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Overview on the spectral combination of integral transformations

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Content:

- Motivation,
- Theoretical background,
- Numerical experiment,
- Results,
- Conclusion and discussion.

Motivation:

- A method for downward continuation without a regularization parameter,
- a method for recovering gravitational field quantities on the Earth surface,
- relative contribution of each component to a combined solution.

Theoretical background:

The solution of gravimetric boundary-value problem (BVP) is defined as:

$$T^V(r_{Top},\Omega) = \frac{1}{4\pi}\int_{\Omega'}\sum_{n}^{N_{\max}}(2n+1)t_{n,V}^{-1}B_n^V P_{n,0}(\cos\psi)T_z(R_{MOS},\Omega')\mathrm{d}\Omega',$$

$$T^H(r_{Top},\Omega) = \frac{1}{4\pi}\int_{\Omega'}\sum_{n}^{N_{\max}}(2n+1)t_{n,H}^{-1}B_n^H P_{n,1}(\cos\psi)\left[-T_x(R_{MOS},\Omega')\cos\alpha' + T_y(R_{MOS},\Omega')\sin\alpha'\right]\mathrm{d}\Omega',$$

where

$$B_n^V = (-1)\frac{r_{Top}\,n!}{(n+1)!},\quad B_n^H = (+1)\frac{r_{Top}(n-1)!}{(n+1)!}$$

$$t_{n,V} = t_{n,H} = (r_{Top}/R_{MOS})^{(n+2)}.$$

The position of the computation point is designed by spherical geocentric coordinates , where r_{Top} is the geocentric radius on the Earth's topography (Top) and Ω = (φ, λ) substitutes the spherical latitude φ and longitude λ. The position of the integration element is specified by spherical geocentric coordinates . The symbol R_{MOS} represents the geocentric radius of the integration element on a mean orbital sphere (MOS) and Ω' = (φ', λ') stands for the geocentric direction with the spherical latitude φ' and longitude λ'. ψ denotes the spherical distance between the computational point and integration element. Symbols T_x, T_y and T_z represent corresponding first-order directional derivative of the disturbing gravitational potential and $P_{n,m}$ are the non-normalized Legendre function of the first kind of degree n and order m.

Theoretical background:

The solution of the spherical gradiometric boundary-value problem for the disturbing potential is

$$T^{VV}(r_{Top},\Omega)=\frac{1}{4\pi}\int_{\Omega'}\sum_{n}^{N_{\max}}(2n+1)B_n^{VV}t_{n,VV}^{-1}P_{n,0}(\cos\psi)T_{zz}(R_{MOS},\Omega')\mathrm{d}\Omega',$$

$$T^{VH}(r_{Top},\Omega)=\frac{1}{4\pi}\int_{\Omega'}\sum_{n}^{N_{\max}}(2n+1)B_n^{VH}t_{n,VH}^{-1}P_{n,1}(\cos\psi)\left[T_{xz}(R_{MOS},\Omega')\cos\alpha'-T_{yz}(R_{MOS},\Omega')\sin\alpha'\right]\mathrm{d}\Omega',$$

$$T^{HH}(r_{Top},\Omega)=\frac{1}{4\pi}\int_{\Omega'}\sum_{n}^{N_{\max}}(2n+1)B_n^{HH}t_{n,HH}^{-1}P_{n,2}(\cos\psi)$$
$$\times\left[\left(T_{xx}(R_{MOS},\Omega')-T_{yy}(R_{MOS},\Omega')\right)\cos 2\alpha'-2T_{xz}(R_{MOS},\Omega')\sin 2\alpha'\right]\mathrm{d}\Omega',$$

where

$$B_n^{VV}=(+1)\frac{r_{Top}{}^2 n!}{(n+2)!},\ B_n^{VH}=(-1)\frac{r_{Top}{}^2(n-1)!}{(n+2)!},B_n^{HH}=(+1)\frac{r_{Top}{}^2(n-2)!}{(n+2)!}$$

$$t_{n,VV}=t_{n,VH}=t_{n,HH}=(r_{Top}/R_{MOS})^{(n+3)}.$$

Symbols T_{xx}, T_{xy}, T_{xz}, T_{yy}, T_{yz} and T_{zz} represent corresponding second-order directional derivative of the disturbing gravitational potential

Theoretical background:

The solution of the spherical gravitational curvature boundary-value problem is defined as

$$T^{VVV}(r_{Top},\Omega)=\frac{1}{4\pi}\int_{\Omega'}\sum_{n}^{N_{\max}}(2n+1)B_n^{VVV}t_{n,VVV}^{-1}P_{n,0}(\cos\psi)T_{zzz}(R_{MOS},\Omega')\mathrm{d}\Omega',$$

$$T^{VVH}(r_{Top},\Omega)=\frac{1}{4\pi}\int_{\Omega'}\sum_{n}^{N_{\max}}(2n+1)B_n^{VVH}t_{n,VVH}^{-1}P_{n,1}(\cos\psi)\left[T_{xzz}(R_{MOS},\Omega')\cos\alpha'-T_{yzz}(R_{MOS},\Omega')\sin\alpha'\right]\mathrm{d}\Omega',$$

$$T^{VHH}(r_{Top},\Omega)=\frac{1}{4\pi}\int_{\Omega'}\sum_{n}^{N_{\max}}(2n+1)B_n^{VHH}t_{n,VHH}^{-1}P_{n,2}(\cos\psi)$$

$$\times\left[\left(T_{xxz}(R_{MOS},\Omega')-T_{yyz}(R_{MOS},\Omega')\right)\cos 2\alpha'-2T_{xyz}(R_{MOS},\Omega')\sin 2\alpha'\right]\mathrm{d}\Omega',$$

$$T^{HHH}(r_{Top},\Omega)=\frac{1}{4\pi}\int_{\Omega'}\sum_{n}^{N_{\max}}(2n+1)B_n^{HHH}t_{n,HHH}^{-1}P_{n,3}(\cos\psi)\begin{bmatrix}\left(T_{xxx}(R_{MOS},\Omega')-3T_{xyy}(R_{MOS},\Omega')\right)\cos 3\alpha'\\+\left(T_{yyy}(R_{MOS},\Omega')-3T_{xxy}(R_{MOS},\Omega')\right)\sin 2\alpha'\end{bmatrix}\mathrm{d}\Omega',$$

where

$$B_n^{VVV}=(-1)\frac{r_{Top}{}^3 n!}{(n+3)!},\ B_n^{VVH}=(+1)\frac{r_{Top}{}^3(n-1)!}{(n+3)!},\ B_n^{VHH}=(-1)\frac{r_{Top}{}^3(n-2)!}{(n+3)!},\ B_n^{VHH}=(-1)\frac{r_{Top}{}^3(n-3)!}{(n+3)!}$$

$$t_{n,VVV}=t_{n,VVH}=t_{n,VHH}=t_{n,HHH}=(r_{Top}/R_{MOS})^{(n+4)}.$$

Symbols T_{xxx}, T_{xxy}, T_{xxz}, T_{xyy}, T_{xyz}, T_{xzz}, T_{yyy}, T_{yyz}, T_{yzz} and T_{zzz} represent corresponding third-order directional derivative of the disturbing gravitational potential

Theoretical background:

Theoretically, each solution of the gravitmetric, gradiometric and gravitational curvature BVPs should provide same value of the disturbing gravitational potential. We can combine solutions of the same order, as well as, with different orders. When combining two integral transformations (solutions), it reads:

$$2T^{jk}(r_{Top},\Omega) = \frac{1}{4\pi}\int_{\Omega'}\left[\sum_{n}^{N_{\max}}(2n+1)t_{n,j}^{-1}B_n^{j}P_{n,m}(\cos\psi)\right]T_j(R_{MOS},\Omega')\mathrm{d}\Omega'$$
$$+\frac{1}{4\pi}\int_{\Omega'}\left[\sum_{n}^{N_{\max}}(2n+1)t_{n,k}^{-1}B_n^{k}P_{n,m}(\cos\psi)\right]T_k(R_{MOS},\Omega')\mathrm{d}\Omega',$$
$$j,k = V,H,VV,VH,HH,VVV,VVH,VHH,HHH.$$

For example:

$$2T^{VV-VVV}(r_{Top},\Omega) = \frac{1}{4\pi}\int_{\Omega'}\sum_{n=0}^{\infty}(2n+1)t_{n,VV}^{-1}B_n^{VV}P_n(\cos\psi)T_{zz}(R_{MOS},\Omega')\mathrm{d}\Omega'$$
$$+\frac{1}{4\pi}\int_{\Omega'}\sum_{n=0}^{\infty}(2n+1)t_{n,VVV}^{-1}B_n^{VVV}P_n(\cos\psi)T_{zzz}(R_{MOS},\Omega')\mathrm{d}\Omega'.$$

Theoretical background:

Let us to modify previous equation by spectral weights which minimize the root-mean square error of estimated disturbing gravitational potential

$$T^{jk}(r_{Top},\Omega) = \frac{1}{4\pi}\int_{\Omega'}\left[\sum_{n}^{N_{\max}}(2n+1)a_n^{jk}B_n^{j}P_{n,m}(\cos\psi)\right]T_j(R_{MOS},\Omega')\mathrm{d}\Omega'$$

$$+\frac{1}{4\pi}\int_{\Omega'}\left[\sum_{n}^{N_{\max}}(2n+1)a_n^{kj}B_n^{k}P_{n,m}(\cos\psi)\right]T_k(R_{MOS},\Omega')\mathrm{d}\Omega',$$

$$j,k = V,H,VV,VH,HH,VVV,VVH,VHH,HHH,$$

with spectral weights in the following form

$$a_n^{jk} = \frac{t_n^{j}\bar{\sigma}_{n,k}^2}{\left(t_n^{k}\right)^2\bar{\sigma}_{n,j}^2 + \left(t_n^{j}\right)^2\bar{\sigma}_{n,k}^2},\ a_n^{kj} = \frac{t_n^{k}\bar{\sigma}_{n,j}^2}{\left(t_n^{k}\right)^2\bar{\sigma}_{n,j}^2 + \left(t_n^{j}\right)^2\bar{\sigma}_{n,k}^2}\ \text{where}\ \bar{\sigma}_{n,j}^2 = (B_n^{j})^2\sigma_{n,j}^2,\ \bar{\sigma}_{n,k}^2 = (B_n^{k})^2\sigma_{n,k}^2.$$

An example for a two component-combined estimator

$$T^{VV-VVV}(r_{Top},\Omega) = \frac{1}{4\pi}\int_{\Omega'}\sum_{n}^{N_{\max}}(2n+1)a_n^{VV-VVV}B_n^{VV}P_n(\cos\psi)T_{zz}(R_{MOS},\Omega')\mathrm{d}\Omega'$$

$$+\frac{1}{4\pi}\int_{\Omega'}\sum_{n}^{N_{\max}}(2n+1)a_n^{VVV-VV}B_n^{VVV}P_n(\cos\psi)T_{zzz}(R_{MOS},\Omega')\mathrm{d}\Omega'.$$

Theoretical background:

The average of three integral transformations is defined as:

$$3T^{jkl}(r_{Top},\Omega) = \frac{1}{4\pi}\int_{\Omega'}\left[\sum_{n}^{N_{\max}}(2n+1)t_{n,j}^{-1}B_n^j P_n(\cos\psi)\right]T_j(R_{MOS},\Omega')\mathrm{d}\Omega'$$

$$+\frac{1}{4\pi}\int_{\Omega'}\left[\sum_{n}^{N_{\max}}(2n+1)t_{n,k}^{-1}B_n^k P_n(\cos\psi)\right]T_k(R_{MOS},\Omega')\mathrm{d}\Omega'$$

$$+\frac{1}{4\pi}\int_{\Omega'}\left[\sum_{n}^{N_{\max}}(2n+1)t_{n,l}^{-1}B_n^l P_n(\cos\psi)\right]T_l(R_{MOS},\Omega')\mathrm{d}\Omega',$$

$$j,k,l = V, H, VV, VH, HH, VVV, VVH, VHH, HHH.$$

For example:

$$3T^{V-VV-VVV}(r_{Top},\Omega) = \frac{1}{4\pi}\int_{\Omega'}\sum_{n}^{N_{\max}}(2n+1)t_{n,V}^{-1}B_n^V P_n(\cos\psi)T_z(R_{MOS},\Omega')\mathrm{d}\Omega'$$

$$+\frac{1}{4\pi}\int_{\Omega'}\sum_{n}^{N_{\max}}(2n+1)t_{n,VV}^{-1}B_n^{VV} P_n(\cos\psi)T_{zz}(R_{MOS},\Omega')\mathrm{d}\Omega'$$

$$+\frac{1}{4\pi}\int_{\Omega'}\sum_{n}^{N_{\max}}(2n+1)t_{n,VVV}^{-1}B_n^{VVV} P_n(\cos\psi)T_{zzz}(R_{MOS},\Omega')\mathrm{d}\Omega'.$$

Theoretical background:

After modification by spectral weights

$$T^{jkl}(r_{Top},\Omega) = \frac{1}{4\pi}\int_{\Omega'}\left[\sum_{n}^{N_{\max}}(2n+1)a_n^{jkl}B_n^{j}P_{n,m}(\cos\psi)\right]T_j(R_{MOS},\Omega')\mathrm{d}\Omega'$$
$$+\frac{1}{4\pi}\int_{\Omega'}\left[\sum_{n}^{N_{\max}}(2n+1)a_n^{kjl}B_n^{k}P_{n,m}(\cos\psi)\right]T_k(R_{MOS},\Omega')\mathrm{d}\Omega'$$
$$+\frac{1}{4\pi}\int_{\Omega'}\left[\sum_{n}^{N_{\max}}(2n+1)a_n^{ljk}B_n^{l}P_{n,m}(\cos\psi)\right]T_l(R_{MOS},\Omega')\mathrm{d}\Omega',$$
$$j,k,l = V,H,VV,VH,HH,VVV,VVH,VHH,HHH.$$

with spectral weights in the following form

$$a_n^{jkl} = \frac{t_n^j\bar{\sigma}_{n,k}^2\bar{\sigma}_{n,l}^2}{\left(t_n^j\right)^2\bar{\sigma}_{n,k}^2\bar{\sigma}_{n,l}^2+\left(t_n^k\right)^2\bar{\sigma}_{n,j}^2\bar{\sigma}_{n,l}^2+\left(t_n^l\right)^2\bar{\sigma}_{n,j}^2\bar{\sigma}_{n,k}^2},\quad a_n^{kjl} = \frac{t_n^k\bar{\sigma}_{n,j}^2\bar{\sigma}_{n,l}^2}{\left(t_n^j\right)^2\bar{\sigma}_{n,k}^2\bar{\sigma}_{n,l}^2+\left(t_n^k\right)^2\bar{\sigma}_{n,j}^2\bar{\sigma}_{n,l}^2+\left(t_n^l\right)^2\bar{\sigma}_{n,j}^2\bar{\sigma}_{n,k}^2},$$
$$a_n^{ljk} = \frac{t_n^l\bar{\sigma}_{n,j}^2\bar{\sigma}_{n,k}^2}{\left(t_n^j\right)^2\bar{\sigma}_{n,k}^2\bar{\sigma}_{n,l}^2+\left(t_n^k\right)^2\bar{\sigma}_{n,j}^2\bar{\sigma}_{n,l}^2+\left(t_n^l\right)^2\bar{\sigma}_{n,j}^2\bar{\sigma}_{n,k}^2}.$$

Theoretical background:

An example for a three component-combined estimator

$$T^{V-VV-VVV}(r_{Top},\Omega) = \frac{1}{4\pi}\int_{\Omega'}\sum_{n}^{N_{\max}}(2n+1)a_n^{V-VV-VVV}B_n^{V}P_n(\cos\psi)T_z(R_{MOS},\Omega')\mathrm{d}\Omega'$$
$$+\frac{1}{4\pi}\int_{\Omega'}\sum_{n}^{N_{\max}}(2n+1)a_n^{VV-V-VVV}B_n^{VV}P_n(\cos\psi)T_{zz}(R_{MOS},\Omega')\mathrm{d}\Omega'$$
$$+\frac{1}{4\pi}\int_{\Omega'}\sum_{n}^{N_{\max}}(2n+1)a_n^{VVV-V-VV}B_n^{VVV}P_n(\cos\psi)T_{zzz}(R_{MOS},\Omega')\mathrm{d}\Omega'.$$

In the same way, we can define four, five, six, seven, eight and nine component estimator. The spectral weights for *N*-component estimator are defined as follows

$$a_n^{j} = \frac{t_{n,j}\left(\prod_{j'=1}^{N}\bar{\sigma}_{n,j'}^{2}\Big/\bar{\sigma}_{n,j}^{2}\right)}{\sum_{j=1}^{N}\left(t_{n,j}\right)^{2}\left\{\prod_{j'=1}^{N}\bar{\sigma}_{n,j'}^{2}\Big/\bar{\sigma}_{n,j}^{2}\right\}}.$$

Numerical experiment: Simulated satellite data

T_x, T_y, T_z, T_{xx}, T_{xy}, T_{xz}, T_{yy}, T_{yz}, T_{zz}, T_{xxx}, T_{xxy}, T_{xxz}, T_{xyy}, T_{xyz}, T_{xzz}, T_{yyy}, T_{yyz}, T_{yzz} and T_{zzz}

- generated from EIGEN6C4 up to the degree 250 at the satellite level r = 6628136.46 m,

- regular equiangular grid limited by $\varphi \in [-89.75°;89.75°]$, $\lambda \in [0.25°;359.75°]$ with the grid step of 0.5°,

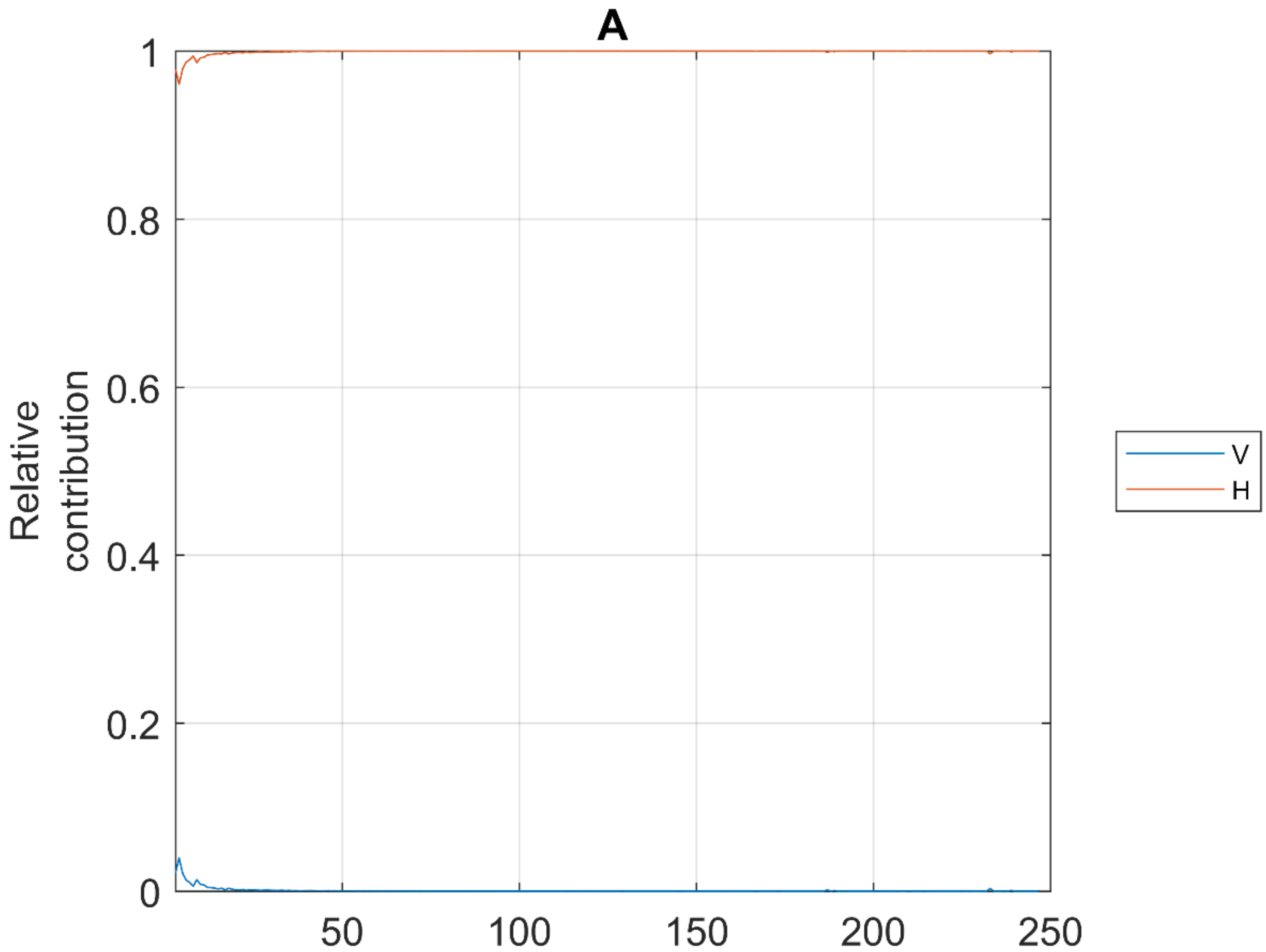
- simulated satellite data polluted by Gaussian noise of 1.7×10^{-14} m s^{-2}, 2.4×10^{-20} s^{-2} and 1.2×10^{-25} m^{-1} s^{-2}, respectively,

- GOAL – to investigate how different components influence combined solutions,

- TEST POINT – Mount Everest.

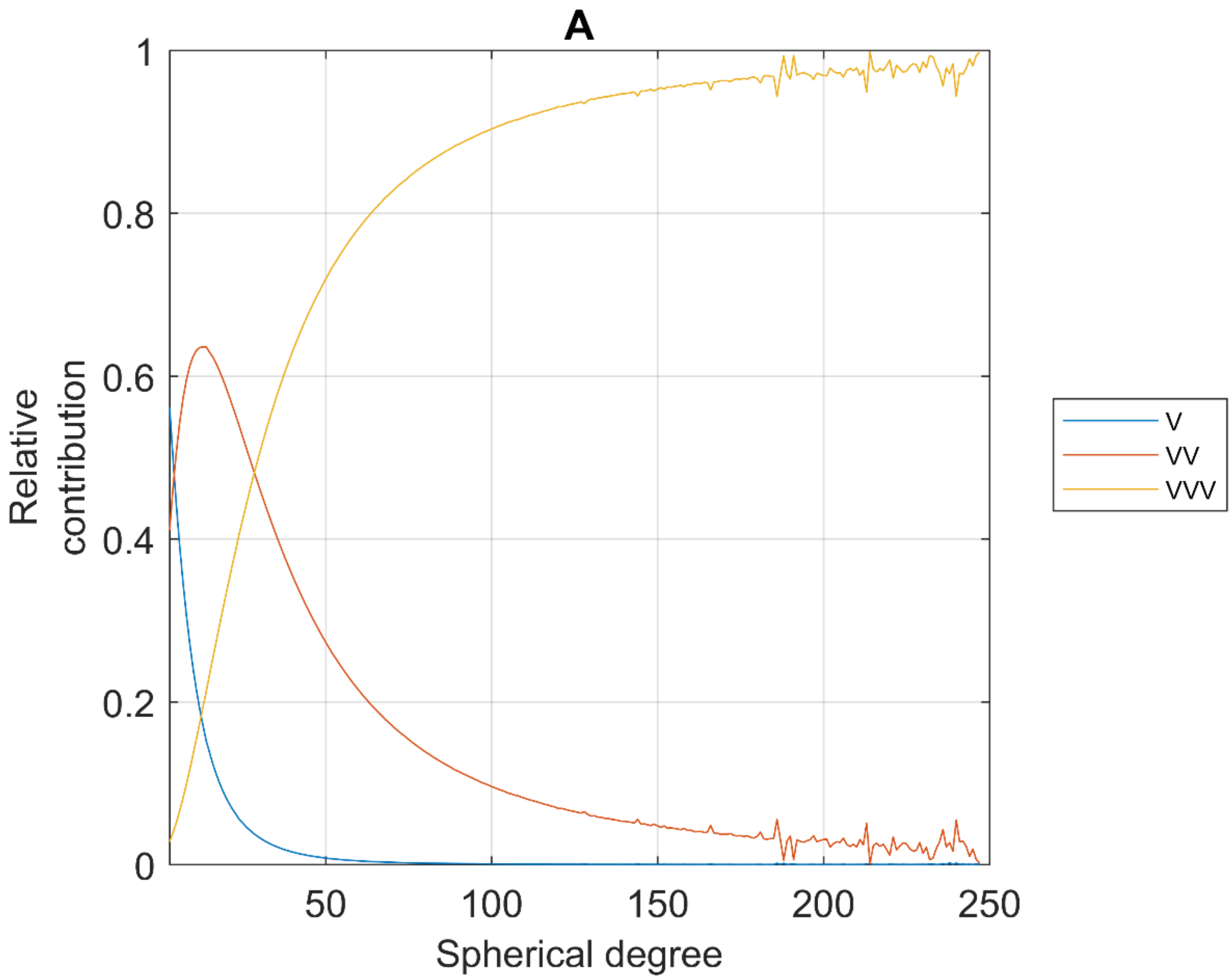
Results: Contribution analysis (Mount Everest)

Two component-combined estimator $T^{V\text{-}H}$



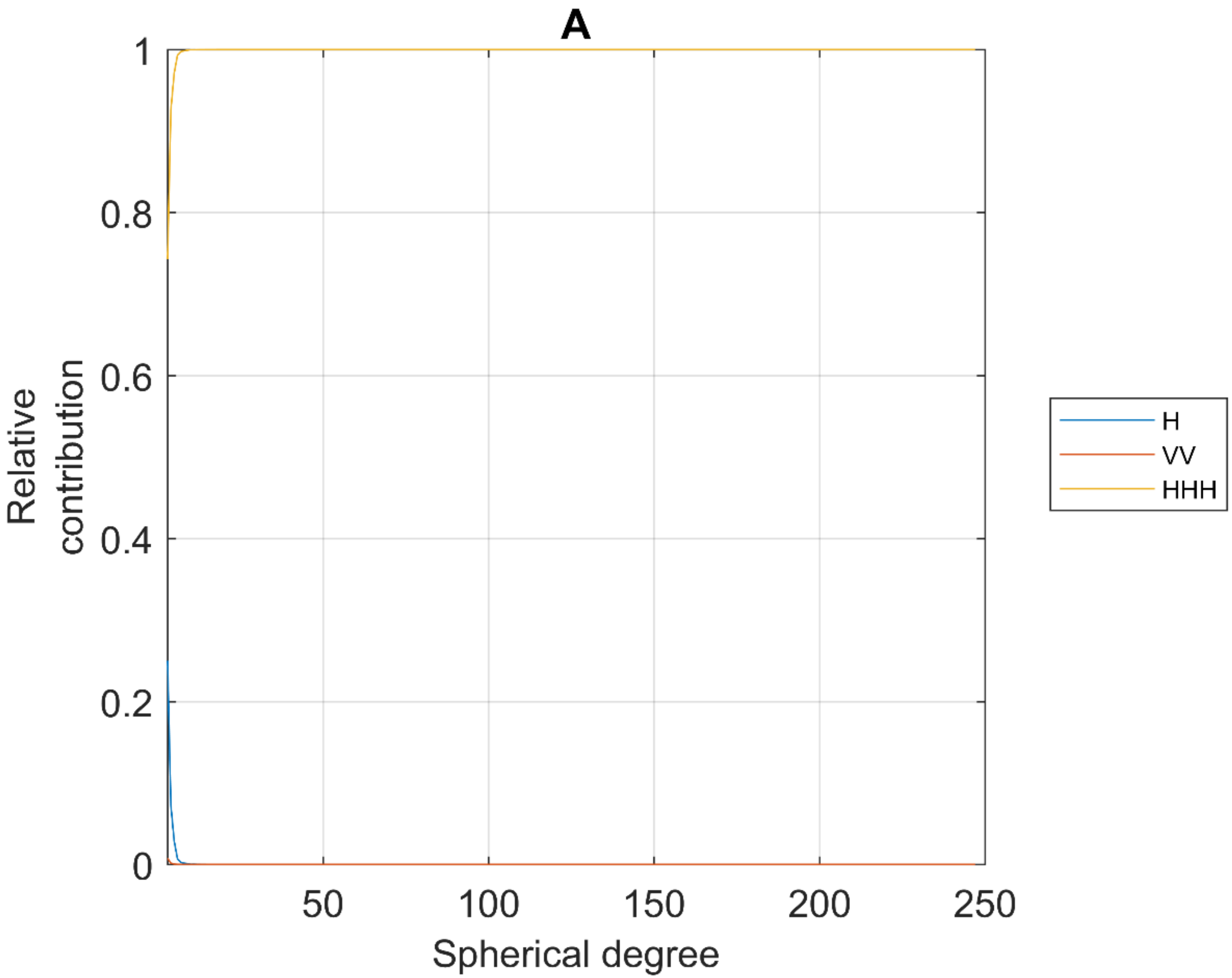
Results: Contribution analysis (Mount Everest)

Three component-combined estimator $T^{V\text{-}VV\text{-}VVV}$



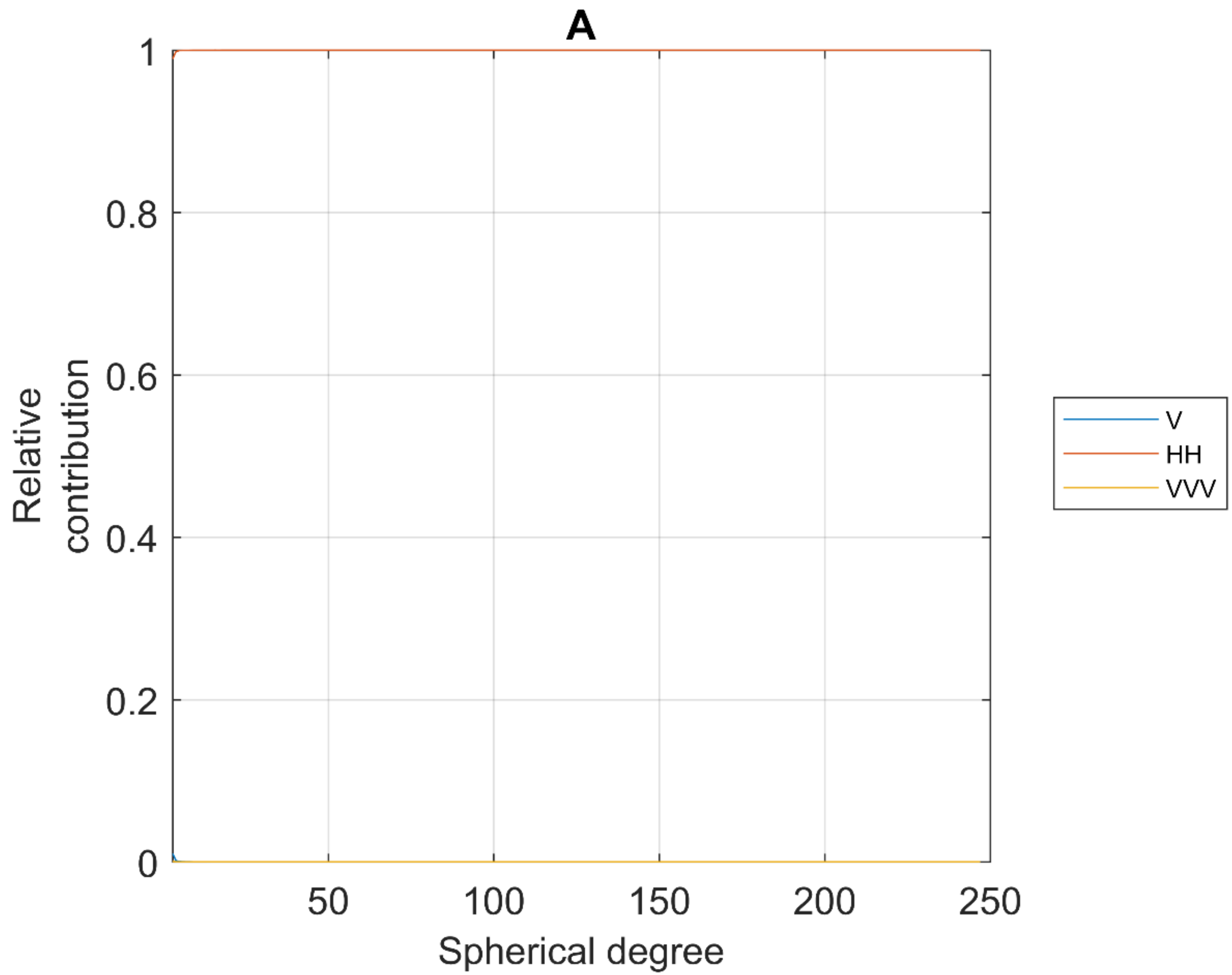
Results: Contribution analysis (Mount Everest)

Three component-combined estimator $T^{H\text{-}VV\text{-}HHH}$



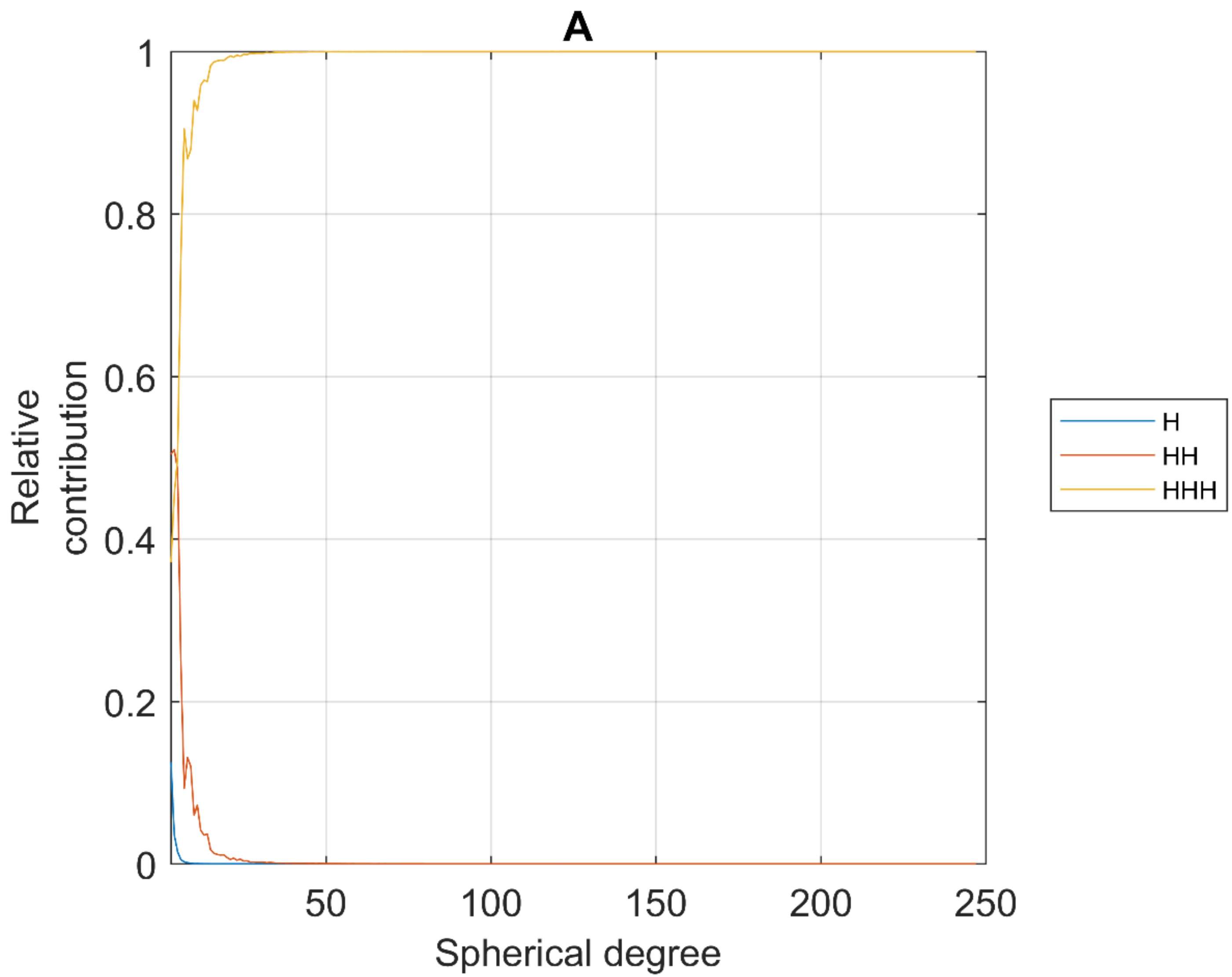
Results: Contribution analysis (Mount Everest)

Three component-combined estimator $T^{V\text{-}HH\text{-}VVV}$



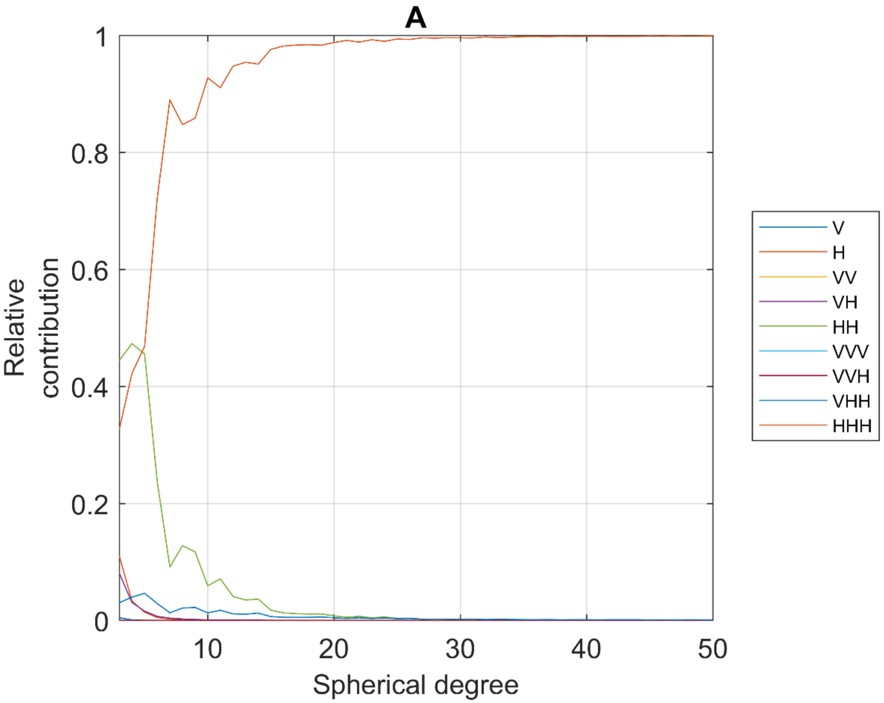
Results: Contribution analysis (Mount Everest)

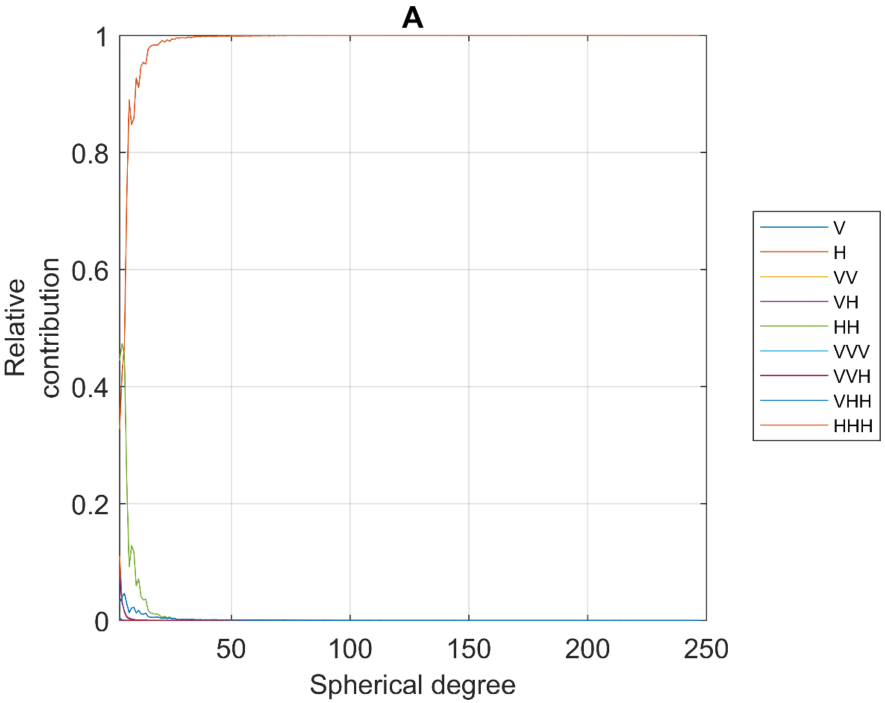
Three component-combined estimator $T^{H\text{-}HH\text{-}HHH}$



Results: Contribution analysis (Mount Everest)

Nine component-combined estimator $T^{V\text{-}H\text{-}VV\text{-}VH\text{-}HH\text{-}VVV\text{-}VVH\text{-}VHH\text{-}HHH}$





Conclusion and discussion:

- A method for downward continuation satellite data on the Earth surface without a regularization parameter was derived and tested,

- Spectral weights have been generalized,

- Horizontal derivatives of disturbing gravitational potential influence results more than vertical derivatives,

- Higher-order derivatives of disturbing gravitational potential influence results more than lover-order derivatives

- For more details about combination of gravitational curvature boundary-value problem and of vertical solutions, respectively, please see

Pitoňák M, Eshagh M, Šprlák M, Tenzer R, Novák P (2018). Spectral combination of spherical gravitational curvature boundary-value problems. Geophysical Journal International 214(2): 773-791

Pitoňák M, Novák P, Eshagh M, Tenzer R, Šprlák M. Downward continuation of gravitational field quantities to an irregular surface by spectral weighting. Submitted to Journal of Geodesy (after first revision).

Thank you for your attention

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