Estimability in Rank-Defect Mixed-Integer Models

A new estimation criterion for mixed integer GNSS models

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G1 – Geodetic Theory and Algorithms --- G1.1 Session: Recent Developments in Geodetic Theory

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Estimability of Parameter Subsets

Definition

$$\mathsf{E}(y) = A_1\, x_1 + A_2\, x_2, \quad x_1 \in \mathbb{R}^{n_1},\ x_2 \in \mathbb{R}^{n_2}$$

Function $F^T x_1$ is said to be **estimable** if it can be **unbiasedly** estimated by a function of the observations y

[Rao 1973]

Necessary and sufficient conditions

$$\begin{cases} (i):\ F \perp A_2 \quad (F^T A_2 = 0) \\ (ii):\ F^T = L^T A_1 \end{cases}$$

(i) ➡ ***Must be x_2-free**

(ii) ➡ **Must be a function of the observations' expectation** $\mathsf{E}(y)$

Integer Estimability

*Existing theory of estimability has to be extended for **mixed integer models**!*

Definition

$$\mathsf{E}(y) = A_1\, z_1 + A_2\, x_2, \quad z_1 \in \mathbb{Z}^{n_1},\ x_2 \in \mathbb{R}^{n_2}$$

Function $\tilde{z} = F^T z_1$ is said to be **integer-estimable** if:
1) it is **estimable** and 2) F guarantees the existence of an integer z_1 for **every integer** $\tilde{z}$

Necessary and sufficient conditions

$$\begin{cases} (i): \ F \perp A_2 \quad (F^T A_2 = 0) \\ (ii): \ F^T = L^T A_1 \\ (iii): \ F^T \mathcal{Z} = [0, I], \quad \text{for some } \mathcal{Z} \end{cases}$$

(i) ➡ **Must be x_2-free**

(ii) ➡ **Must be a function of the observations' expectation** $\mathsf{E}(y)$

(iii) ➡ ***Can be derived from any original integers**

$\mathcal{Z}$: *The **admissible ambiguity transformations** whose entries and those of their inverse are integer-valued* [Teunissen 1995]

Theorem 1 (Integer-Estimability) *Let* $\mathsf{E}(y) = Az + Bb$ *be a mixed-integer model, where* $y \in \mathbb{R}^m$ $z \in \mathbb{Z}^n$ *and* $b \in \mathbb{R}^v$. *Then, the necessary and sufficient conditions for p linearly independent functions* $\tilde{z} = F^T z$ *to be estimable or integer-estimable are as follows*:

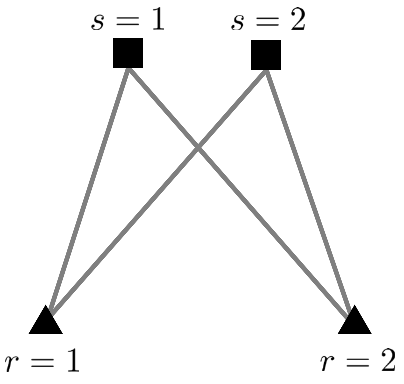
1. $\tilde{z} = F^T z$ *is estimable* iff $F = A^T B^{\perp} X$ *for some* X, *where* $B^{\perp}$ *is a basis matrix of the orthogonal complement of the range space of* B.

2. $\tilde{z} = F^T z$ *is integer-estimable* iff $F = A^T B^{\perp} X$ *for some* X *and* $F^T \mathcal{Z} = [I_p, 0]$ *for some admissible ambiguity transformation* $\mathcal{Z}$.

Integer estimability of the DD ambiguities

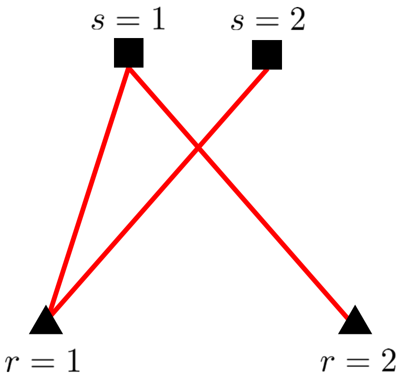
What about the **standard DD ambiguities**? Are they integer-estimable?

Yes, **graph theory** can be used to form DD ambiguities and to show that they are indeed integer-estimable.



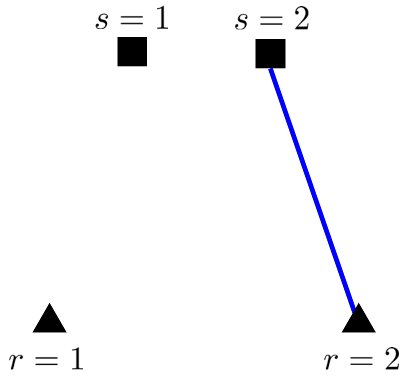
Network ambiguity graph

➡



Spanning tree
(Choosing pivot ambiguities)

➡



Integer-estimable ambiguity

Incidence-matrix of the spanning tree

$$\begin{pmatrix} 0, & -1, & 0, & 0\\ 0, & 0, & -1, & 0\\ 1, & -1, & 0, & 0\\ 1, & 0, & -1, & 1\end{pmatrix}^{-1} = \begin{pmatrix} -1, & 0, & 1, & 0\\ -1, & 0, & 0, & 0\\ 0, & -1, & 0, & 0\\ 1, & -1, & -1, & 1\end{pmatrix}$$

$\mathcal{Z}$

➡

$$\underbrace{[1,\ -1,\ -1,\ 1]}_{F^T}\underbrace{\begin{pmatrix} 0, & -1, & 0, & 0\\ 0, & 0, & -1, & 0\\ 1, & -1, & 0, & 0\\ 1, & 0, & -1, & 1\end{pmatrix}}_{\mathcal{Z}} = [0,\ 0,\ 0,\ 1]$$

Example 1 (*Wide-lane narrow-lane integer fixing*) The wide-lane and narrow-lane ambiguities, $z_w = z_1 - z_2$ and $z_n = z_1 + z_2$, are two well-known combinations of GPS DD ambiguities (Goad 1992; Teunissen 1995). However, as the following shows, they may not be used in paired form for integer ambiguity resolution:

$$\begin{bmatrix} z_w \\ z_n \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{F^T} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \frac{1}{2} \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{F^{-T} \neq \mathbb{Z}} \begin{bmatrix} z_w \\ z_n \end{bmatrix} \tag{10}$$

Since $F \in \mathbb{Z}^{2\times2}$, both the wide-lane and narrow-lane are integer whenever the DD ambiguities are integer. The converse is not true, however. Since the inverse of F is not admissible, the DD ambiguities z_1 and z_2 are not anymore guaranteed to be integer, for every integer values of z_w and z_n. Hence, would one integer resolve $(z_w, z_n)^T$, one may implicitly have fixed the integer DD ambiguities to *non-integer* values and thereby thus have forced the model to inconsistent and wrong constraints.

Multi-frequency integer-combinations

Recently, **integer-combined carrier-phase observations** are proposed to reduce the impact of the ionosphere and/or to minimize the variance of the resultant combinations.

[Feng, 2008; Cocard et al. 2008; Shu et al. 2017]

$$\underbrace{[2,\ 4,\ -6]}_{F^T}\underbrace{\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{Z} = [0,\ 0,\ 2] \neq [0,\ 0,\ 1]$$

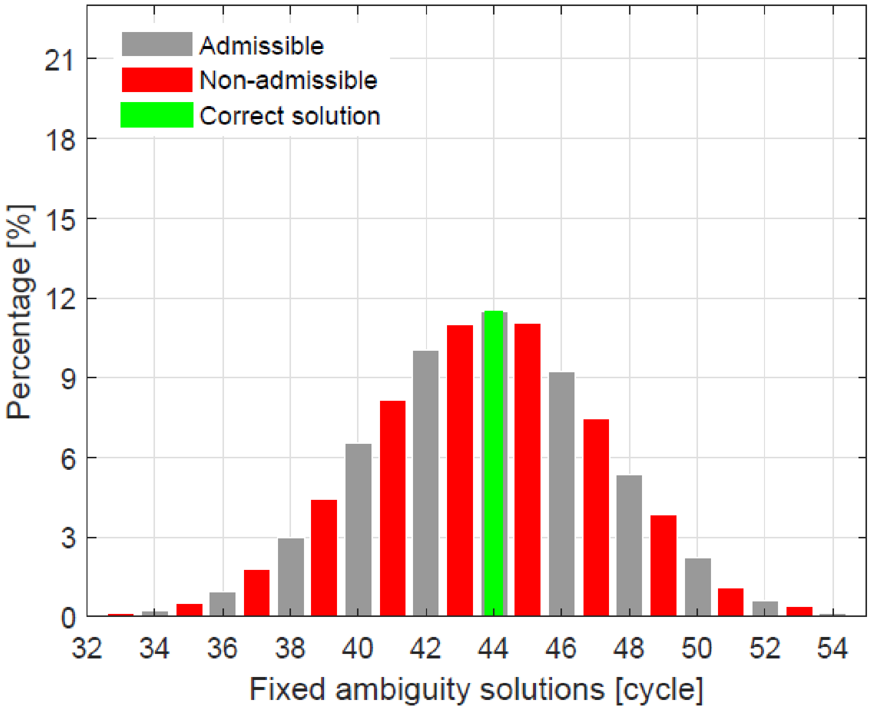
***Example*: Galileo triple-frequency**

$$\tilde{z} = \underset{\substack{\uparrow \\ \text{E1}}}{2\tilde{z}_1} + \underset{\substack{\uparrow \\ \text{E5b}}}{4\tilde{z}_2} - \underset{\substack{\uparrow \\ \text{E6}}}{6\tilde{z}_3}$$

20,000** DD ambiguity samples collected by a zero-baseline, of which **49.9%** are odd numbers, thereby **not admissible!

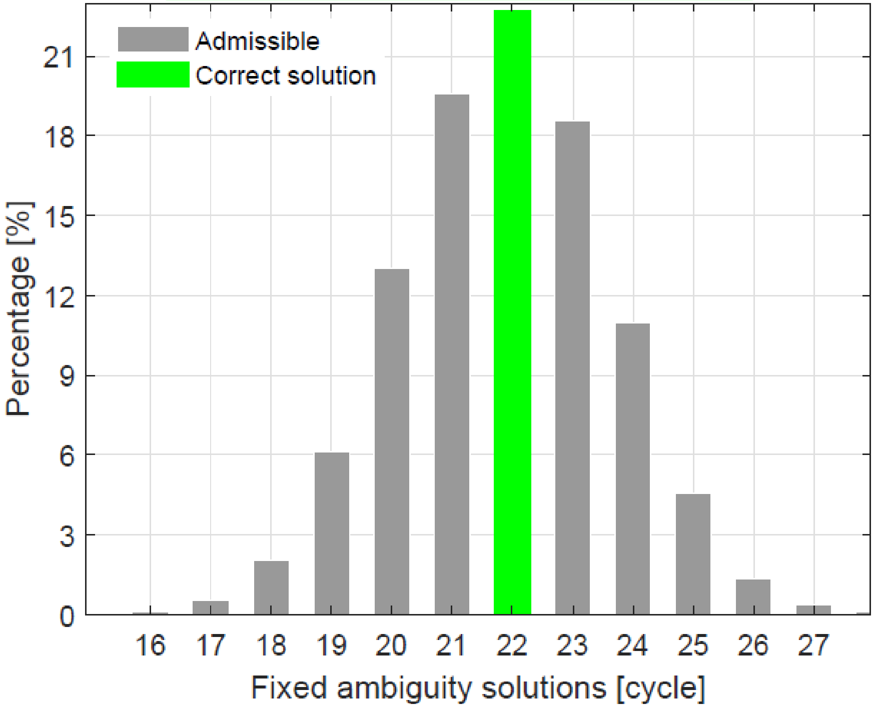
$$\tilde{z} = 2\tilde{z}_1 + 4\tilde{z}_2 - 6\tilde{z}_3$$

Estimable and integer* *(but not integer estimable)



$$\tilde{\tilde{z}} = \tilde{z}_1 + 2\tilde{z}_2 - 3\tilde{z}_3$$

Integer Estimable



Example 4 (*Integer combination of DD ambiguities*) Another integer combination that is estimable is

$$\eta_{,j} = a_{12} z_{1r,j}^{13} - a_{13} z_{1r,j}^{12} \tag{14}$$

It is an integer combination of GLONASS DD ambiguities, which can be written in terms of the undifferenced ambiguities as $\eta_{,j} = F^T z$, where $F^T = [a_{23}, -a_{13}, a_{12}]$ and $z = [z_{1r,j}^1, z_{1r,j}^2, z_{1r,j}^3]^T$. As we have the decomposition

$$\underbrace{[a_{23}, -a_{13}, a_{12}]}_{F^T} \underbrace{\begin{bmatrix} \alpha & a_{13}/g & 1 \\ \beta & a_{23}/g & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathcal{Z}} = [g, 0, 0] \tag{15}$$

with $\alpha a_{23} - \beta a_{13} = g$ and $g = \text{GCD}(a_{23}, a_{13})$, it directly follows that $\eta_{,j}$ is not integer-estimable in general. It is integer-estimable if $a_{23} = 1$, $a_{13} = 1$ or $a_{12} = 1$, since then $g = 1$. Note that $\text{GCD}(a_{23}, a_{13}) = \text{GCD}(a_{23}, a_{12})$.

Example 7 (*Wide-lane or narrow-lane integer fixing*) The wide-lane and narrow-lane ambiguities are separately integer-estimable, since

$$\underbrace{[1,-1]}_{F_w^T}\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{Z_w} = [1,0], \quad \underbrace{[1,1]}_{F_n^T}\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}}_{Z_n} = [1,0] \tag{26}$$

But although they are both separately integer-estimable, they are not jointly integer-estimable as Example 1 has shown. Thus, one will always be able to find integer L1 and L2 DD ambiguities for each integer wide-lane ambiguity $z_w \in \mathbb{Z}$, and also separately for each integer narrow-lane ambiguity $z_n \in \mathbb{Z}$, but not necessarily for each integer pair $(z_w, z_n)^T \in \mathbb{Z}^2$.

GLONASS FDMA integer estimable ambiguities

*The FDMA GLONASS wavelengths are **different** for different **channel numbers***

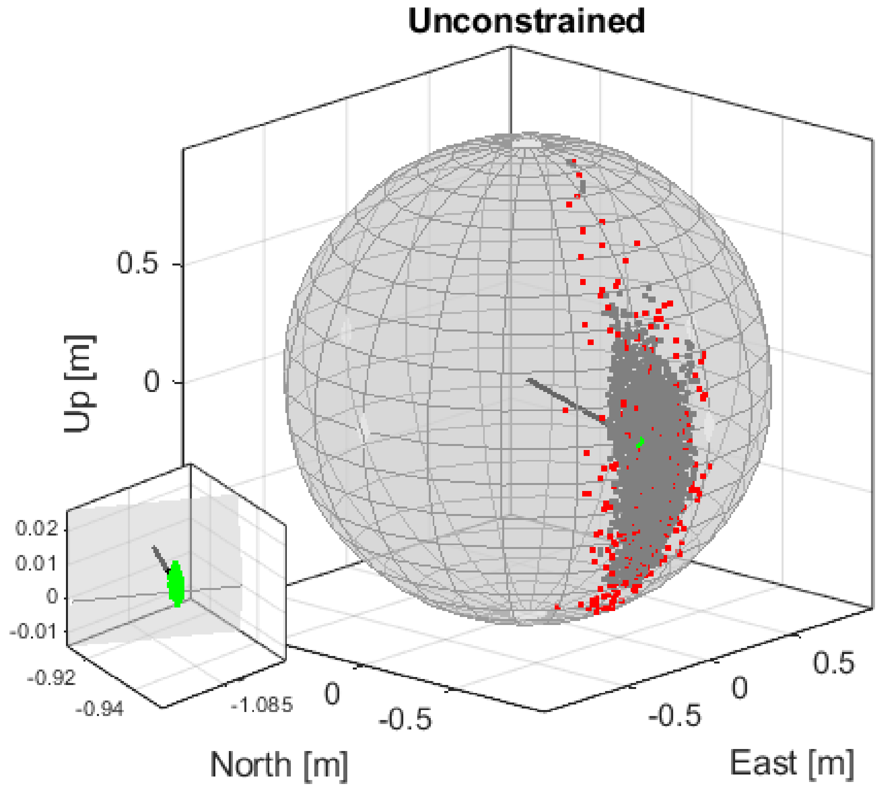
$$\lambda^s = \frac{z_\epsilon}{z_\epsilon + \kappa^s}\,\lambda_o,\;\; s = 1, \ldots, 24 \quad \Rightarrow \quad \mathsf{E}(\phi_{12}^{12}) = \lambda^2\, z_{12}^2 - \lambda^1\, z_{12}^1 = [\frac{z_\epsilon\, g}{(z_\epsilon + \kappa^1)(z_\epsilon + \kappa^2)}\lambda_o]\,\tilde{z}$$

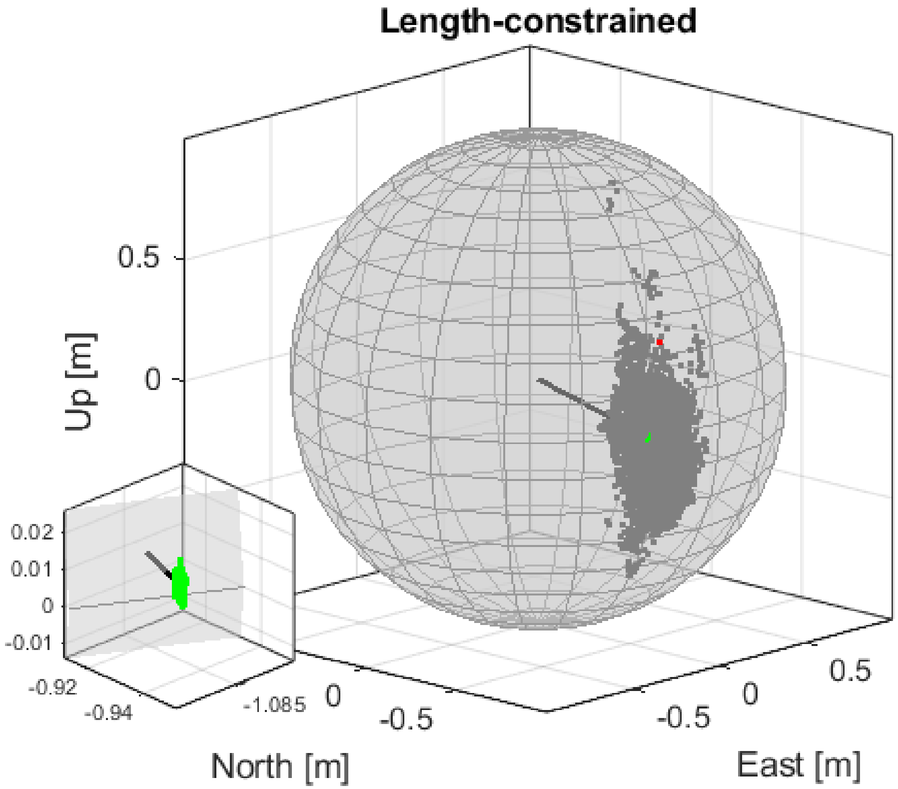
$$z_\epsilon = 2848,\;\; g = \mathrm{GCD}(z_\epsilon + \kappa^1, z_\epsilon + \kappa^2)$$

$$\boxed{\tilde{z} = \frac{z_\epsilon + \kappa^1}{g}\, z_{12}^2 - \frac{z_\epsilon + \kappa^2}{g}\, z_{12}^1}$$

[Teunissen 2019; Teunissen and Khodabandeh 2019]

Unaided GLONASS direction-finding





Low-cost receiver

Without ambiguity fixing

CORRECT ambiguity fixing

WRONG ambiguity fixing

GLONASS FDMA integer estimable ambiguities

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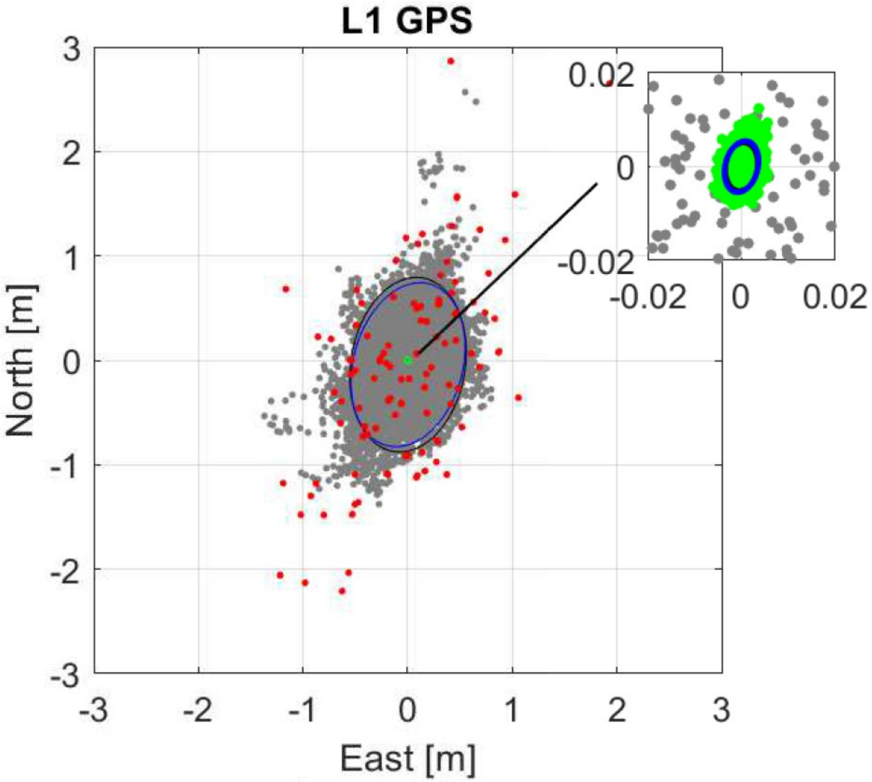
$$\lambda^s = \frac{z_\epsilon}{z_\epsilon + \kappa^s}\lambda_o, \;\; s = 1, \dots, 24 \quad \Rightarrow \quad \mathsf{E}(\phi_{12}^{12}) = \lambda^2 z_{12}^2 - \lambda^1 z_{12}^1 = [\frac{z_\epsilon\, g}{(z_\epsilon + \kappa^1)(z_\epsilon + \kappa^2)}\lambda_o]\, \tilde{z}$$

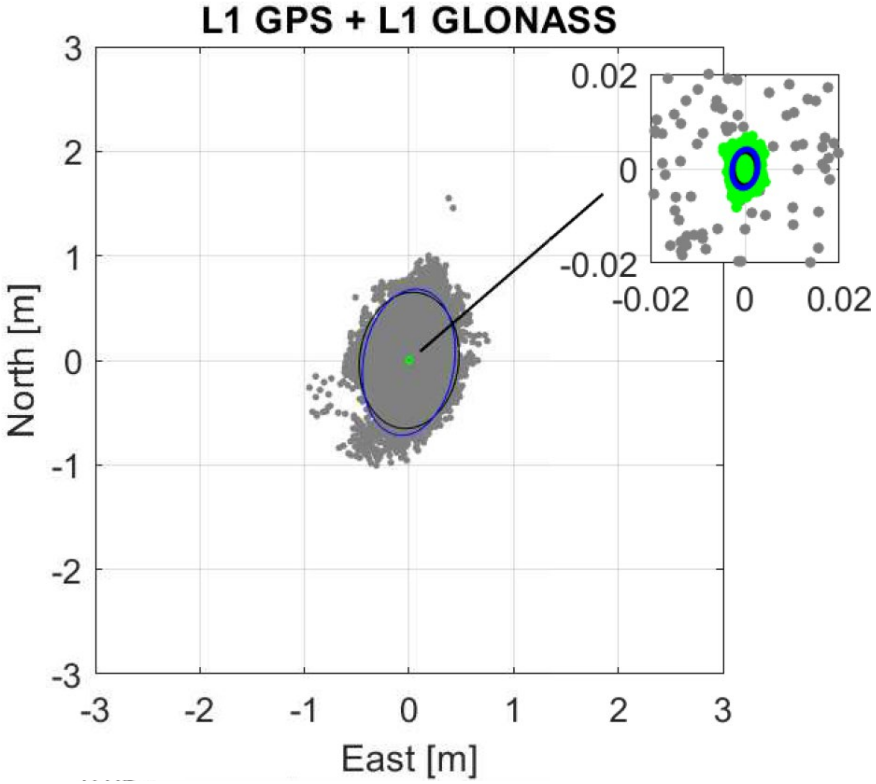
$$z_\epsilon = 2848, \;\; g = \mathrm{GCD}(z_\epsilon + \kappa^1, z_\epsilon + \kappa^2)$$

$$\tilde{z} = \frac{z_\epsilon + \kappa^1}{g} z_{12}^2 - \frac{z_\epsilon + \kappa^2}{g} z_{12}^1$$

[Teunissen 2019; Teunissen and Khodabandeh 2019]

GLONASS + GPS single-frequency RTK





Low-cost receiver

Without ambiguity fixing

CORRECT ambiguity fixing

WRONG ambiguity fixing

Concluding remarks

The standard concept of estimability is too limited when dealing with rank-defect mixed integer models

that would result in pitfalls in which float ambiguity solutions are **integer-mean**, but do not necessarily correspond to the original ambiguities

Integer Estimability provides tools for

- **Network-derived ambiguities**
- **GLONASS Ambiguity Resolution**
- **Non-GNSS mixed-integer models**

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