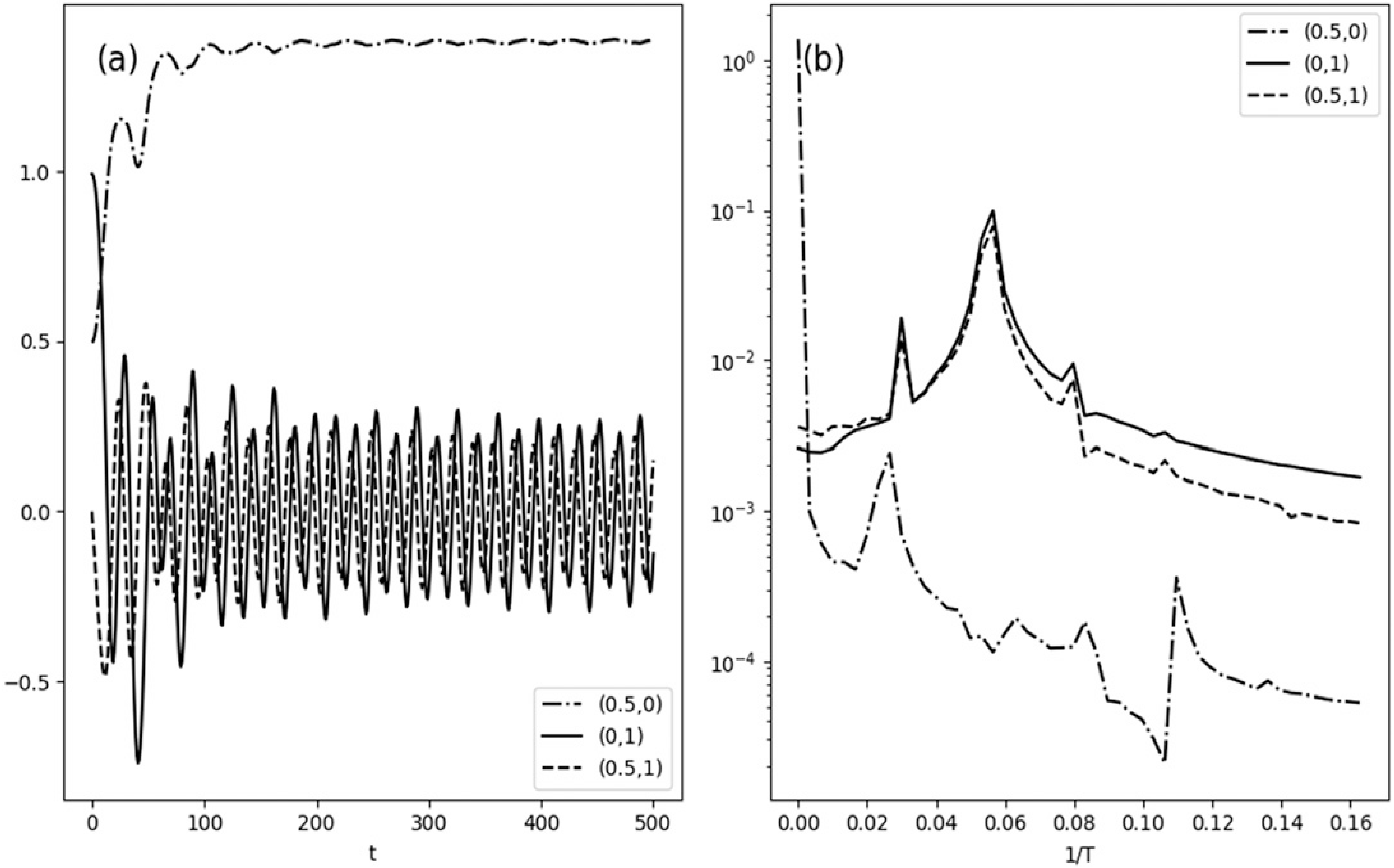
$$\psi_{xx} + \psi_{yy} + \psi_{zz} = 0, \quad z > 0.$$

$$z = 0: \qquad \psi_{zt} + [\psi, \psi_z] = 0$$

At the initial moment, a periodic flow plus disturbance are set.



Time dependence of the amplitudes of the main Fourier harmonics



Danger: full energy is not conserved in numerical codes.

For numerical calculations, we used the pseudo-spectral code for solving surface quasi-geostrophic equations (SQG-equations) taken from the PyQG package (https://pyqg.readthedocs.io/en/latest/index.html).

Periodic flow with two boundaries; linear stability

$$\psi_{xx} + \psi_{yy} + \psi_{zz} = 0,\ z > 0$$

$$z = 0,1: \qquad \psi_{zt} + [\psi, \psi_z] = 0$$

Exact solution

$$\overline{\psi} = -\frac{\sinh(lz)}{l\sinh(l)}\sin(ly) \qquad \overline{u} = \frac{\sinh(lz)}{\sinh(l)}\cos(ly) \qquad \overline{\sigma} = -\frac{\cosh(lz)}{\sinh(l)}\sin(ly)$$

corresponds to a spatially periodic flow concentrated at the upper boundary.

The dynamics of small perturbations is described by solutions of the Laplace equation with boundary conditions:

$$z = 0,1: \qquad (\partial/\partial t + \overline{u}\partial/\partial x)\psi_z - \overline{u}_z\psi_x = 0$$

Following the Galerkin method, a solution is sought $\psi = C(x,z,t) + B(x,z,t)\cos(ly)$

From the Laplace eq. and boundary conditions: $B_{xx} + B_{zz} - l^2 B = 0$ $\quad C_{xx} + C_{zz} = 0$

$$z = 0,1: \qquad B_{zt} + h(z)C_{xz} - h'(z)C_x = 0, \qquad C_{zt} + (1/2)h(z)B_{xz} - (1/2)h'(z)B_x = 0$$

Finding a solution in the form of normal modes

$$B = (a_1\sinh(\mu z) + b_1\cosh(\mu z))e^{ik(x-ct)} \qquad C = (a_0\sinh(kz) + b_0\cosh(kz))e^{ik(x-ct)} \qquad \mu^2 = k^2 + l^2$$

gives an expression for the instability increment $\lambda = kc_i$

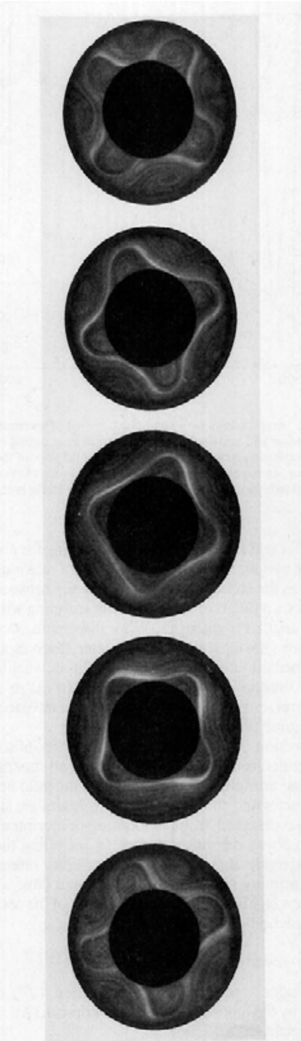
Result: the instability if $0 < k < l$

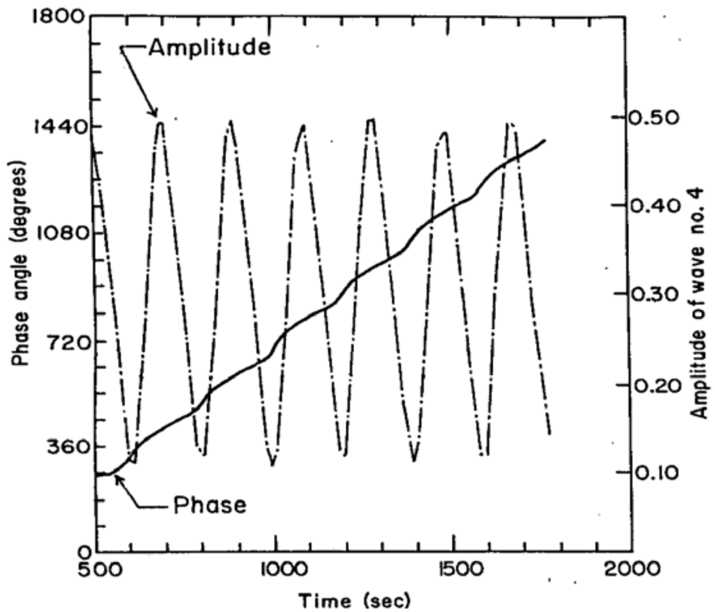
Nonlinear vacillations in laboratory experiments by Hide (1953, 1958)

No turbulence: everything works like a clock.

An idealized weakly nonlinear theory is developed by Pedlosky (1963,1970) .

When weak supercriticality, then oscillations.





Question: what is the result of the development of instability of quasi-two dimensional flows of an ideal fluid?
Answer: if we take into account that energy is conserved, then (with large probability) it leads to the occurrence of oscillations.

Thank you for attention!

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