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Laplacian Structure, Solution Domain Geometry and Successive Approximations in Gravity Field Studies

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1. Introduction

Green's functions are an important tool in solving *problems of mathematical physics*. Equally this holds in gravity field studies.

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Green's function is an integral kernel, which, convolved with input values, gives the solution of the particular problem considered.

●

Regarding its construction, there exist elegant and powerful methods for one or two dimensional problems.

●

However, *only very few of these methods carried over to higher dimensions*. The higher the dimension of the Euclidean space the simpler the boundary of the region of interest had to be.

●

In order to preserve the benefit of the Green's function method an *approximation procedure is discussed*. *Our aim is* to implement the procedure with the particular focus on the solution of the *LGBVP (Linear gravimetric boundary value problem)* and on the *related functional analytic aspects*.

2. Linear Gravimetric Boundary-Value Problem

For this problem the *solution domain* Ω is the *exterior of the Earth* and the problem means to find T such that

$$\Delta T = div\, \mathbf{grad}\, T = 0 \quad \text{in} \quad \Omega \, ,$$

$$\frac{\partial T}{\partial \mathbf{s}} = \langle \mathbf{s}, \mathbf{grad}\, T \rangle = -\,\delta g \quad \text{on} \quad \partial\Omega \, ,$$

where

$$\mathbf{s} = -\frac{1}{\gamma}\, \mathbf{grad}\, U \, ,$$

$\langle\, ,\rangle$ is the inner product, Δ means *Laplace's operator* and $\partial\Omega$ is the boundary of Ω.

Note that $T = W - U$ is the *disturbing potential* and $\delta g = g - \gamma$ the *gravity disturbance*, where with W and U we identify the gravity and a normal potential of the Earth.

To be more specific, we recall the ***classical theory of normal gravity*** referred to a ***level ellipsoid*** with semiaxes a and b, $a \geq b$ and the linear eccentricity $E = \sqrt{a^2 - b^2}$.

- In this case we can introduce ***ellipsoidal coordinates*** u, β, λ related to ***Cartesian coordinates*** x_1, x_2, x_3 by the equations

$$x_1 = \sqrt{u^2 + E^2}\cos\beta\cos\lambda \; , \quad x_2 = \sqrt{u^2 + E^2}\cos\beta\sin\lambda \; , \quad x_3 = u\sin\beta \; .$$

- In addition we will suppose that a ***function*** $h(\beta,\lambda)$ ***describes the boundary*** $\partial\Omega$ of Ω with respect to the level ellipsoid $u = b$, i.e. $\partial\Omega$ ***is represented by***

$$\Rightarrow \quad \begin{aligned} x_1 &= \sqrt{[b + h(\beta,\lambda)]^2 + E^2}\cos\beta\cos\lambda \; , \\ x_2 &= \sqrt{[b + h(\beta,\lambda)]^2 + E^2}\cos\beta\sin\lambda \; , \\ x_3 &= [b + h(\beta,\lambda)]\sin\beta \; . \end{aligned}$$

Now we return to the *LGBVP (Linear Gravimetric Boundary-Value Problem)*. We can interpret the boundary condition

$$\frac{\partial T}{\partial \mathbf{s}} = \langle \mathbf{s}, \mathbf{grad}\, T \rangle = -\,\delta g \quad \text{on} \quad \partial\Omega$$

in terms of a derivative of T with respect to u, i.e.

$$\frac{\partial T}{\partial u} = -\,w(b+h,\beta)\,\delta g \quad \text{on} \quad \partial\Omega \;,$$

where

$$w(u,\beta) = \sqrt{\frac{u^2 + E^2 \sin^2\beta}{u^2 + E^2}} \;.$$

In solving the *LGBVP* a *transformation of coordinates* will be applied.

This will open a *way for an alternative* between the *boundary complexity* and the *complexity of the coefficients* of the partial differential equation governing the solution.

3. Transformation of Coordinates and an Attenuation Function

Our starting point will be the mapping as above, i.e.

$$x_1 = \sqrt{u^2 + E^2}\cos\beta\cos\lambda \quad ,$$

$$x_2 = \sqrt{u^2 + E^2}\cos\beta\sin\lambda \quad ,$$

$$x_3 = u\sin\beta \quad ,$$

but with

$$u = z + \omega(z)\,h(\beta,\lambda) \quad ,$$

where z is a *new coordinate* and $\omega(z)$ is a *twice continuously differentiable attenuation function* defined for $z \in [\,b, \infty)$, such that

$$\omega(z)h(\beta,\lambda) > -\,b \ ,$$

$$\omega(b)=1 \ , \qquad \frac{d\omega}{dz}(b)=0$$

and

$$\omega(z)=0 \quad \text{for} \quad z\in[\,z_{ext}\,,\infty\,)\ , \quad \text{where} \quad b<z_{ext} \ .$$

Stress **that the assumption concerning the continuity of** ω **and its 1st and the 2nd derivatives implies**

$$\lim \omega(z)=0\ ,\quad \lim\frac{d\omega(z)}{dz}=0\ ,\quad \lim\frac{d^2\omega(z)}{dz^2}=0 \quad \text{for} \quad z\to z_{ext}^{-} \ .$$

Note: $z,\ \beta,\ \lambda$ **form a** ***system of new curvilinear coordinates*** **and for**

$$\frac{du}{dz}=1+\frac{d\omega}{dz}h>0 \ ,$$

the transformation is a *one-to-one mapping* between Ω and the *outer space* Ω_{ell} of our oblate ellipsoid of revolution.

The construction of the *attenuation function* $\omega(z)$ in the interval $[b, z_{ext})$, i.e. for $b \leq z < z_{ext}$, deserves some attention. *Here we give an example* (applied in this work). We put

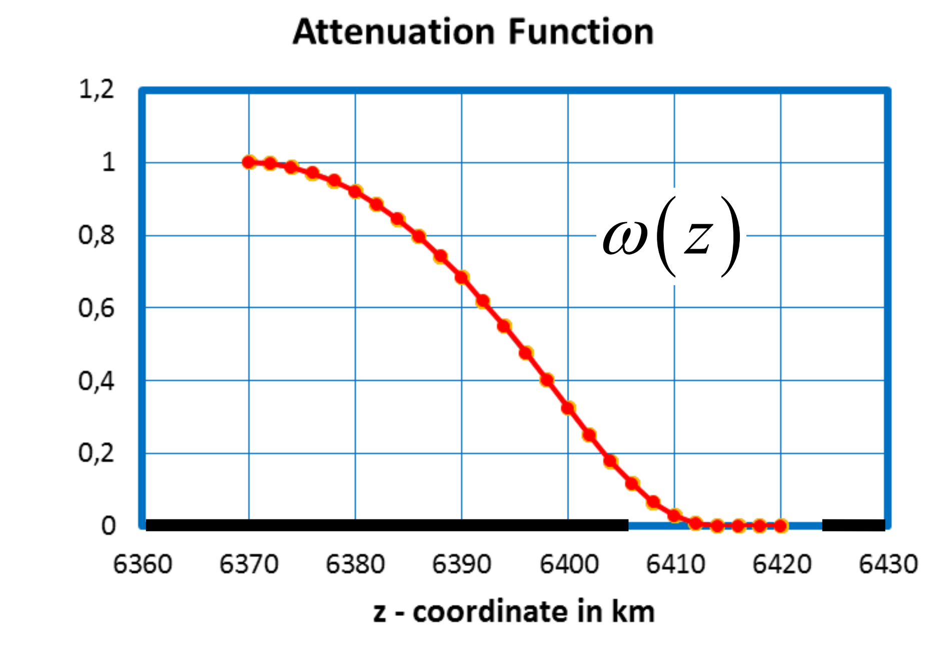
$$\omega(z) = \exp\left[2 - \frac{2(\Delta z)^2}{(\Delta z)^2 - (z-b)^2}\right] \text{ , } \quad \textbf{where} \quad \Delta z = z_{ext} - b \text{ ,}$$

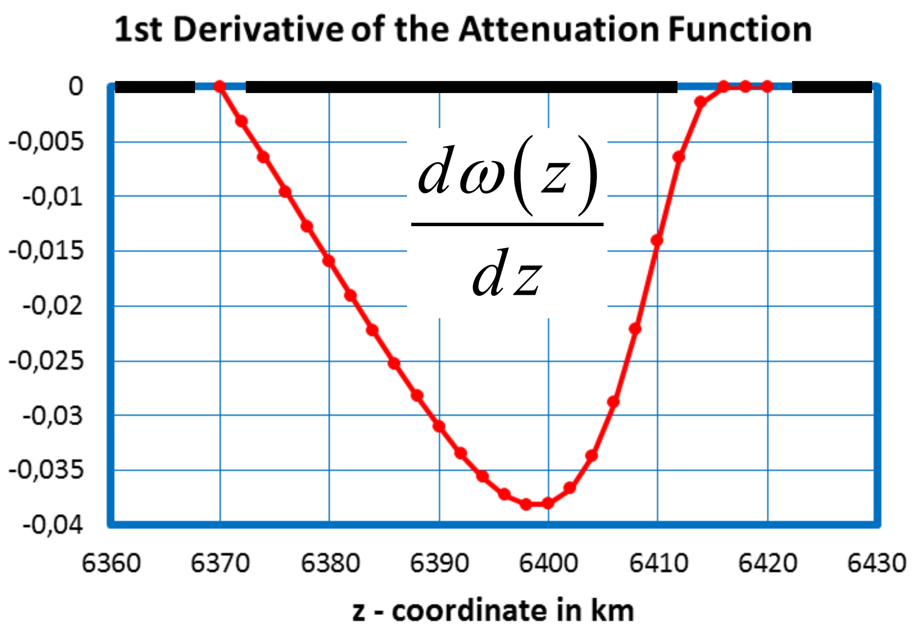
i.e.

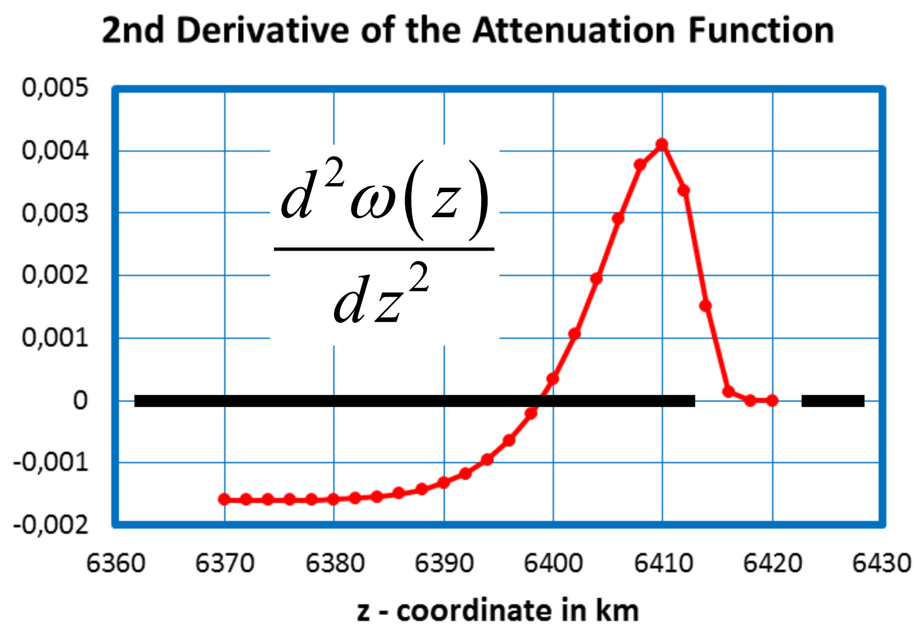
$$\omega(z) = \mathrm{e}^{\left[2 - \frac{2(\Delta z)^2}{(\Delta z)^2 - (z-b)^2}\right]} \quad \textbf{with} \quad \mathrm{e} \approx 2{,}71828 \text{ .}$$

The attenuation function and its derivatives are *demonstrated on the following slide*.

P. Holota, O. Nesvadba







4. Transformation of the Boundary Condition

In the coordinates z, β, λ *the boundary* $\partial\Omega$ is defined by $z = b$ and its image $\partial\Omega_{ell}$ coincides with our oblate ellipsoid.

In addition the transformation changes the formal representation of the *LGBVP*. Indeed, *the boundary condition turns into*

$$\frac{\partial T}{\partial z} = -w\left[z + \omega(z)h(\beta,\lambda)\right]\delta g \quad \text{for} \quad z = b\,.$$

Hence, denoting by $\partial/\partial n$ the derivative in the direction of the unit (outer) normal $\boldsymbol{n}$ of $\partial\Omega_{ell}$, we obtain

$$\frac{\partial T}{\partial n} = -\sqrt{1+\varepsilon}\,\delta g \quad \text{on} \quad \partial\Omega_{ell}\,,$$

where

$$\varepsilon = \frac{E^2(2bh + h^2)\cos^2\beta}{(a^2\sin^2\beta + b^2\cos^2\beta)[(b+h)^2 + E^2]}$$

may practically be neglected.

5. Laplacian and Topography-Dependent Coefficients

It is somewhat more complicated to express *Laplace's operator* of T in terms of the coordinates z, β, λ, which do not form an orthogonal system. *We will use the tensor calculus*. Recalling

$$x_1 = \sqrt{[z+\omega(z)h(\beta,\lambda)]^2 + E^2}\cos\beta\cos\lambda \ ,$$

$$x_2 = \sqrt{[z+\omega(z)h(\beta,\lambda)]^2 + E^2}\cos\beta\sin\lambda \ ,$$

$$x_3 = [z+\omega(z)h(\beta,\lambda)]\sin\beta$$

and putting $y_1 = z$, $y_2 = \beta$, $y_3 = \lambda$, we easily deduce that the *Jacobian*

$$J = \left|\frac{\partial x_i}{\partial y_j}\right| = -\left(1+\frac{d\omega}{dz}h\right)\left[\left(z+\omega h\right)^2 + E^2\sin^2\beta\right]\cos\beta < 0 \ ,$$

apart from its zero values for $\beta = -\pi/2$ and $\pi/2$. Thus, the transformation is a *one-to-one mapping*.

Moreover, for *Laplace's operator* Δ applied on T we generally have

$$\Delta T = \frac{1}{\sqrt{g}} \frac{\partial}{\partial y_i} \left(\sqrt{g}\, g^{ij} \frac{\partial T}{\partial y_j} \right) = g^{ij} \frac{\partial^2 T}{\partial y_i \partial y_j} + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}\, g^{ij}}{\partial y_i} \frac{\partial T}{\partial y_j} \,.$$

Here $g = \left| g_{ij} \right| = J^2$ is the *determinant* related to the respective *metric tensor* g_{ij} and g^{ij} means the *associated metric tensor*.

After some algebra and neglecting the difference

$$w^2(z+\omega h, \beta) - w^2(z, \beta) \leq \frac{E^2}{z^2} \left[2\omega \frac{h}{z} + \left(\omega \frac{h}{z} \right)^2 \right] \cos^2\beta \;,$$

we can deduce

$$\Delta T = \frac{z^2 + E^2 \sin^2\beta}{(z+\omega h)^2 + E^2 \sin^2\beta} \left[\Delta_{ell} T - \delta(T, h) \right] \;,$$

where

$$\Delta_{ell}T = \frac{1}{z^2+E^2\sin^2\beta}\left[\left(z^2+E^2\right)\frac{\partial^2 T}{\partial z^2}+2z\frac{\partial T}{\partial z}+\frac{\partial^2 T}{\partial \beta^2}-\frac{\sin\beta}{\cos\beta}\frac{\partial T}{\partial \beta}+\frac{z^2+E^2\sin^2\beta}{(z^2+E^2)\cos^2\beta}\frac{\partial^2 T}{\partial \lambda^2}\right],$$

$$\delta(T,h) = A_1\frac{\partial T}{\partial z}+A_2\frac{\partial^2 T}{\partial z^2}+A_3\frac{1}{\sqrt{z^2+E^2\sin^2\beta}}\frac{\partial^2 T}{\partial z\partial \beta}+A_4\frac{w(z+\omega h,\beta)}{\sqrt{z^2+E^2}\cos\beta}\frac{\partial^2 T}{\partial z\partial \lambda}$$

and A_i **are *topography dependent coefficients* given by**

$$A_1 = \left(1+\frac{d\omega}{dz}h\right)^{-1}\left[2\left(\frac{d\omega}{dz}-\frac{\omega}{z}\right)\frac{zh}{z^2+E^2\sin^2\beta}+\omega\Delta_E h\right]-$$

$$-2\left(1+\frac{d\omega}{dz}h\right)^{-2}\omega\frac{d\omega}{dz}\left|\mathbf{grad}_E h\right|^2+\left(1+\frac{d\omega}{dz}h\right)^{-3}\left[\frac{(z+\omega h)^2+E^2}{z^2+E^2\sin^2\beta}+\omega^2\left|\mathbf{grad}_E h\right|^2\right]\frac{d^2\omega}{dz^2}h,$$

$$A_2 = \left(1+\frac{d\omega}{dz}h\right)^{-2}\left\{2\left(\frac{d\omega}{dz}-\frac{\omega z}{z^2+E^2}\right)h+\left[\left(\frac{d\omega}{dz}\right)^2-\frac{\omega^2}{z^2+E^2}\right]h^2\right\}\frac{z^2+E^2}{z^2+E^2\sin^2\beta}-$$

$$-\left(1+\frac{d\omega}{dz}h\right)^{-2}\omega^2\left|\boldsymbol{grad}_E h\right|^2 ,$$

$$A_3 = \left(1+\frac{d\omega}{dz}h\right)^{-1}\frac{2\omega}{\sqrt{z^2+E^2\sin^2\beta}}\frac{\partial h}{\partial\beta}\,,\quad A_4 = \left(1+\frac{d\omega}{dz}h\right)^{-1}\frac{2\,\omega\,w(z+\omega h,\beta)}{\sqrt{z^2+E^2}\cos\beta}\frac{\partial h}{\partial\lambda}$$

with

$$\left|\boldsymbol{grad}_E h\right|^2 = \frac{1}{z^2+E^2\sin^2\beta}\left[\left(\frac{\partial h}{\partial\beta}\right)^2+\frac{z^2+E^2\sin^2\beta}{(z^2+E^2)\cos^2\beta}\left(\frac{\partial h}{\partial\lambda}\right)^2\right]$$

and

$$\Delta_E h = \frac{1}{z^2+E^2\sin^2\beta}\left[\frac{\partial^2 h}{\partial\beta^2}-\frac{\sin\beta}{\cos\beta}\frac{\partial h}{\partial\beta}+\frac{z^2+E^2\sin^2\beta}{(z^2+E^2)\cos^2\beta}\frac{\partial^2 h}{\partial\lambda^2}\right]$$

being the *1st and the 2nd Beltrami differential operators*.

6. Linear GBVP and Neumann's Function

The disturbing potential T is a *harmonic function in the original solution domain* Ω. In the space of the curvilinear coordinates $z,\ \beta,\ \lambda$, therefore, T satisfies Laplace's equation $\Delta T = 0$ for $z > b$. *This yields*

$$\Delta_{ell} T = \delta(T, h) \quad \text{for} \quad z > b \ ,$$

where

$$\delta(T,h) = A_1 \frac{\partial T}{\partial z} + A_2 \frac{\partial^2 T}{\partial z^2} + A_3 \frac{1}{\sqrt{z^2 + E^2 \sin^2 \beta}} \frac{\partial^2 T}{\partial z \partial \beta} + A_4 \frac{w(z + \omega h, \beta)}{\sqrt{z^2 + E^2} \cos \beta} \frac{\partial^2 T}{\partial z \partial \lambda} \ .$$

Hence the *linear gravimetric boundary value problem* attains the form

$$\Delta_{ell} T = f \quad \text{in} \quad \Omega_{ell} \ ,$$

$$\frac{\partial T}{\partial n} = -\sqrt{1+\varepsilon}\ \delta g \quad \text{on} \quad \partial \Omega_{ell} \ ,$$

where $f = \delta(T, h)$ and ε may be omitted.

Neglecting the fact that $f = \delta(T, h)$ depends on T , we can represent the solution of the problem *formally* by means of a classical apparatus of mathematical physics.

Indeed, *we may construct the respective Green's function of the second kind* $N(\boldsymbol{x}, \boldsymbol{y})$, which solves *Neumann's boundary value problem* for the domain Ω_{ell} . Then (*formally*)

$$T(\boldsymbol{y}) = \frac{1}{4\pi} \int_{\partial\Omega_{ell}} \delta g(\boldsymbol{x}) N(\boldsymbol{x}, \boldsymbol{y}) d_x S - \frac{1}{4\pi} \int_{\Omega_{ell}} f(\boldsymbol{x}) N(\boldsymbol{x}, \boldsymbol{y}) d_x V \ .$$

7. Iteration Process

Nevertheless, *Neumann's function* $N(\boldsymbol{x}, \boldsymbol{y})$ can also be used *to solve the transformed LGBVP* , where $f = \delta(T, h)$ *depends on* T . In this case the integral formula above represents an *integro-differential equation* for T .

For clarity we can put

$$F(\boldsymbol{y}) = \frac{1}{4\pi} \int_{\partial\Omega_{ell}} \delta g(\boldsymbol{x}) N(\boldsymbol{x}, \boldsymbol{y}) \, d_x S$$

$$K T(\boldsymbol{y}) = -\frac{1}{4\pi} \int_{\Omega_{ell}} \delta\left[T(\boldsymbol{x}), h(\boldsymbol{x})\right] N(\boldsymbol{x}, \boldsymbol{y}) \, d_x V \text{ ,}$$

where $F(\boldsymbol{y})$ is a ***harmonic function*** and $KT(\boldsymbol{y})$ is an ***integro-differential operator applied on*** T , such that

$$\Delta_{ell} K T = \delta(T, h) \quad \text{in} \quad \Omega_{ell} \quad \text{and} \quad \frac{\partial K T}{\partial n} = 0 \quad \text{on} \quad \partial\Omega_{ell} \text{ .}$$

Under this notation the problem is to find T from

$$T = F + K T \text{ .}$$

We apply the ***method of successive approximations***, i.e.

$$T = \lim_{n} T_n \text{ ,} \qquad T_n = F + K T_{n-1} \text{ ,}$$

where T_0 is the starting approximation, e.g. $T_0 = F$.

8. Function Spaces and Estimates of a Particular Solution of Poisson's Equation

Our aim is to examine whether the iteration process converges. The investigation will be based on *Banach's fixed point theorem* interpreted for functions from a *Sobolev weight space*.

Let $W_2^{(1)}(\Omega_{ell})$ be a *Sobolev weight space* equipped by the norm

$$\| u \|_1 = \left[\int_{\Omega_{ell}} \frac{1}{z^2} u^2 \, dV + \int_{\Omega_{ell}} |\, \mathbf{grad}\, u \,|^2 \, dV \right]^{1/2} ,$$

Similarly, let $W_2^{(2)}(\Omega_{ell})$ be a *Sobolev weight space* with the norm

$$\| u \|_2 = \left[\| u \|_1 + \sum_{i=1}^{3} \int_{\Omega_{ell}} |\, \mathbf{grad} \left(grad_i u \right) |^2 \, dV \right]^{1/2} . \qquad (1)$$

It is obvious that $W_2^{(2)}(\Omega_{ell}) \subset W_2^{(1)}(\Omega_{ell})$.

The crucial point **for the use of Banach's theorem *is to show that the operator*** K ***is a contraction mapping*. In our case it means to show that there is a constant** $\alpha < 1$ **such that the inequality**

$$\| Ku - Kv \|_2 \leq \alpha \| u - v \|_2$$

holds for arbitrary functions u **and** v **from** $W_2^{(2)}(\Omega_{ell})$**, provided that also** F **belongs to** $W_2^{(2)}(\Omega_{ell})$**. Considering the linearity of the operator** K **, we can see that it is enough to show that**

$$\| KT' \|_2 \leq \alpha \| T' \|_2$$

with $\alpha < 1$ **holds for any** T' **from** $W_2^{(2)}(\Omega_{ell})$ **, still provided that** F **belongs to** $W_2^{(2)}(\Omega_{ell})$ **.**

Recall that in our investigation **we particularly have**

$$u(\mathbf{y}) = KT'(\mathbf{y}) = -\frac{1}{4\pi}\int_{\Omega_{ell}} \delta\left[T'(\mathbf{x}), h(\mathbf{x})\right] N(\mathbf{x},\mathbf{y})\, d_x V \qquad \textbf{(2)}$$

We will study first how the estimate u ***by means of*** δ **.**

From the fundamental properties of Neumann's function **it can be deduced that** u **is a solution of the following boundary-value problem**

$$\Delta_{ell} u = \delta(T', h) \quad \textbf{for} \quad z > b, \ \textbf{i.e.} \ \mathbf{y} \in \Omega_{ell} \qquad \textbf{(3)}$$

and

$$\frac{\partial u}{\partial n} = 0 \quad \textbf{for} \quad z = b\,, \ \textbf{i.e.} \ \mathbf{y} \in \partial\Omega_{ell}\ . \qquad \textbf{(4)}$$

It is thus clear that u **can be estimated as the solution of the above problem in** $W_2^{(2)}(\Omega_{ell})$ **.**

In the 1st step we will estimate u as a function from $W_2^{(1)}(\Omega_{ell})$, i.e. *as the so-called weak solution of the problem* mentioned above. We multiply Eq. (3) by an arbitrary function v from $W_2^{(1)}(\Omega_{ell})$ and integrate this product over Ω_{ell}. We obtain

$$\int_{\Omega_{ell}} v\,\Delta_S u\,dV = \int_{\Omega_{ell}} v\,\delta\left(T',h\right)dV \ .$$

Moreover, the *integration by parts (Green's first identity)* on the left hand side together with the boundary condition (4) results in

$$A(v,u) = -\int_{\Omega_{ell}} v\,\delta\left(T',h\right)dV \ , \quad \text{where} \qquad (5)$$

$$A(v,u) = \int_{\Omega_{ell}} \left\langle \mathbf{grad}\, v, \mathbf{grad}\, u \right\rangle dV$$

Similarly, we denote the right hand side of Eq. (5) by Fv, i.e.

$$Fv = -\int_{\Omega_{ell}} v\,\delta\left(T',h\right)dV \ .$$

Following now the ***Hölder inequality***, we have

$$\left|Fv\right| \le \left(\int_{\Omega_{ell}} z^2 \delta^2 (T',h)\, dV \right)^{1/2} \left(\int_{\Omega_{ell}} \frac{1}{z^2} v^2 \, dV \right)^{1/2} \text{, i.e.}$$

$$\left|Fv\right| \le \left\| z\,\delta(T',h) \right\|_{L_2(\Omega_{ell})} \left\| v \right\|_1$$

Thus Fv is a continuous functional and for its norm the following estimate holds

$$\left\| F \right\| \le \left\| z\,\delta(T',h) \right\|_{L_2(\Omega_{ell})} .$$

To estimate the norm of u we still have to examine whether $A(v,u)$ is a coercive bilinear form. We have to show that

$$A(v,v) \ge \alpha \left\| v \right\|_1^2 , \quad \alpha = const. > 0 ,$$

is valid for all v from $W_2^{(1)}(\Omega_{ell})$. According to (Holota, 1991, 1997) we can put

$$\alpha = \frac{1}{5} .$$

Our further considerations are based on the simple *Lax-Milgram generalization of the famous Riesz representation theorem*.

The assumptions of the Lax-Milgram theorem are met. Indeed, the bilinear form $A(v,u)$ is coercive, as we have shown above. Moreover, it is also continuous, i.e., there exists a constant $(const.)$ such that

$$\left| A(v,u) \right| \le (const.) \left\| v \right\|_1 \left\| u \right\|_1$$

holds for all v and u from $W_2^{(1)}(\Omega_{ell})$. A simple proof is left for the reader. Recalling our notation, we have

$$A(v,u) = Fv$$

in view of Eq. (5) and we can apply the *Lax-Milgram theorem* immediately. We arrive at

$$\left\| u \right\|_1 \le 5 \left\| z\,\delta\left(T', h\right) \right\|_{L_2(\Omega_{ell})} ,$$

which is our desired estimate.

9. Calderon-Zygmund Inequality

Now we will estimate u as a function from $W_2^{(2)}(\Omega_{ell})$, i.e. in the norm

$$\| u \|_2 = \left[\| u \|_1 + \sum_{i=1}^{3} \int_{\Omega_{ell}} \left| \boldsymbol{grad}\left(grad_i u \right) \right|^2 dV \right]^{1/2} .$$

As to $\| u \|_1$, we can use the result of the preceding section. Hence it remains to estimate the term

$$D = \sum_{i=1}^{3} \int_{\Omega_{ell}} \left| \boldsymbol{grad}\left(grad_i u \right) \right|^2 dV = \sum_{i=1}^{3} \int_{b<z} \left| \boldsymbol{grad}\left(grad_i u \right) \right|^2 dV .$$

Let us approach first $D' = \sum_{i=1}^{3} \int_{b<z<b_e} \left| \boldsymbol{grad}\left(grad_i u \right) \right|^2 dV$

Putting $D_e = \sum_{i=1}^{3} \int_{z<b_e} \left| \boldsymbol{grad}\left(grad_i u \right) \right|^2 dV$ **, we have** $D' \le D_e$ **.**

***Applying now the Green identity twice*, we obtain**

$$D_e = \sum_{i=1}^{3} \int_{z=b_e} (grad_i u) \frac{\partial}{\partial n} (grad_i u)\, dS - \sum_{i=1}^{3} \int_{z<b_e} (grad_i u)\, \Delta_{ell} (grad_i u)\, dV =$$

$$= \sum_{i=1}^{3} \int_{z=b_e} (grad_i u) \frac{\partial}{\partial n} (grad_i u)\, dS - \int_{z<b_e} \left\langle \mathbf{grad}\, u, \mathbf{grad} \left(\Delta_{ell} u \right) \right\rangle dV =$$

$$= \sum_{i=1}^{3} \int_{z=b_e} (grad_i u) \frac{\partial}{\partial n} (grad_i u)\, dS - \int_{z=b_e} \left(\Delta_{ell} u \right) \frac{\partial u}{\partial n}\, dS + \int_{z<b_e} \left(\Delta_{ell} u \right)^2 dV$$

and recalling the integral representation u, Eq. (2), we easily deduce that asymptotically

$$grad_i u = O\left(z^{-2}\right), \quad \frac{\partial}{\partial n} grad_i u = O\left(z^{-3}\right) \quad \text{and} \quad \delta(T,h) = O\left(z^{-4}\right)$$

as $z \to \infty$ (Note. O means the Landau symbol).

Hence $D = D' \leq \int\limits_{0<z} \left(\Delta_S u \right)^2 dV$ **for** $b_e \to \infty$ **,**

which yields the desired estimate

$$D \leq \int\limits_{\Omega_{ell}} \delta^2(T,h)\, dV = \left\| \delta(T,h) \right\|^2_{L_2(\Omega_{ell})}$$

in view of Eq. (3). Combining now

$$\| u \|_1 \leq 5 \left\| z\,\delta(T',h) \right\|_{L_2(\Omega_{ell})}$$

with our last result, we obtain

$$\| u \|_2^2 \leq 25 \left\| z\,\delta(T',h) \right\|^2_{L_2(\Omega_{ell})} + \frac{1}{b^2} \left\| z\,\delta(T',h) \right\|^2_{L_2(\Omega_{ell})} \text{ ,}$$

which enables us to write conclusively that

$$\Rightarrow \quad \| KT' \|_2 = \| u \|_2 \leq \left(25 + \frac{1}{b^2} \right)^{1/2} \left\| z\,\delta(T',h) \right\|_{L_2(\Omega_{ell})}$$

10. Contraction Mapping

Recall that our original aim is to prove the contractivity of the operator K. Therefore, in view of

$$\| KT' \|_2 \le \left(25 + \frac{1}{b^2} \right)^{1/2} \left\| z\,\delta\left(T',h\right) \right\|_{L_2(\Omega_{ell})}$$

it remains to estimate the right hand side by means of $\| T' \|_2$, i.e. to get

$$\left\| z\,\delta\left(T',h\right) \right\|_{L_2(\Omega_{ell})} \le (const.)\ \| T' \|_2 \ ,$$

where

$$\delta\left(T,h\right) = A_1 \frac{\partial T}{\partial z} + A_2 \frac{\partial^2 T}{\partial z^2} + A_3 \frac{1}{\sqrt{z^2 + E^2 \sin^2\beta}} \frac{\partial^2 T}{\partial z \partial \beta} + A_4 \frac{w\left(z+\omega h, \beta\right)}{\sqrt{z^2 + E^2}\cos\beta} \frac{\partial^2 T}{\partial z \partial \lambda}$$

and A_i, $i = 1, 2, 3, 4$, are *topography dependent coefficients*.

In consequence the factor α in

$$\| KT' \|_2 \leq \alpha \| T' \|_2$$

depends on *essential supreme values*

$$\operatorname{supess} | A_i | \quad \text{of} \quad A_i \,, \; i = 1, 2, 3, 4 \quad \text{in the domain} \quad \Omega_{ell}$$

***Note.* Since α depends on *the essential supreme values* of A_i, $i = 1, 2, 3, 4$, we can expect that the contraction nature of K will be kept also for larger slopes, provided that they do not occur too frequently.**

The *inequality* $\alpha < 1$ (sufficient for K to be a contraction mapping) has been proved for the topography of a *realistic range of heights and relatively gentle slopes*.

11. Operator with Reduced Degree of Derivatives

It is convenient to modify the operator K in order *to reduce the degree of derivatives involved* in $\delta(T,h)$ and to display the mutual interplay of individual terms in $\delta(T,h)$ more explicitly. *Integrating by parts*, we get

$$(KT)_P = -\frac{1}{4\pi}\int_{z=b} NA_2\,\delta g\,dS - \frac{1}{4\pi}\int_{b<z<b_{ext}} NA_5\frac{\partial T}{\partial z}\,dV +$$

$$+\frac{1}{4\pi}\int_{b<z<b_{ext}}\left(A_2\frac{\partial N}{\partial z}+A_3\frac{1}{z}\frac{\partial N}{\partial \beta}+A_4\frac{1}{z\cos\beta}\frac{\partial N}{\partial\lambda}\right)\frac{\partial T}{\partial z}\,dV\ ,$$

where

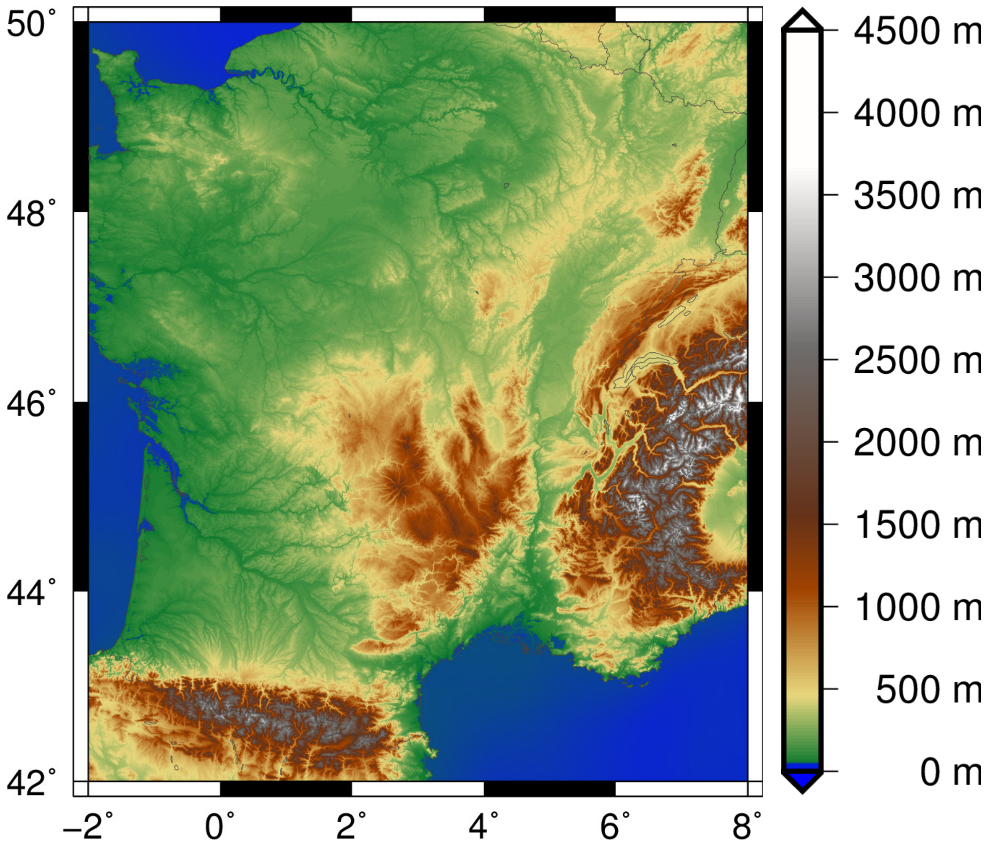
$$A_5 = A_1 - \frac{\partial A_2}{\partial z} - \frac{2}{z}A_2 - \frac{1}{z}\frac{\partial A_3}{\partial\beta} + \frac{\sin\beta}{z\cos\beta}A_3 - \frac{1}{z\cos\beta}\frac{\partial A_4}{\partial\lambda}\ .$$

Note. the quantities with and without the subscript P are referred to the *computation* and the *variable point* of the integration.

P. Holota, O. Nesvadba

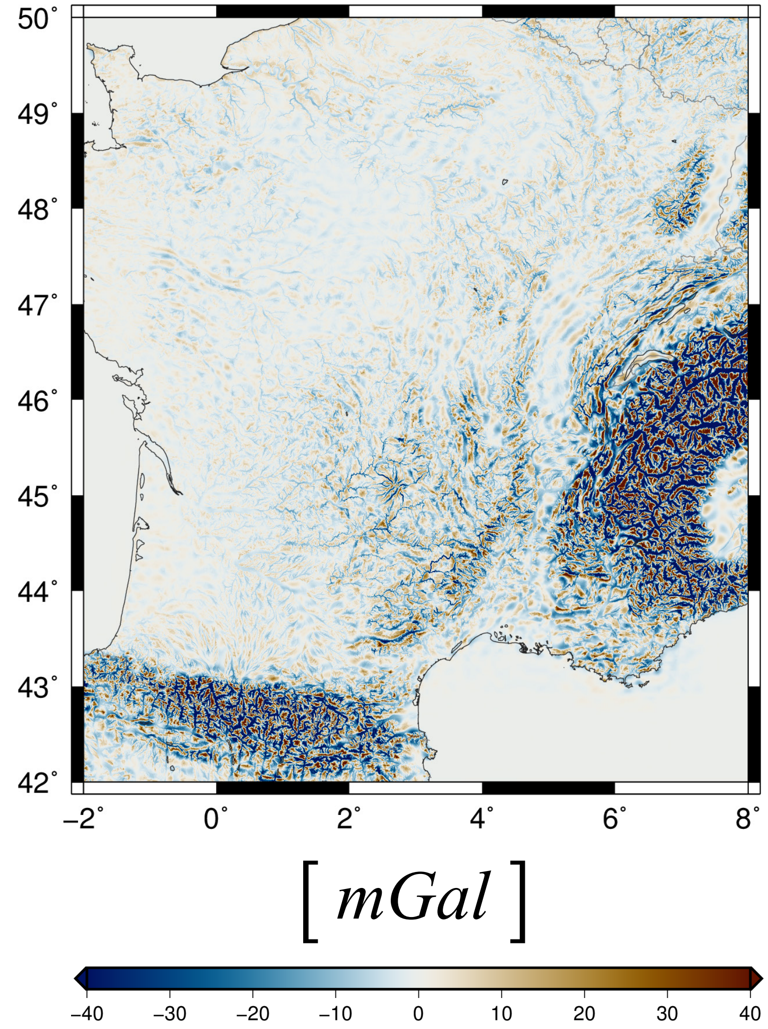
13. Experiment France - Auvergne

Shuttle Radar Topography Mission



$1\,arcsec$ **resolution (*SRTM1*)**

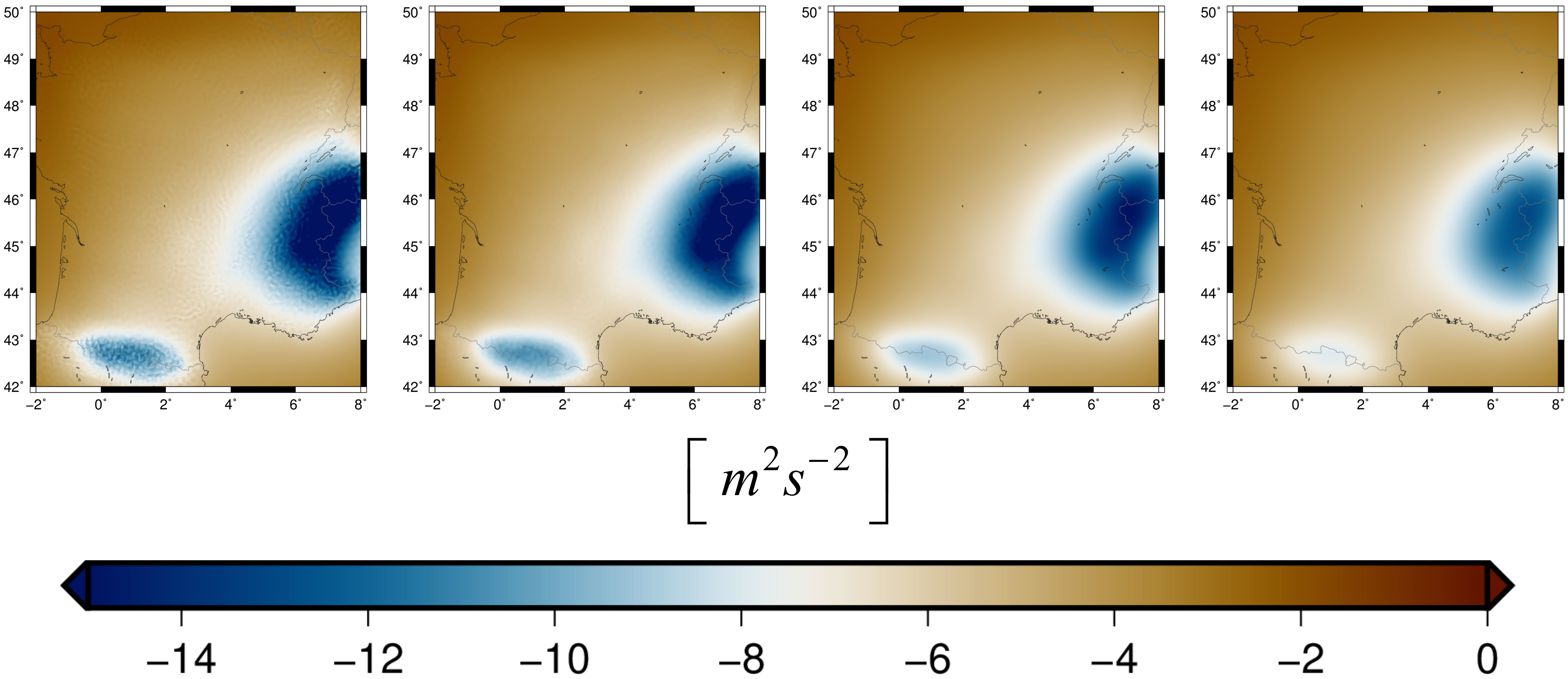
Gravity Data δg



Starting Approximation

$$\left(T_0\right)_P = \tilde{F}_P = \frac{1}{4\pi}\int\limits_{z=b} N\left(1 - A_2\right)\delta g \, dS$$

$h = 0\ m$ $\quad h = 10\,000\ m$ $\quad h = 25\,000\ m$ $\quad h = 50\,000\ m$



$\left[m^2 s^{-2} \right]$

Computation of T_1

Topography Dependent Coefficients for z = b

$$A_1 = \frac{1}{b^2 + E^2 \sin^2\beta}\left[-2h - \frac{\sin\beta}{\cos\beta}\frac{\partial h}{\partial\beta} + \frac{\partial^2 h}{\partial\beta^2} + \frac{b^2 + E^2\sin^2\beta}{(b^2+E^2)\cos^2\beta}\frac{\partial^2 h}{\partial\lambda^2}\right]$$

$$A_2 = -\frac{2bh + h^2}{b^2 + E^2\sin^2\beta} - \frac{1}{b^2 + E^2\sin^2\beta}\left[\left(\frac{\partial h}{\partial\beta}\right)^2 + \frac{b^2 + E^2\sin^2\beta}{(b^2+E^2)\cos^2\beta}\left(\frac{\partial h}{\partial\lambda}\right)^2\right]$$

$$A_3 = \frac{2}{\sqrt{b^2 + E^2\sin^2\beta}}\frac{\partial h}{\partial\beta}\ , \qquad A_4 = \frac{2}{\sqrt{b^2+E^2}\cos\beta}\frac{\partial h}{\partial\lambda}$$

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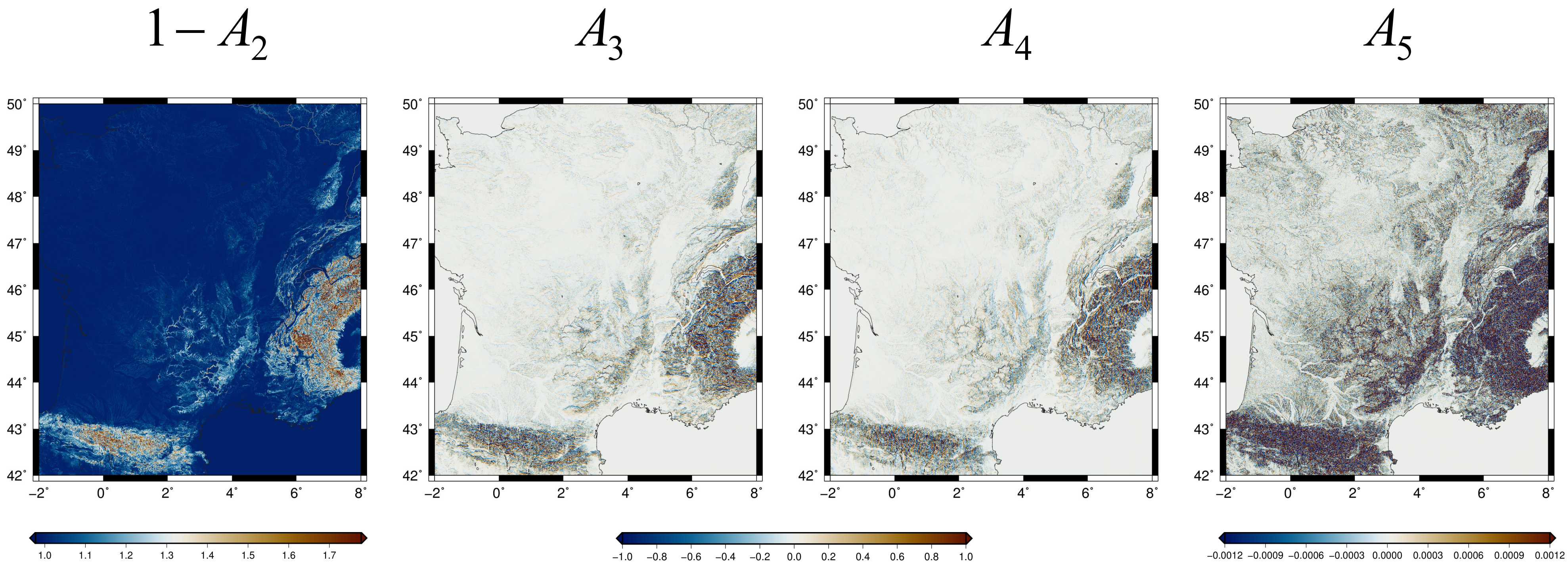
Computation of T_1

Topography Dependent Coefficients for *France - Auvergne* and z = b (SRTM - 1 arcsec)



$$A_5 = A_1 - \frac{\partial A_2}{\partial z} - \frac{2}{z} A_2 - \frac{1}{z}\frac{\partial A_3}{\partial \beta} + \frac{\sin\beta}{z\cos\beta} A_3 - \frac{1}{z\cos\beta}\frac{\partial A_4}{\partial \lambda}$$

The approach thus rests: on *the solution of the transformed Laplace differential equation* for the input *gravity disturbances* given on the surface of the *oblate ellipsoid of revolution*.

***Green's function representation of the solution* and the *method of successive approximations* are used:**

$$T = \lim_{n} T_n \quad , \quad T_n = F + KT_{n-1}$$

where

$$F(\mathbf{y}) = \frac{1}{4\pi} \int_{\partial\Omega_{ell}} \delta g(\mathbf{x}) N(\mathbf{x}, \mathbf{y})\, d_x S$$

is a *harmonic function* and

$$KT_{n-1}(\mathbf{y}) = -\frac{1}{4\pi} \int_{\Omega_{ell}} \delta\left[T_{n-1}(\mathbf{x}) h(\mathbf{x})\right] N(\mathbf{x}, \mathbf{y})\, d_x V$$

is an *integro-differential operator* applied on T_{n-1}.

Thank you for your attention !

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