Energy Transfers Between Balanced and Unbalanced Motions in Geophysical Flows

Mesoscale eddies (balanced)

Internal gravity waves (unbalanced)

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Balanced and unbalanced motions

Geophysical flows

two timescale system

Internal gravity waves (unbalanced)

Unbalanced flows

» no balance
» high frequency

fast mode
gravity mode
Gravity manifold

Balanced flows

Mesoscale eddies (balanced)

» Geostrophic balance
» low frequency

slow mode
vortex/vortical mode
Rossby manifold

Non-linear interactions
Different dynamical regimes

Rossby number

\[
\text{Rossby number } \quad \text{Ro} = \frac{\text{flow frequency}}{\text{frequency of rotation}}
\]

- **Weakly stratified**
  - e.g. mixed layer

- **Strongly stratified**
  - e.g. ocean interior

\( \text{Ro} < 1 \)

**Ageostrophic**

- **Strongly coupled**
  - Timescale separation is complex

- **Weakly coupled**
  - Timescales well-separated

\( \text{Ro} \ll 1 \)

**Quasi-geostrophic**
Different dynamical regimes in the Ocean

**Ro < 1**

(Weakly stratified)

**Ro << 1**

(Strongly stratified)
Flow decomposition methods

Non-linear normal mode initialization (NNMI)

» Machenhauer (1977): Initially developed for Numerical Weather Prediction
  » The idea is to initially suppress gravity waves to minimize excitation
  » Leith (1980): Quasi-geostrophic balanced state (1st order in Ro), first iteration

» Warn et. al (1995): higher order in Ro (nth order)

Optimal balance

» Masur and Oliver, 2020, JGAFD : Optimal potential vorticity (OPV) balance based on Viúdez and Dritschel (2004) for a quasi-geostrophic balanced state
  » Iterative procedure
  » Ramp time to match a target potential vorticity
Modal decomposition

Single layer model (scaled):

\[ \partial_t u + u + \nabla h = -Ro \ u \cdot \nabla u \]
\[ \partial_t h + c^2 \nabla \cdot u = -Ro \ \nabla \cdot hu \]

Fourier space:

\[ \partial_t \hat{z} - i A \cdot \hat{z} = Ro \ \hat{n} \]

Linear  \hspace{1cm} \text{Non-linear}

vector \( \hat{z}(k) = (\hat{u}, \hat{v}, \hat{h})^T \)

\[ A = \begin{pmatrix}
0 & -i & -k_x \\
i & 0 & -k_y \\
-c^2k_x & -c^2k_y & 0
\end{pmatrix} \]
Modal decomposition

Single layer model (scaled):

\[ \partial_t u + \frac{u}{\rho} + \nabla h = -\rho_0 u \cdot \nabla u \quad \partial_t h + c^2 \nabla \cdot u = -\rho_0 \nabla \cdot hu \]

Fourier space:

\[ \partial_t \hat{z} - iA \cdot \hat{z} = \rho_0 \hat{n} \]

Linear \hspace{1cm} Non-linear

Balanced mode

Eigenvalues: \[ \omega^0 = 0 \]

Eigenvectors: \[ q^0, p^0 \]

Projection: mode amplitude \[ g^s = p^s \cdot \hat{z} \text{ with } s = 0, \pm \]

\[ \partial_t g^s - i\omega^s g^s = \rho_0 p^s \cdot \hat{n} = -i\rho_0 l^s(g^0, g^\pm) \]

Unbalanced mode

Eigenvalues: \[ \omega^\pm = \pm \sqrt{1 + c^2 k^2} \]

Eigenvectors: \[ q^\pm, p^\pm \]

Projection: mode amplitude \[ g^s = p^s \cdot \hat{z} \text{ with } s = 0, \pm \]

\[ l^s(g^0, g^\pm) = l^s(g^0, 0) + l^s(0, g^\pm) + K^s(g^0, g^\pm) \]

from C-grid discrete operators
Modal decomposition

Modal representation: \[ \partial_t g^s - i\omega^s g^s = Ro p^s \cdot \hat{n} = -iRo l^s(g^0, g^\pm) \]
\[ (Ro \partial_T + \partial_{t^*}) g^s - i\omega^s g^s = -iRo (l^s(g^0, 0) + l^s(0, g^\pm) + K^s(g^0, g^\pm)) \]

Weak interaction assumption: weakly growing waves \( g^\pm = Ro f_1^\pm + Ro^2 f_2^\pm + \ldots \)

expansion in Ro as e.g. in Warn (1996), Kafiabad and Bartello (2017)

introduce fast and slow time scale with \( T = Ro t^* \) and \( \partial_t = Ro \partial_T + \partial_{t^*} \)

slow mode \( g^0 \) varies on \( T \) only, while fast mode \( g^\pm \) has two time scales \( t^* \) and \( T \)
Modal decomposition

» Modal representation:
\[ \partial_t g^s - i \omega^s g^s = Ro \, p^s \cdot \hat{n} = -iRo \, l^s(g^0, g^\pm) \]
\[ (Ro \, \partial_T + \partial_{t^*}) g^s - i \omega^s g^s = -iRo \, (l^s(g^0, 0) + l^s(0, g^\pm) + K^s(g^0, g^\pm)) \]

» Weak interaction assumption: weakly growing waves \( g^\pm = Ro \, f^\pm_1 + Ro^2 \, f^\pm_2 + \ldots \)
» expansion in Ro as e.g. in Warn (1996), Kafiabad and Bartello (2017)
» introduce fast and slow time scale with \( T = Ro \, t^* \) and \( \partial_t = Ro \, \partial_T + \partial_{t^*} \)
» slow mode \( g^0 \) varies on \( T \) only, while fast mode \( g^\pm \) has two time scales \( t^* \) and \( T \)

» SLOW MODE \( s=0 \)
\[ \partial_T g^0 = -il^s(g^0, 0) \]
\[ \partial_T g^0 = -il^s(g^0, f^\pm_1) + il^s(0, f^\pm_1) \]
\[ \partial_T g^0 = -il^s(g^0, f^\pm_2) + il^s(0, f^\pm_2) - il^0(0, f^\pm_1) \]

» FAST MODE \( s=\pm \)
\[ \partial_{t^*} f^\pm_1 - i \omega^\pm f^\pm_1 = -il^\pm(g^0, 0) \]
\[ \partial_T f^\pm_1 + \partial_{t^*} f^\pm_2 - i \omega^\pm f^\pm_2 = -iK^\pm(g^0, f^\pm_1) \]
\[ \partial_T f^\pm_2 + \partial_{t^*} f^\pm_3 - i \omega^\pm f^\pm_3 = -il^\pm(0, f^\pm_1) - iK^\pm(g^0, f^\pm_2) \]

» suppress any wave generation by \( \partial_{t^*} f^\pm_n = 0 \) \( \rightarrow \) ‘slaved’ modes \( f^\pm_n \)

» Machenhauer(1977)
» QG balanced state

» first order slaved mode

\[ f^\pm_1 = l^\pm(g^0, 0)/\omega^\pm, \quad f^\pm_2 = (K^\pm(g^0, f^\pm_1) - i \partial_T f^\pm_1)/\omega^\pm, \ldots \]
Wave emission at higher orders: single layer model

Ro = 0.1

Balanced velocity

Residual (unbalanced) velocity

» double periodic domain 10x5 (dimensionless)
» initialized with an unstable zonal jet

» wave signal seen: is not related to spontaneous emission by shear instability of the balanced flow

» zonal jet meanders and dissolves into eddies
» range of Ro = 0.02 to 0.3, i.e. from mesoscale to sub-mesoscale conditions

» rather, the wave signal could be related to local Rossby numbers with $\text{Ro}_{\text{local}} > 1$: symmetric instability becomes possible

from Eden, Chouksey, and Olbers, JPO, 2019: Gravity wave emission by shear instability
(apparent) Wave emission at higher orders: primitive equation model

1st

- $w_1 \times 10^2$
- $w \times 10^2$

2nd

- $w_2 \times 10^3$
- $(w-w_1) \times 10^3$

3rd

- $w_3 \times 10^4$

4th

- $w_4 \times 10^5$
- $(w-w_1-w_2) \times 10^4$
- $(w-w_1-w_2-w_3) \times 10^5$

Apparent spontaneous emission or 1st or 2nd order slaved modes?

Balanced velocity

Residual (unbalanced) velocity

Weak wave signal is seen only at 4th order.
Convectively generated waves at higher orders: primitive equation model

1st

- a) $w_1 \times 10^2$
- b) $w \times 10^2$

Balanced velocity
Residual (unbalanced) velocity

2nd

- c) $w_2 \times 10^2$
- d) $(w-w_1) \times 10^2$

3rd

- e) $w_3 \times 10^2$
- f) $(w_1-w_2) \times 10^2$

Stronger wave signal seen already at 2nd order.
Wave generation scales exponentially at higher orders for large Rossby number, $Ro$.

Total residual wave energy normalized with the total energy integrated over the model domain.

Dashed lines: different power laws

Exponential scaling law $\exp(-2/Ro)$

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*Eden, Chouksey, and Olbers, 2019: Gravity wave emission by shear instability, JPO*
The non-linear decomposition of balanced and unbalanced motions is achieved and implemented in model in different dynamical regimes up to fourth order.

- Machenhauer (1977) and Warn et al. (1995)

Balanced state is diagnosed in a single layer and primitive equation model using higher order Ro expansion.

- The use of C-grid discrete operators is important for the obtained balanced state.

**Summary**

- Convective instability generates gravity waves rather than spontaneous emission.

- Gravity wave generation scales exponentially at higher orders for large Ro.

- Spontaneous wave emission by shear instability is negligible.
balanced or unbalanced?!

It's as much numerics, as it's the realm of philosophy.