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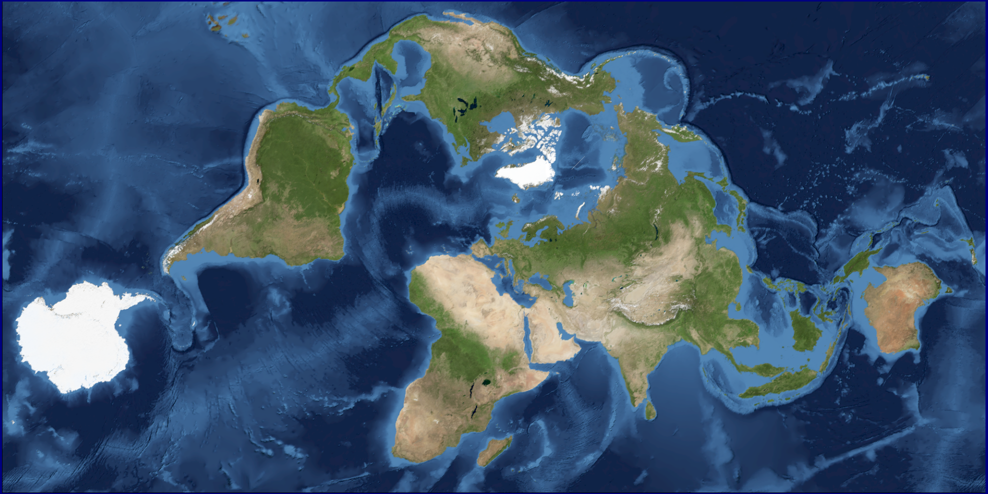
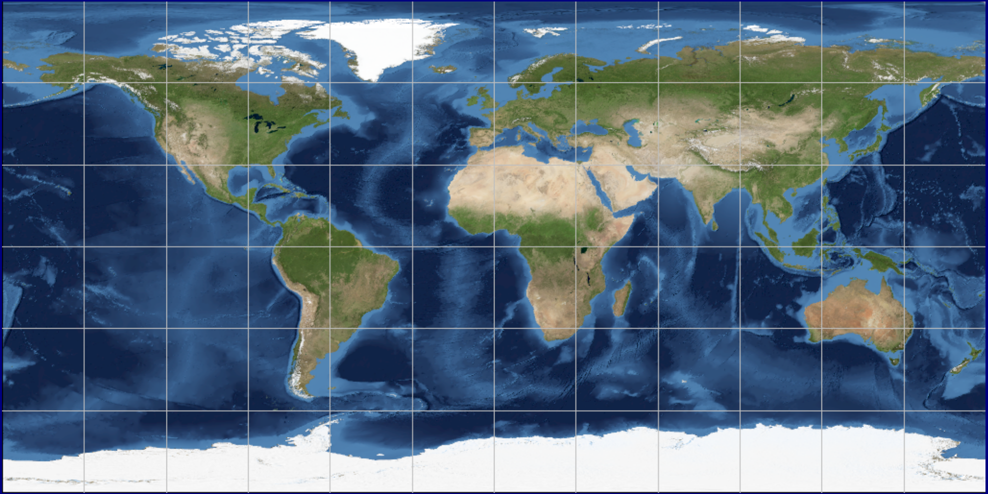
Optimized global map projections for specific applications: The triptychial projection and the Spilhaus projection

Björn Grieger, European Space Astronomy Centre, Madrid



Equidistant cylindrical projection



- Neither shape nor area preserving.
- Widely used because of apparent simplicity.
- But things are not as simple as they may appear!

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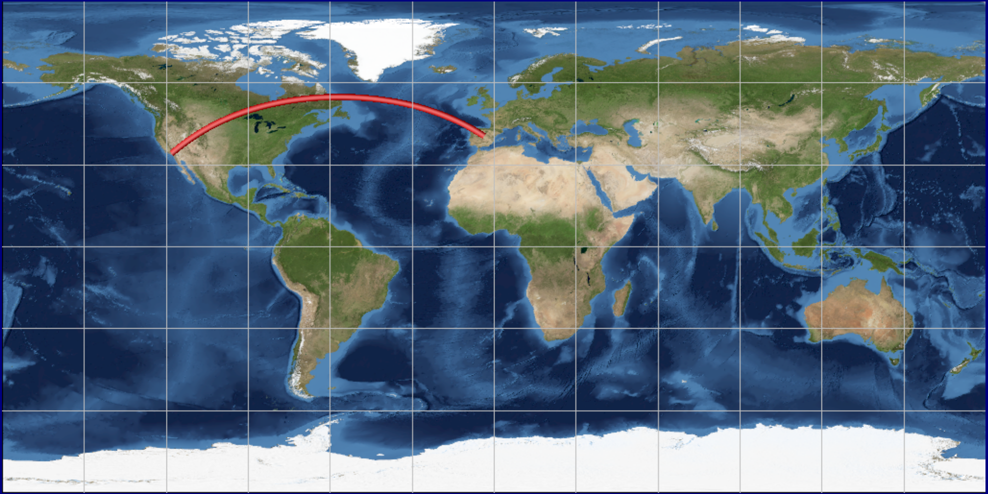
"Straight line" (great circle) in the real world



- Beware: straight lines are **not** straight lines!
- ... e. g., your flight from Madrid to Los Angeles.
- However, a great format to exchange surface data.

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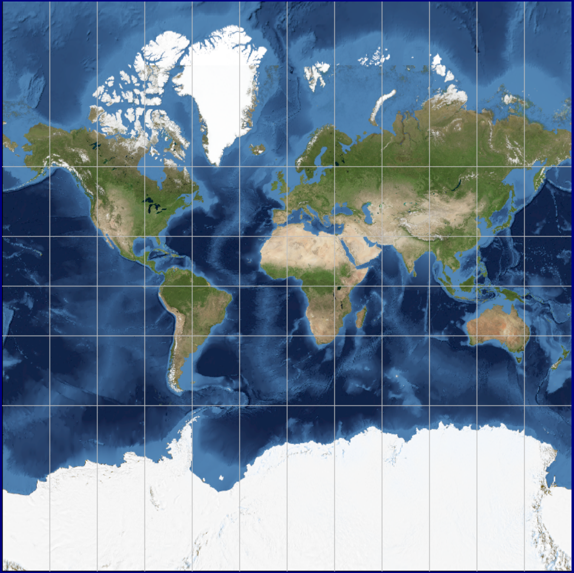
Mercator projection



- Conformal, i. e., shape preserving on (infinitesimal) small scales.
- **Large** scale distorsions (Greenland appears larger than Africa), poles are at infinity.
- Previously used by Google maps.
- Never intended for global world maps.

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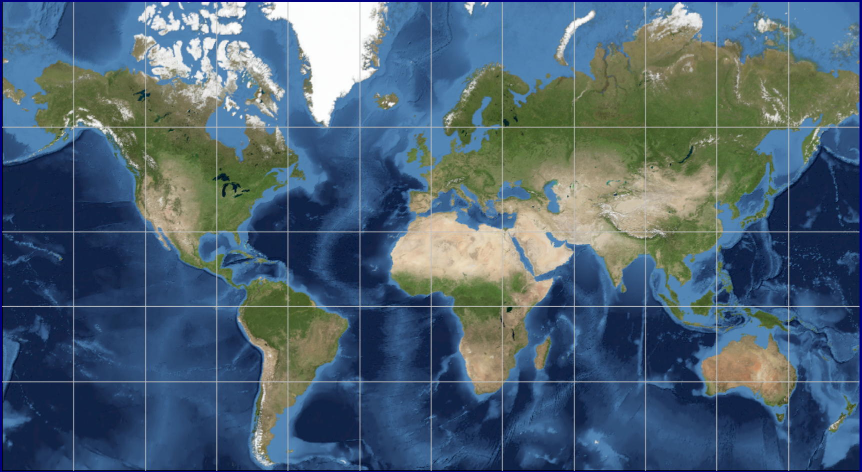
Abused as world map



- Very bad polar areas just cut off.
- Saw it with Greenland photoshopped out at the wall of a travel agency office (they obviously didn't sell trips to Greenland).

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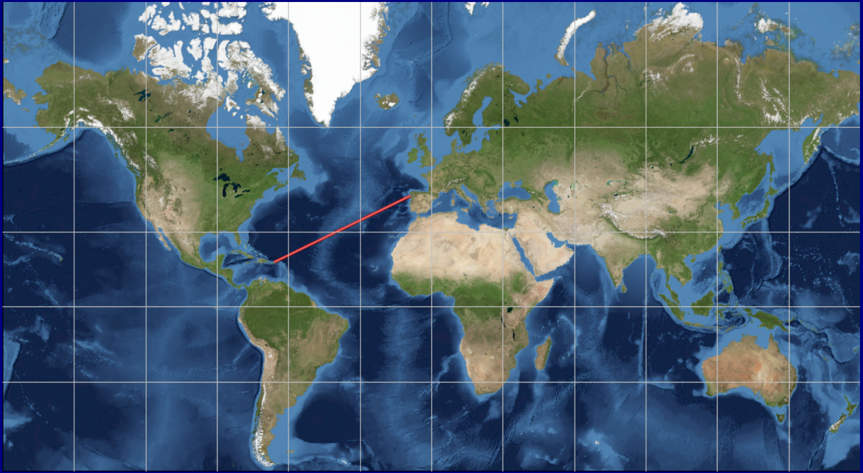
Really the seafarer's map



- The real purpose is navigation.
- Steering a fixed course comes out as straight line!

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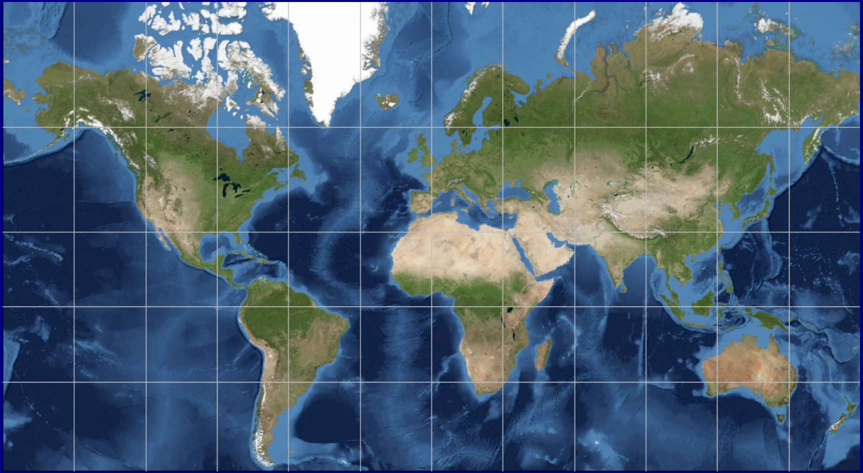
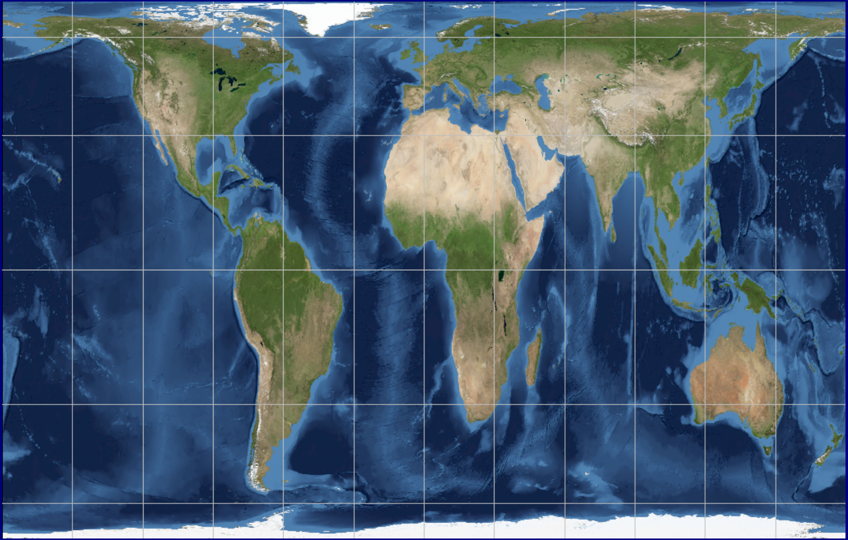
If size matters . . .



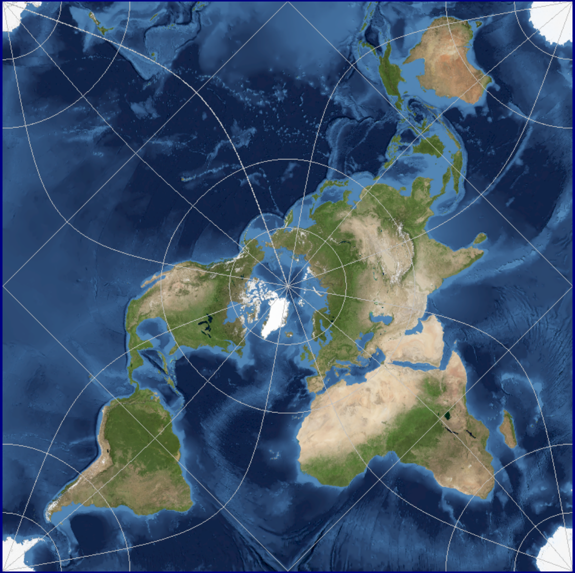
- Criticized for showing countries near the Equator as too small when compared to Europe and North America.

Gall-Peters projection



- Perfectly area preserving, but large shape distortions.

Peirce quincuncial projection (1879)



- Conformal (with the exception of four singular points at the centers of the edges).
- Approximately area preserving over the continents.
- . . . but Antarctica is not treated nicely! (Peirce said that nobody lives there, so nobody will complain.)

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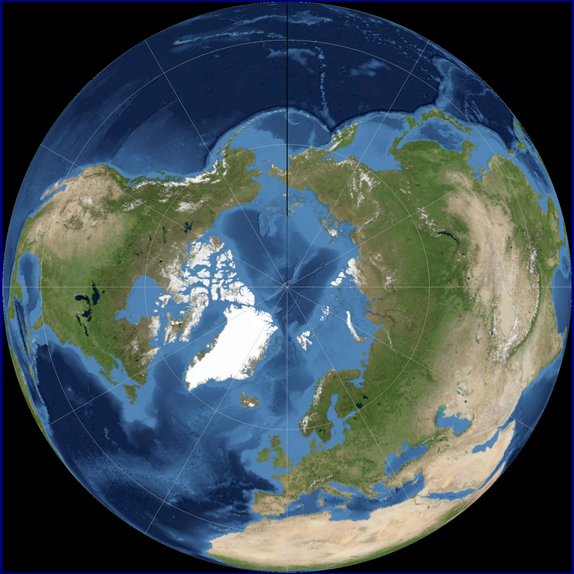
The Northern hemisphere



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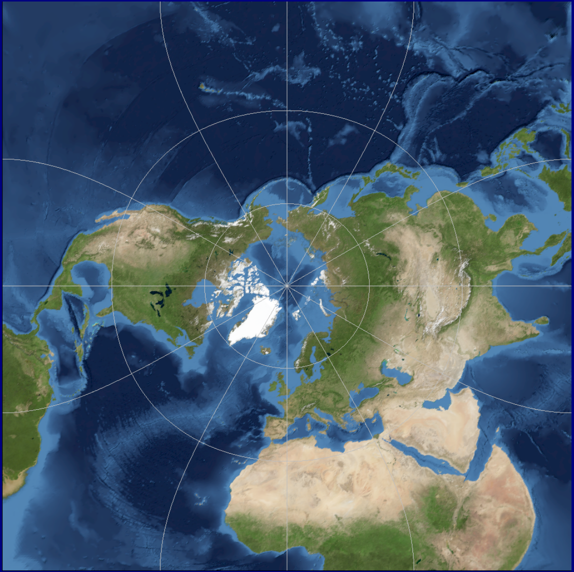
. . . conformally mapped to a square



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Elliptic isometric coordinates

Longitude λ, latitude φ, elliptic integral $F(\alpha) = \int_0^\alpha \frac{1}{\sqrt{1-\frac{1}{2}\sin^2\alpha'}} \mathrm{d}\alpha'$:

$$\xi_1 = \arccos\left(\cos\varphi\,\cos\left(\tfrac{\pi}{4}+\lambda\right)\right) \qquad \eta_1 = \arccos\left(\cos\varphi\,\sin\left(\tfrac{\pi}{4}+\lambda\right)\right)$$

$$\xi_2 = \arcsin\left(\sqrt{2}\,\cos\left(\frac{\xi_1+\eta_1}{2}\right)\right) \qquad \eta_2 = \arcsin\left(\sqrt{2}\,\sin\left(\frac{\xi_1-\eta_1}{2}\right)\right)$$

$$x = \operatorname{sgn}(\xi_2)\,F(|\xi_2|) \qquad y = \operatorname{sgn}(\eta_2)\,F(|\eta_2|)$$

$$x \in [-1.84533, 1.84533] \qquad y \in [-1.84533, 1.84533]$$

This maps a point (λ, φ) on the Northern hemisphere to a point in a square with rectangular coordinates (x, y).

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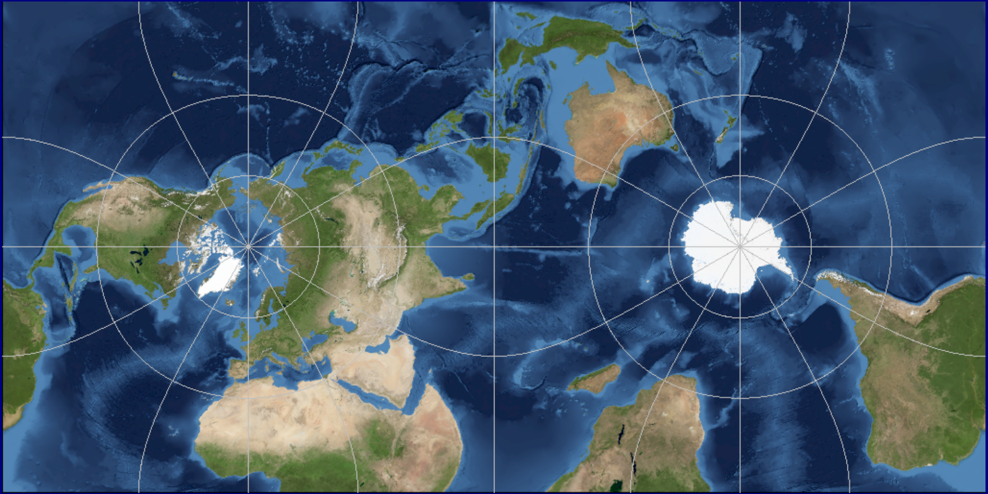
Two hemispheres mapped one by one



- The two hemispheres shown side-by-side.
- Conformal with the exception of the four corner points.
- Area distortions get large close to these points.

Slightly traverse version

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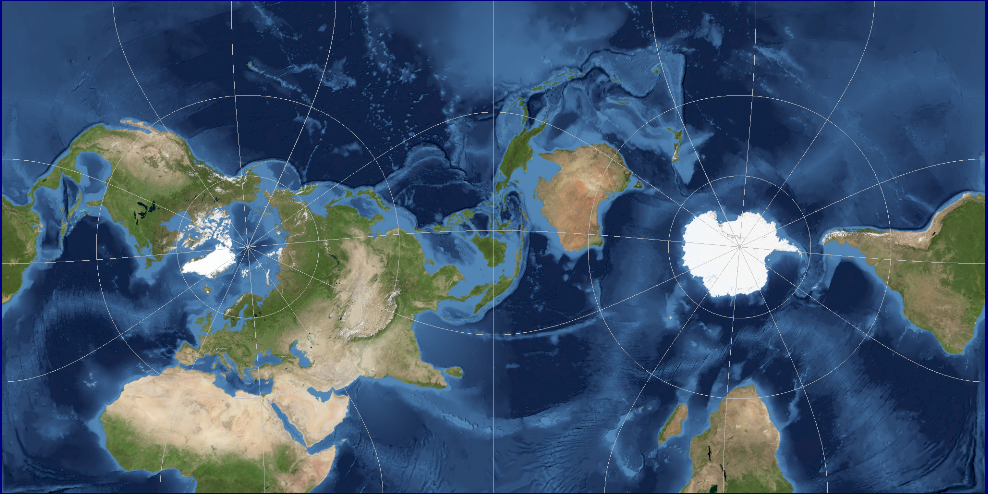

- Rotated 25° around the z-axis to get the continents away from the critical points.

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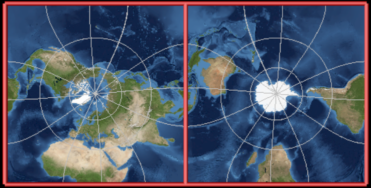
Tessellation

Two hemispheres side by side.



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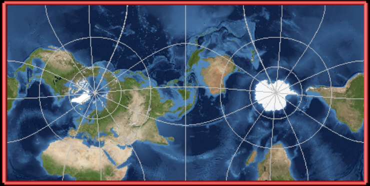
Tessellation

Two hemispheres side by side.

This can be tessellated.



Tessellation

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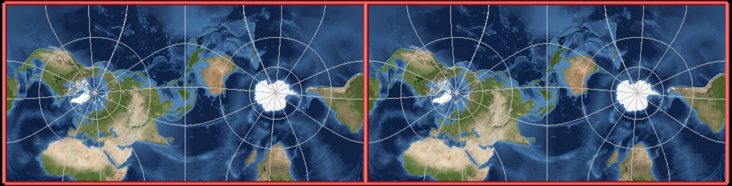
Two hemispheres side by side.

This can be tessellated.



The short edges match perfectly.

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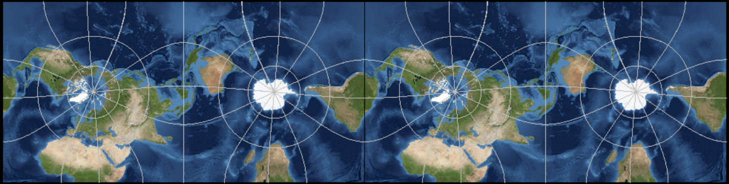
Tessellation

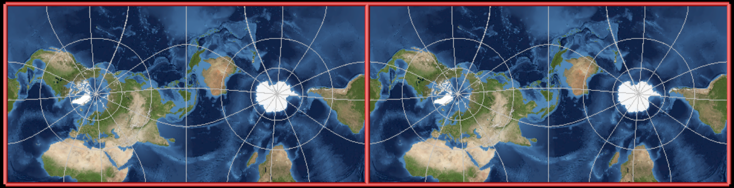
Two hemispheres side by side.

This can be tessellated.



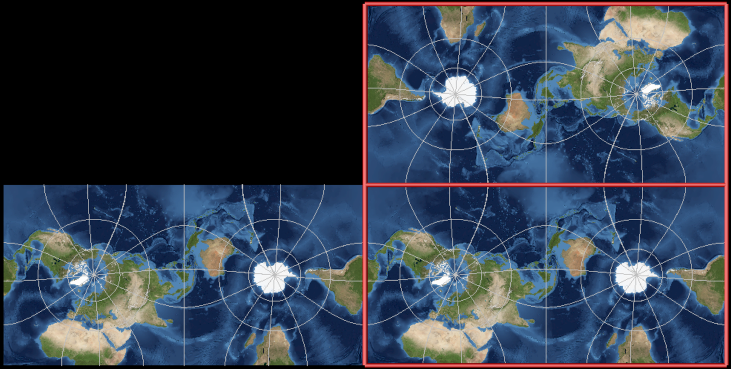
The short edges match perfectly.

Tessellation



The short edges match perfectly.

Tessellation



The short edges match perfectly.

The long edges match after a rotation by 180°.

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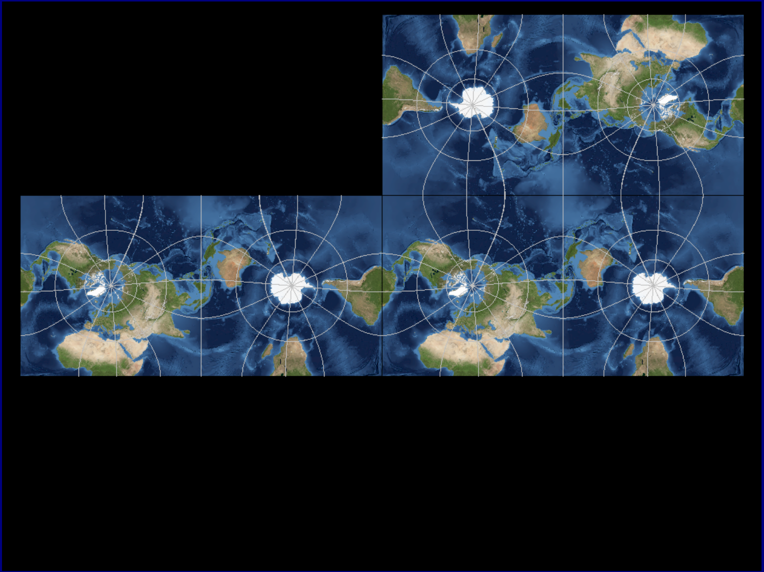
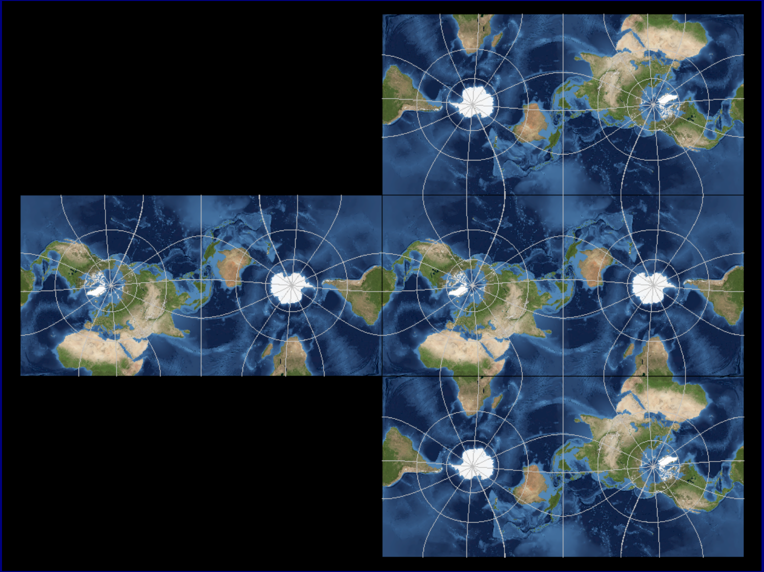
Tessellation



Tessellation



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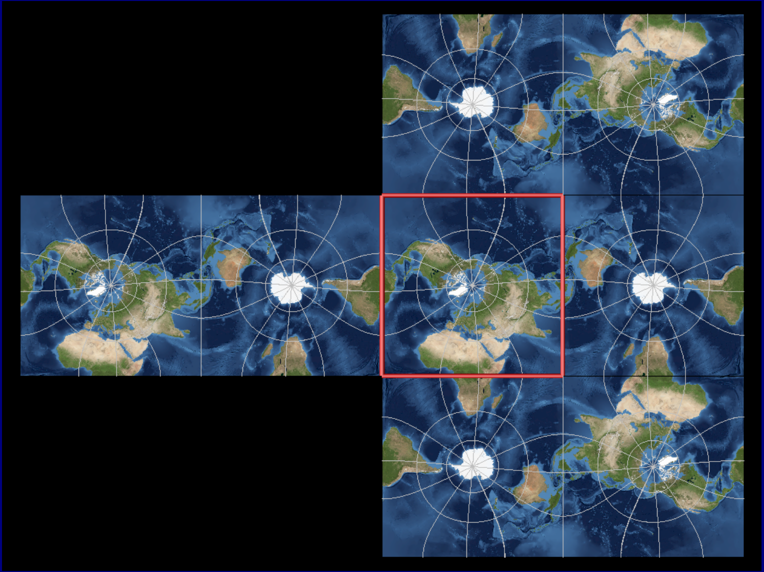
Quincuncial layout



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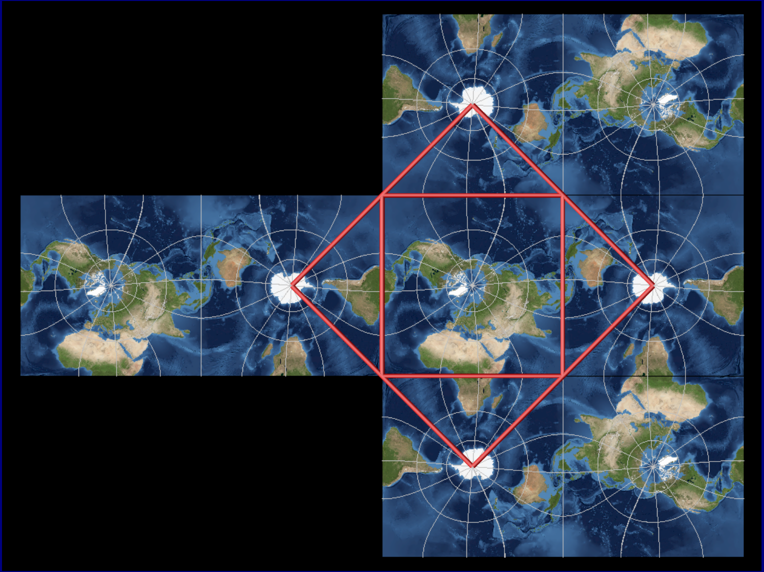
Quincuncial layout



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Quincuncial layout

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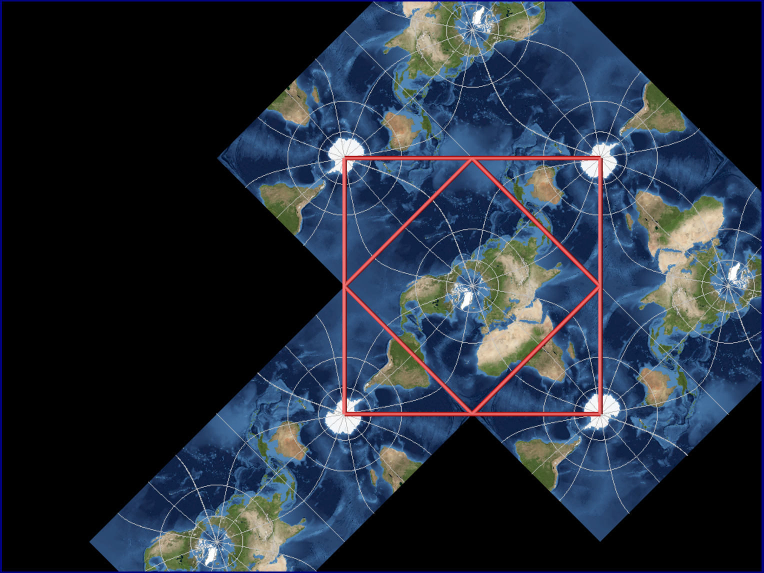

Quincuncial layout

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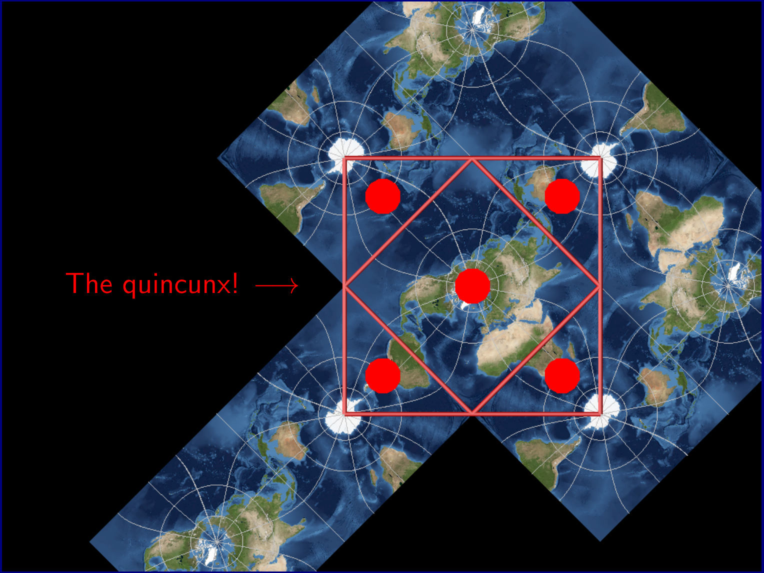
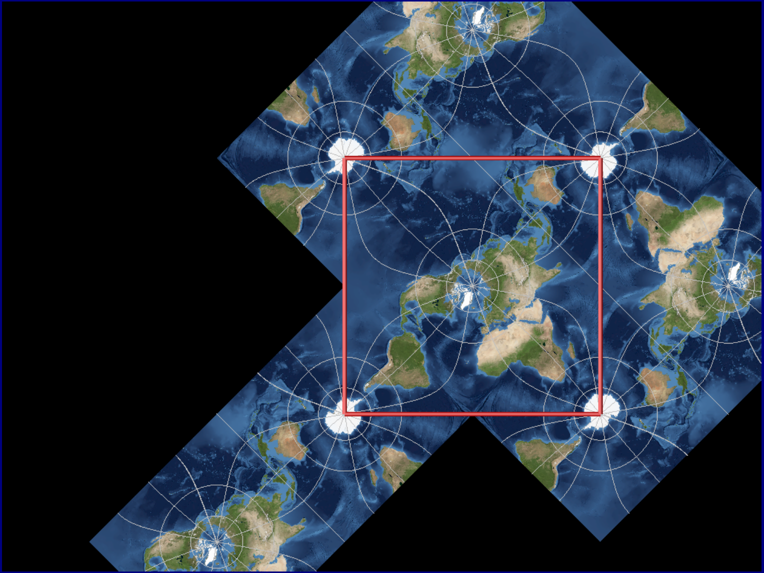


Quincuncial layout



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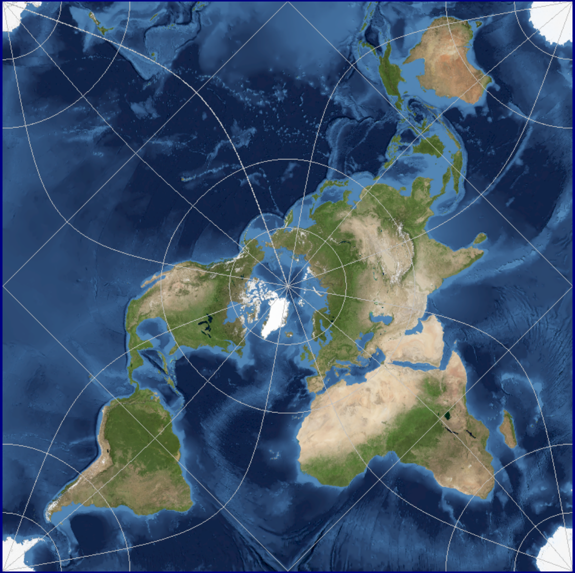
Peirce quincuncial projection (1879)



- Conformal (with the exception of four singular points at the centers of the edges).
- Approximately area preserving over the continents.
- ...but Antarctica is not treated nicely! (Peirce said that nobody lives there, so nobody will complain.)

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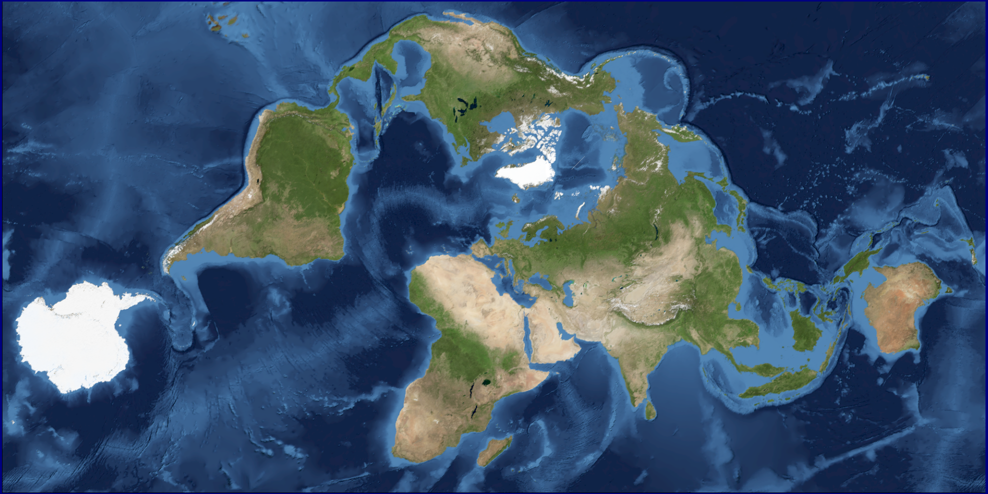
The triptychial projection (Grieger, 2019)



A map of the whole world showing all continents including Antarctica with minimal distortion and without any intersection

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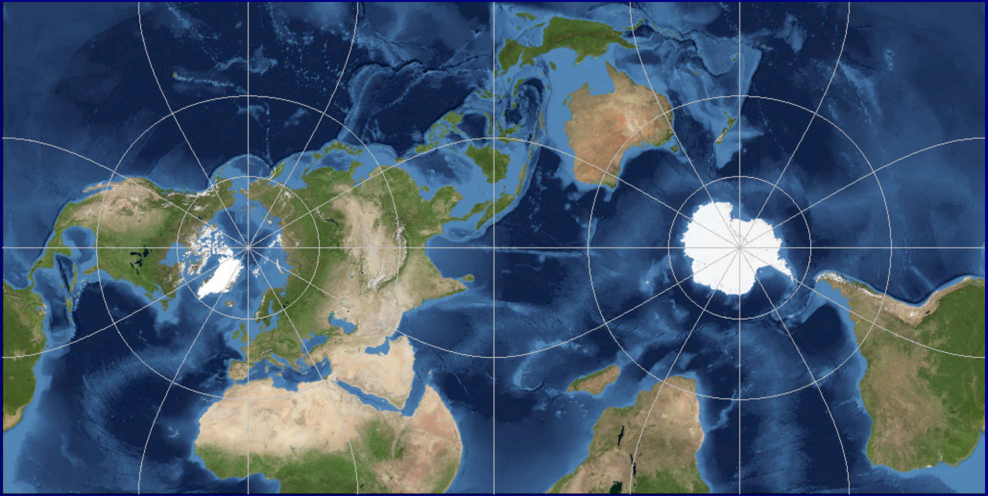
Standard "quincuncial" projection



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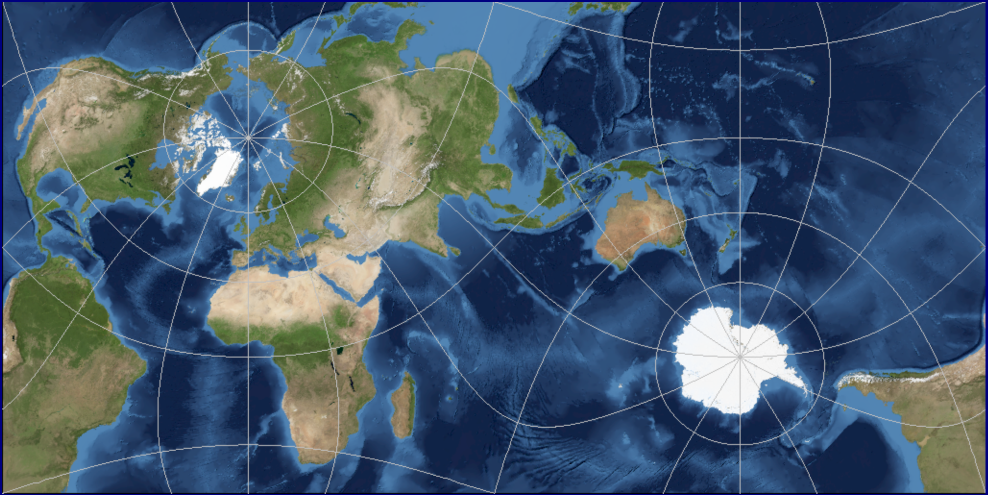
Rotated 45° counterclockwise around the y-axis . . .



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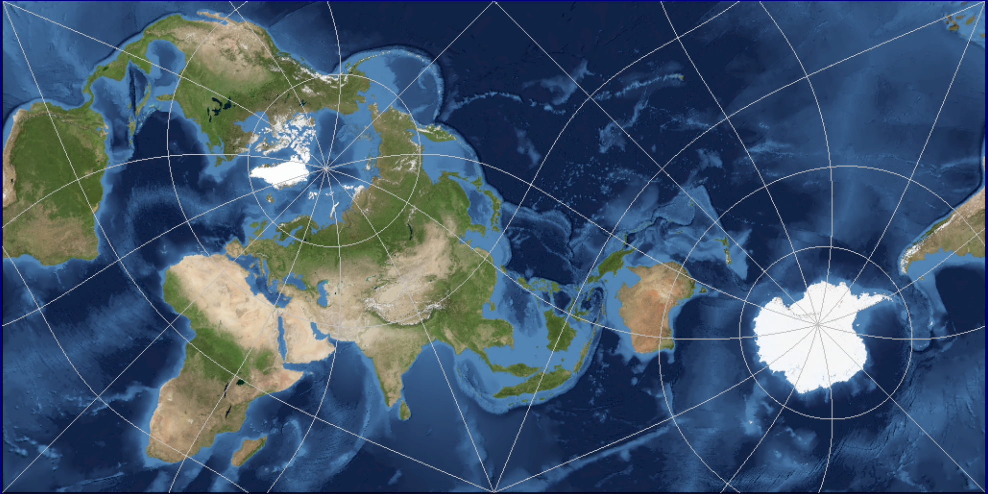
. . . and 45° counterclockwise around the z-axis



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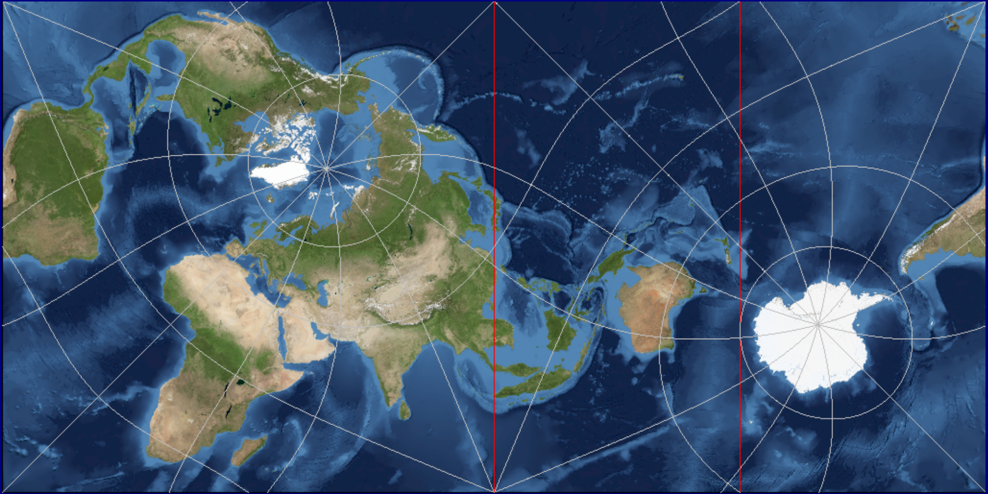
Cut the right hemisphere into two halfs



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. . . and move the right half over to the very left

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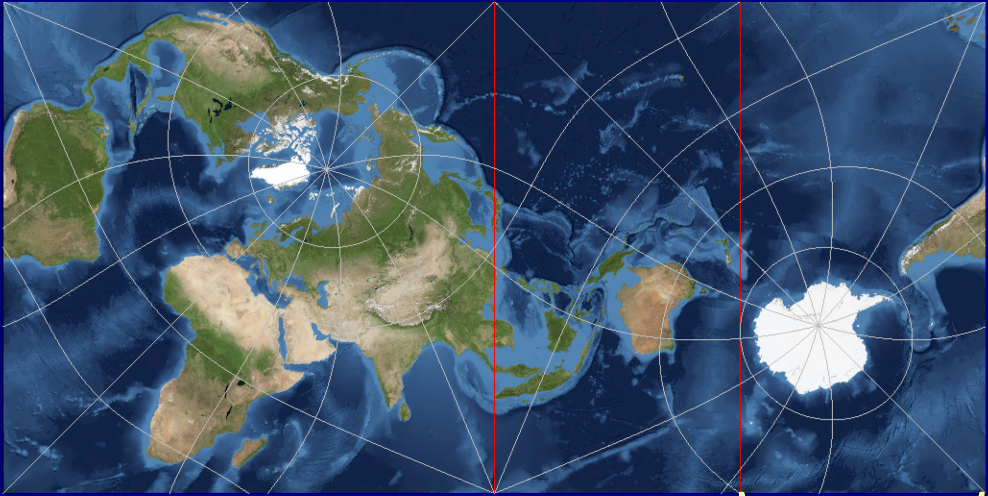
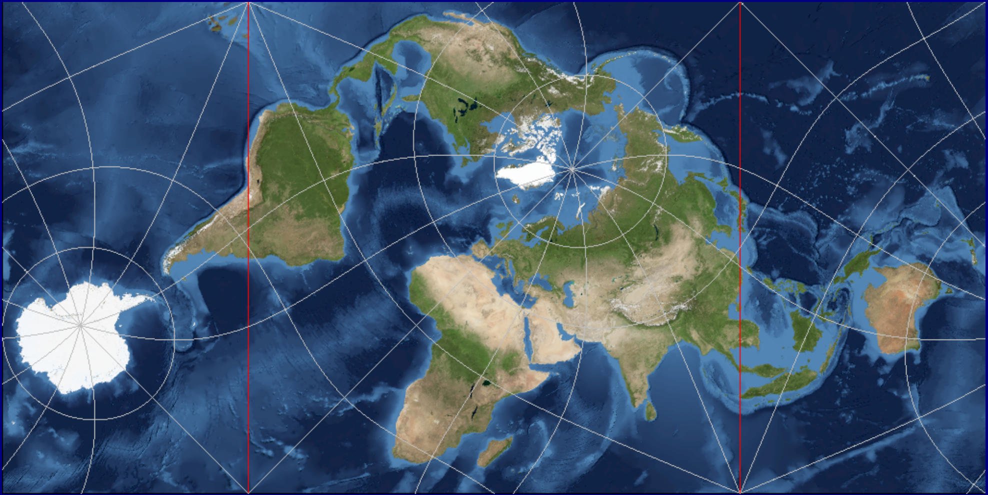


The triptychial projection of the world



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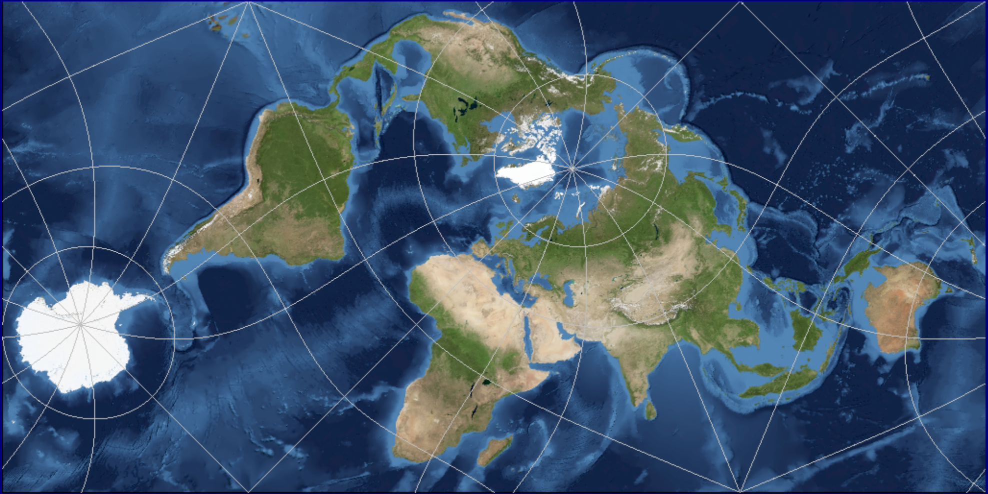
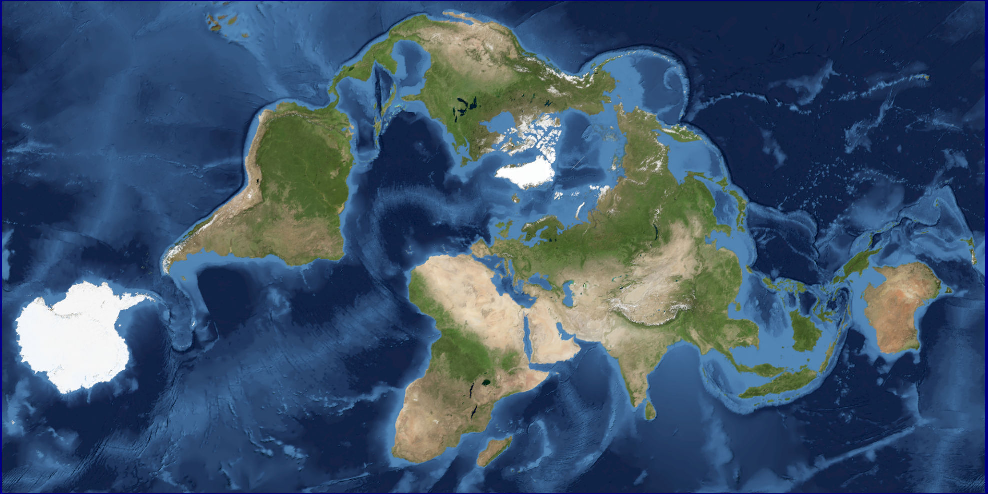
The triptychial projection of the world



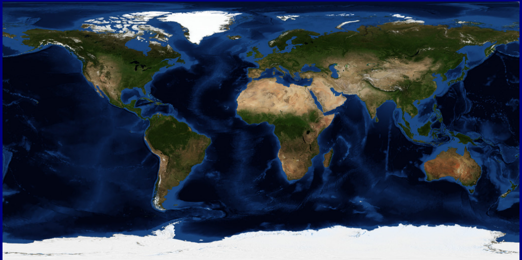
- Conformal (shape preserving on small scales).
- Approximately area preserving over the continents.
- No intersection of continents, in particular Antartica.

The triptychial projection (Grieger, 2019)



A map of the whole world showing all continents including Antarctica with minimal distortion and without any intersection

Surface data



Rectangular map with $2n \times n$ pixels:

$$i_x, i_y \longrightarrow \text{Surface data}$$

Equidistant cylindrical projection:

$$\lambda \longleftrightarrow i_x$$
$$\varphi \longleftrightarrow i_y$$

So we have a mapping:

$$\lambda, \varphi \longrightarrow \text{Surface data} \tag{1}$$

Note: Center of pixel $(1,1)$:

$$\lambda = -180\,\frac{n-\frac{1}{2}}{n}, \varphi = 90\,\frac{\frac{n}{2}-\frac{1}{2}}{\frac{n}{2}}$$

Center pixel $(2n, n)$:

$$\lambda = 180\,\frac{n-\frac{1}{2}}{n}, \varphi = -90\,\frac{\frac{n}{2}-\frac{1}{2}}{\frac{n}{2}}$$

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Applying the projection

Triptychial projection:

$$\lambda, \varphi \longrightarrow j_x, j_y$$

Direct application is not recommened, as it makes trouble at the edges of the map. Instead, use the inverse (needs numerical inversion):

$$j_x, j_y \longrightarrow \lambda, \varphi \tag{2}$$

For each pixel (j_x, j_y) of the projected map:

1. Get (λ, φ) from the mapping (2).
2. Get the surface data from the mapping (1).

Readymade tables providing mapping (2) are available online from the author, see References.

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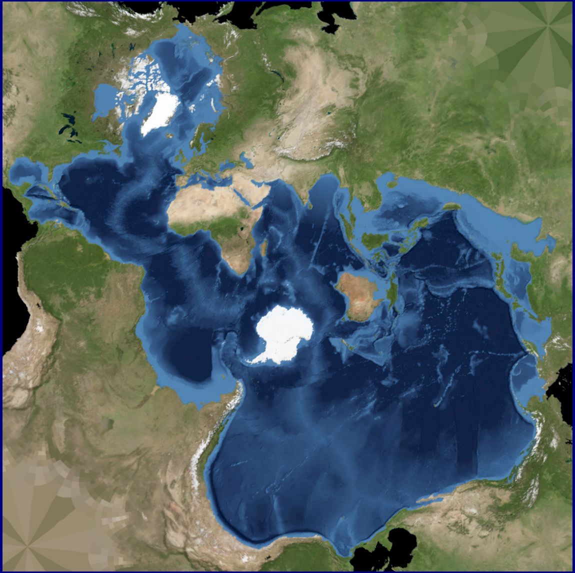
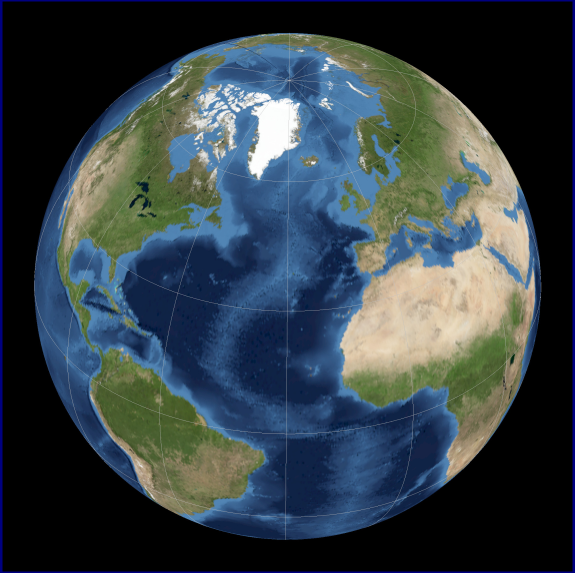
The Spilhaus projection (1979, not 1942)



A map showing the whole world ocean without any intersection and with only moderate distortion

The whole sphere . . .



Longitude λ_1, latitude φ_1

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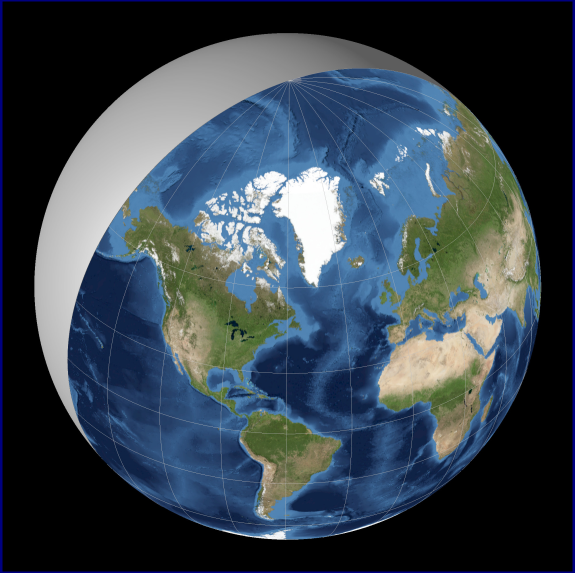
. . . contracted conformally to a hemisphere . . .



New longitude and latitude:

$$\lambda_2 = \frac{1}{2}\,\lambda_1$$

$$\varphi_2 = \frac{\pi}{2} - 2\arctan\sqrt{\tan\left(\frac{\pi}{2} - \varphi_1\right)}$$

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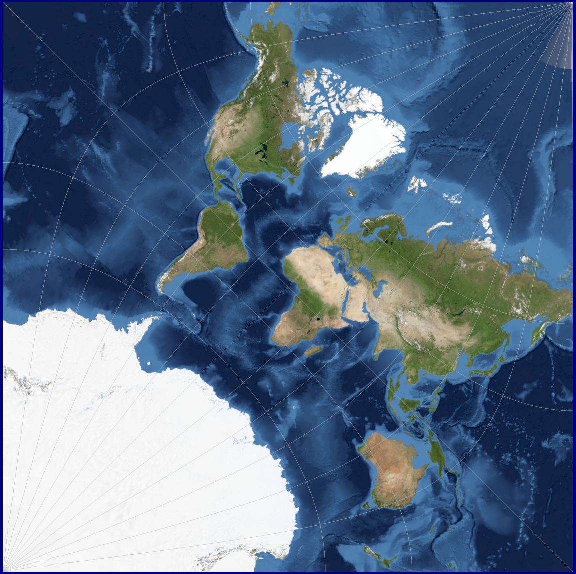
. . . and then conformally mapped to a square



Like before one hemisphere for the quincuncial projection:

$$(\lambda_2, \varphi_2) \longrightarrow (x, y)$$

But here the hemispere contained already the whole (contracted) world!

The Adams projection of the world in a square II (1929)

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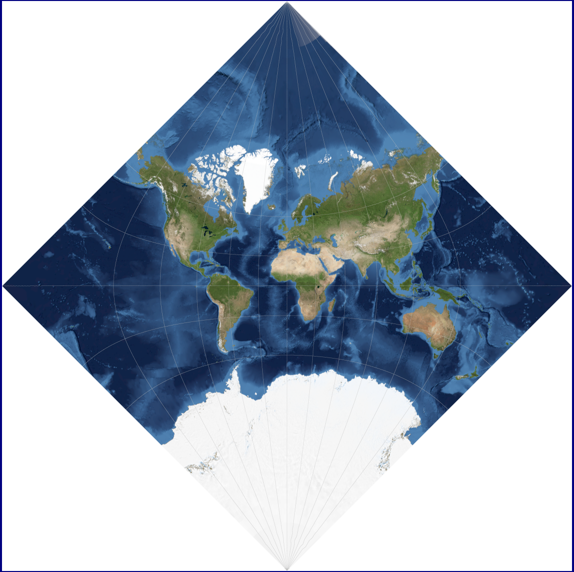

The Spilhaus projection as oblique aspect

1. Rotate the poles well into China (near Hankou) and South America (near Cordoba):
$$-60^\circ \text{ around } (\cos 205^\circ, \sin 205^\circ, 0)$$
2. Rotate the point $(169^\circ\mathrm{W}, 65.3^\circ)$ near the Bering Strait to the edge:
$$-88.02^\circ \text{ around the new } z\text{-axis}$$

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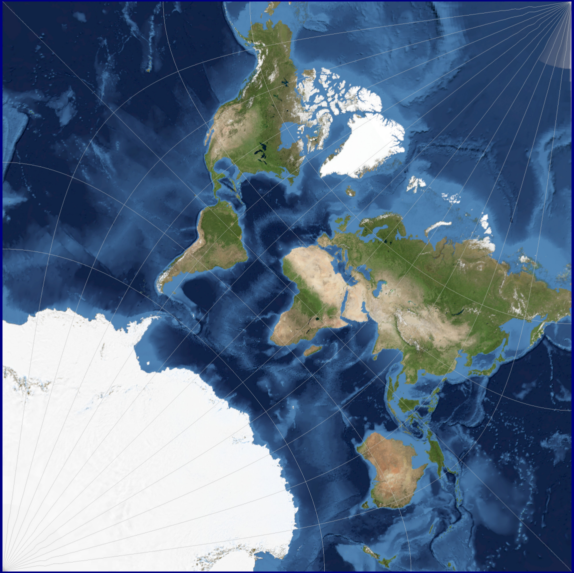
Rotating the poles into China and South America . . .



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Rotating the poles into China and South America . . .

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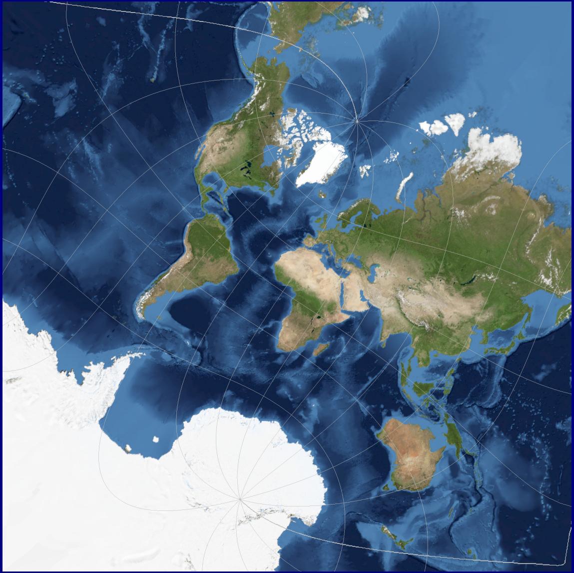

Rotating the poles into China and South America . . .

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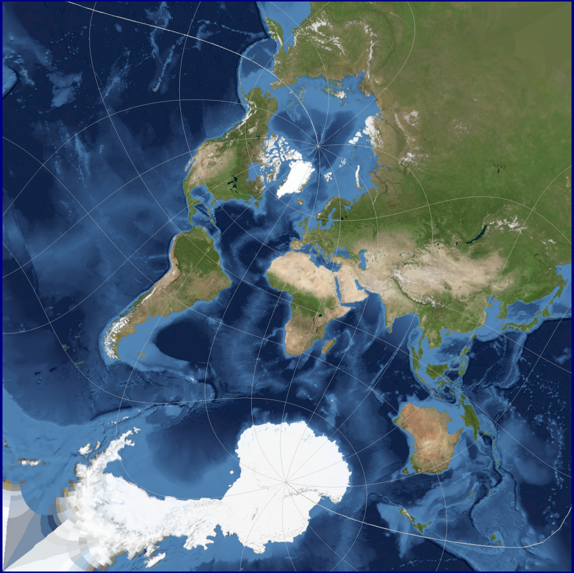

Rotating the poles into China and South America . . .

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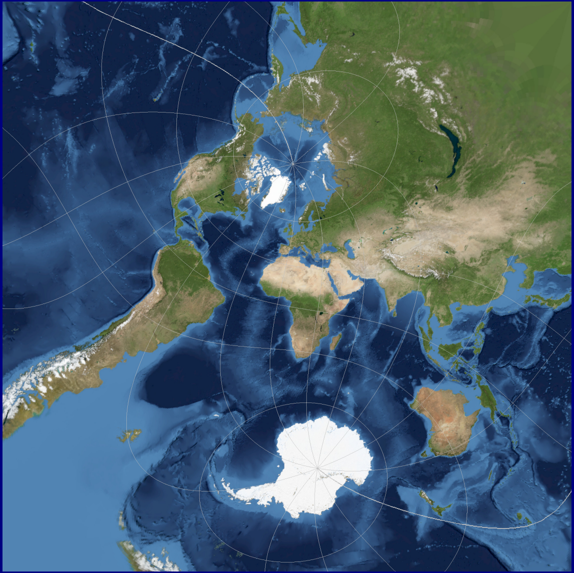

Rotating the poles into China and South America . . .

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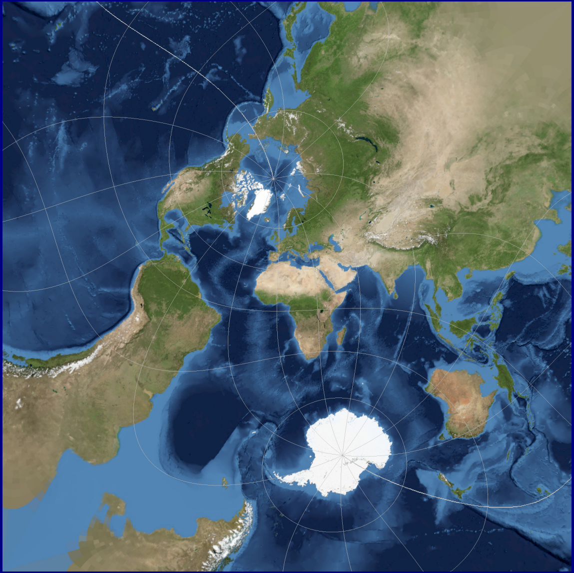
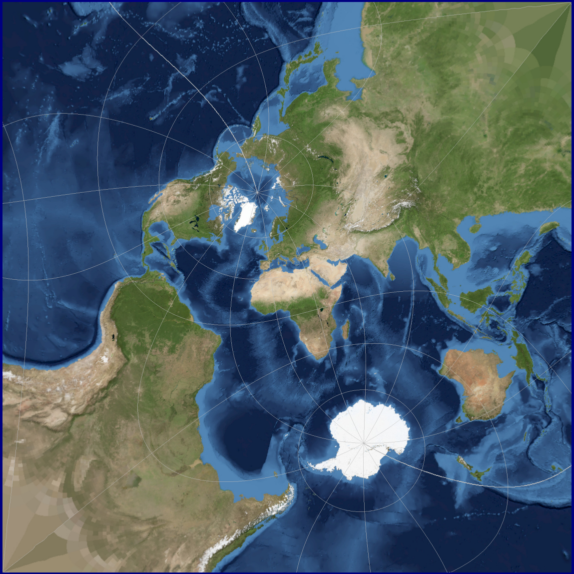


. . . and rotating the Bering Strait to the edge . . .



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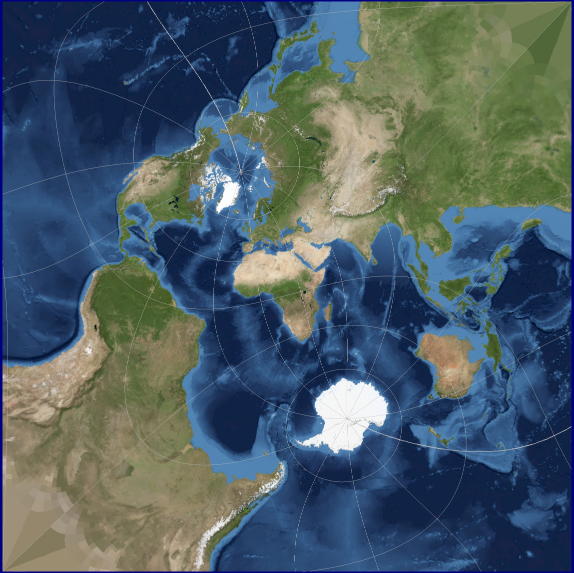

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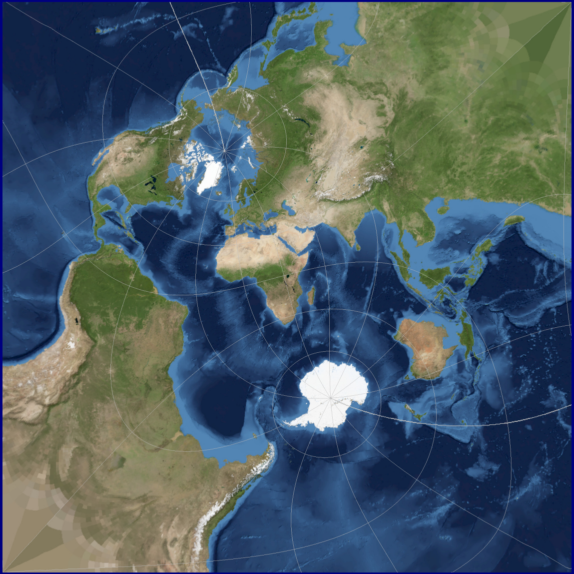

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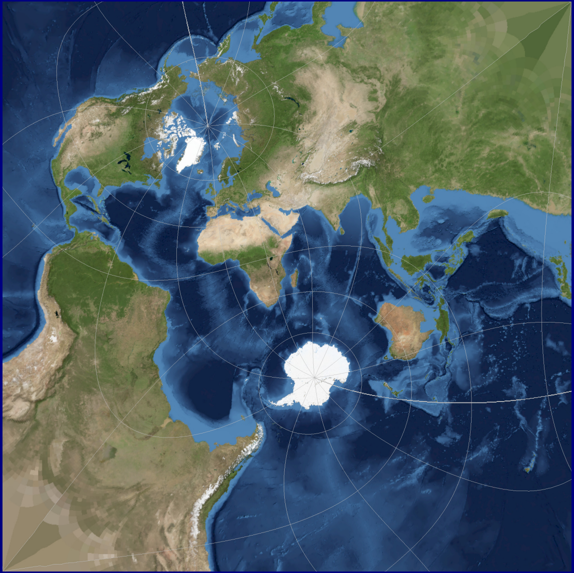

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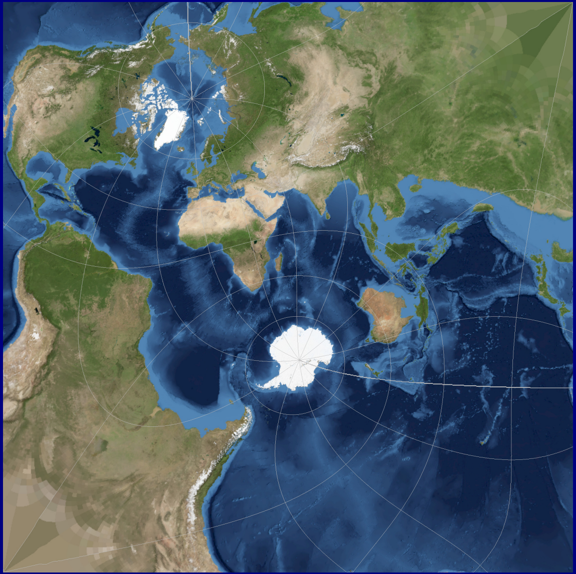

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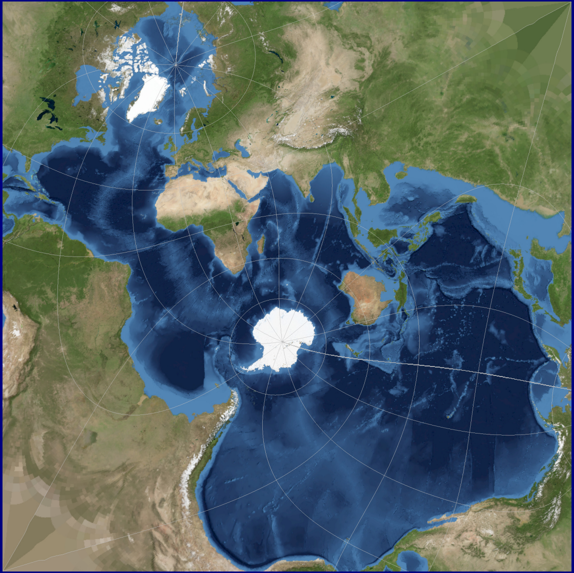
The Spilhaus projection



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Widened view (replications from tesselation)

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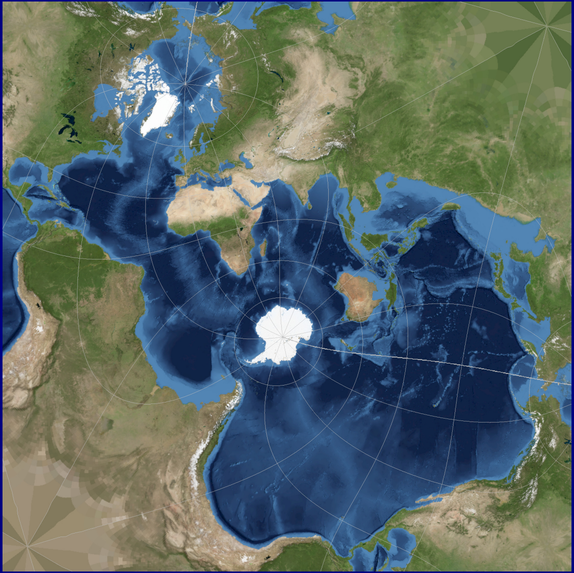
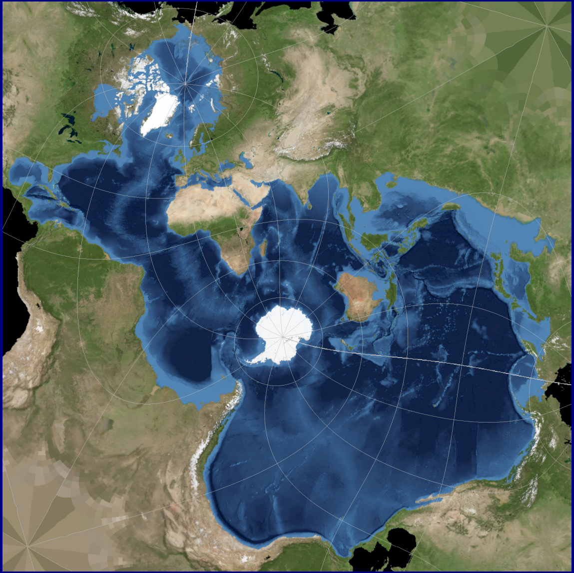


Replicated ocean blacked out



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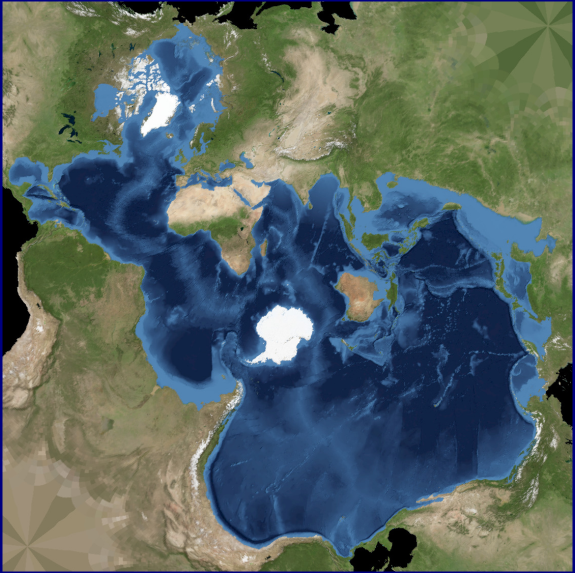

A map showing the whole world ocean without any intersection and with only moderate distortion

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References (1)

Qunicuncial projection

- Peirce, C. S. (1879). "A quincuncial projection of the sphere". American Journal of Mathematics 2 (4): 394–396. doi:10.2307/2369491. Available at https://www.jstor.org/stable/2369491
- Maplesoft, Miscellaneous Projections — Peirce Quincuncial: https://www.maplesoft.com/applications/view.aspx?SID=3589&view=html

Triptychial projection

- Grieger, B. (2019). "Quincuncial adaptive closed Kohonen (QuACK) map for the irregularly shaped comet 67P/Churyumov-Gerasimenko". A&A 630. doi: 10.1051/0004-6361/201834841 (The triptychial projection is described in section 4.4.)
- Tables to do it yourself: http://comsim.esac.esa.int/rossim/bgrieger/triptychial

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References (2)

Spilhaus projection

- ArcGIS, The Spilhaus World Ocean Map in a Square:
  https://storymaps.arcgis.com/stories/756bcae18d304a1eac140f19f4d5cb3d
- Adams, O. S. (1929). "Conformal Projection of the Sphere within a Square". U. S. Coast and Geodetic Survey, Special Publication No. 153. Available at
  https://books.google.com/books/about/Conformal_Projection_of_the_Sphere_Withi.html?id=NI1ZwCAJpD8C

Surface data used

- File
  world.topo.bathy.200407.3x5400x2700.png

  in NASA's Blue Marble collection at
  https://visibleearth.nasa.gov/collection/1484/blue-marble

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