FVM approach for solving the oblique derivative BVP on unstructured meshes above the real Earth's topography

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Outline

- We present a finite volume method (FVM) for the general Poisson problem with the Dirichlet and oblique derivative boundary condition
- We present local gravity field modelling in Slovakia based on the FVM approach considered on unstructured meshes above the real Earth's topography

1. Mathematical formulation

- nonlinear satellite fixed geodetic boundary value problem

$$\begin{aligned} -\Delta T(\boldsymbol{x}) &= 0, & \boldsymbol{x} &\in \Omega, \\ \nabla T(\boldsymbol{x}) \cdot \boldsymbol{V}(\boldsymbol{x}) &= g(\boldsymbol{x}), & \boldsymbol{x} &\in \Gamma, \\ T_{SAT}(\boldsymbol{x}) &= 0, & \boldsymbol{x} &\in \partial\Omega \backslash \Gamma, \end{aligned}$$

- where $\boldsymbol{V}(\boldsymbol{x}) = \boldsymbol{n}(\boldsymbol{x}) + \boldsymbol{W}(\boldsymbol{x})$
- T - unknown disturbed potential
- $\boldsymbol{V}(\boldsymbol{x}) = \frac{\nabla U(\boldsymbol{x})}{|\nabla U(\boldsymbol{x})|}$, where U is normal potential
- $g(\boldsymbol{x})$ - gravity disturbances

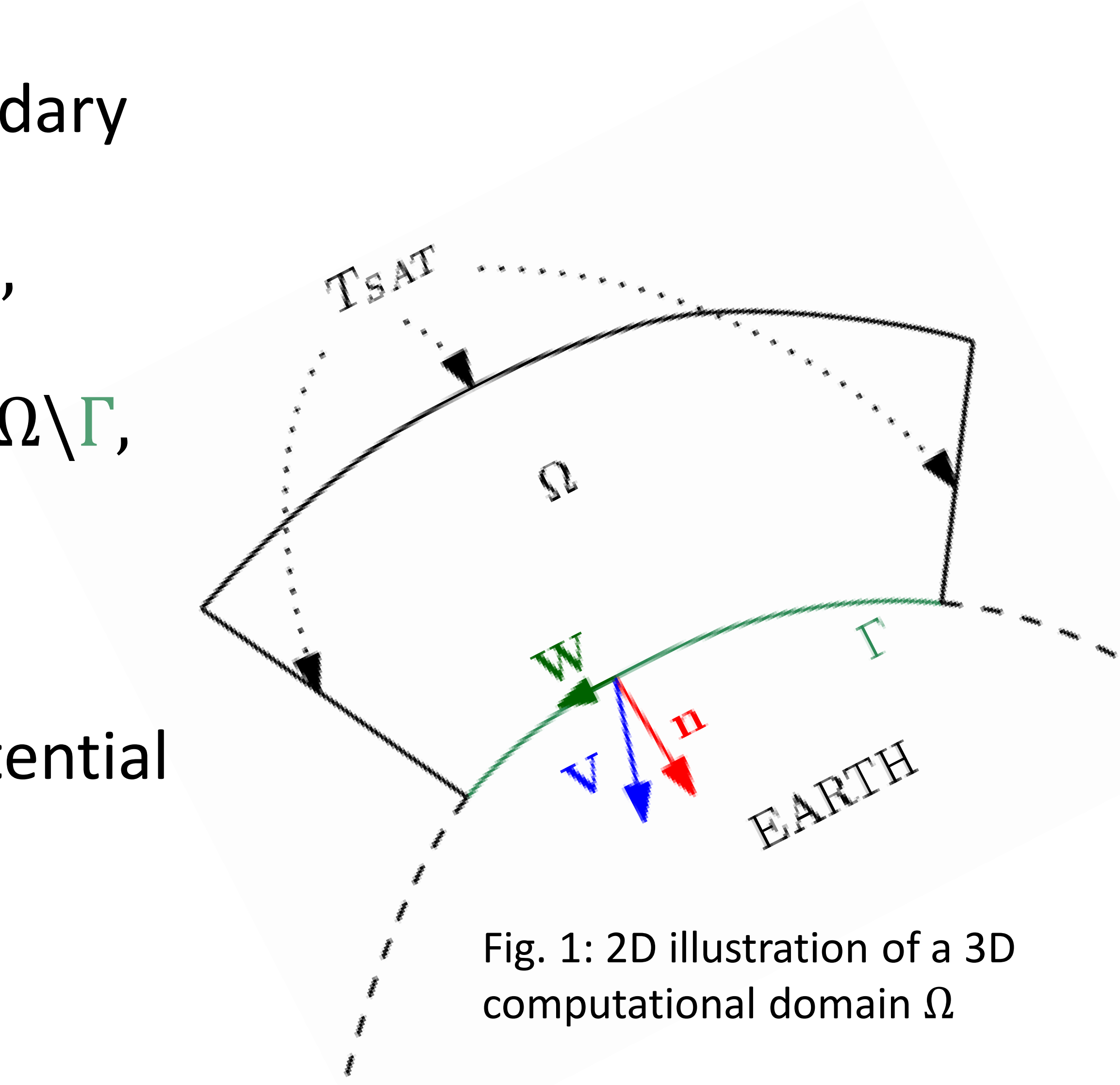


Fig. 1: 2D illustration of a 3D computational domain Ω

2. Generic Finite Volume method (FVM)

- Divide the computational domain Ω into the set of finite volumes p

$$0 = \iiint_p -\Delta T$$

$$= -\sum_{\sigma\in\mathfrak{S}(p)\setminus\mathfrak{S}_\Gamma} \iint_\sigma \nabla T \cdot \boldsymbol{n}_{p,\sigma} - \sum_{\sigma\in\mathfrak{S}(p)\cap\mathfrak{S}_\Gamma} \iint_\sigma \nabla T \cdot \left(\boldsymbol{n}_{p,\sigma} + \boldsymbol{W} - \boldsymbol{W}\right) = (*)$$

- Where $\nabla T \cdot \left(\boldsymbol{n}_{p,\sigma} + \boldsymbol{W}\right) = \nabla T \cdot \boldsymbol{V} = g$
- Where inner fluxes are approximated by some FV scheme $\mathcal{F}^{\Omega}_{p,\sigma}(T) \approx \iint_\sigma \nabla T \cdot \boldsymbol{n}_{p,\sigma}$

$$(*) = -\sum_{\sigma\in\mathfrak{S}(p)\setminus\mathfrak{S}_\Gamma} \mathcal{F}^{\Omega}_{p,\sigma}(T) - \sum_{\sigma\in\mathfrak{S}(p)\cap\mathfrak{S}_\Gamma} \iint_\sigma \mathrm{g} - \nabla T \cdot \boldsymbol{W}$$

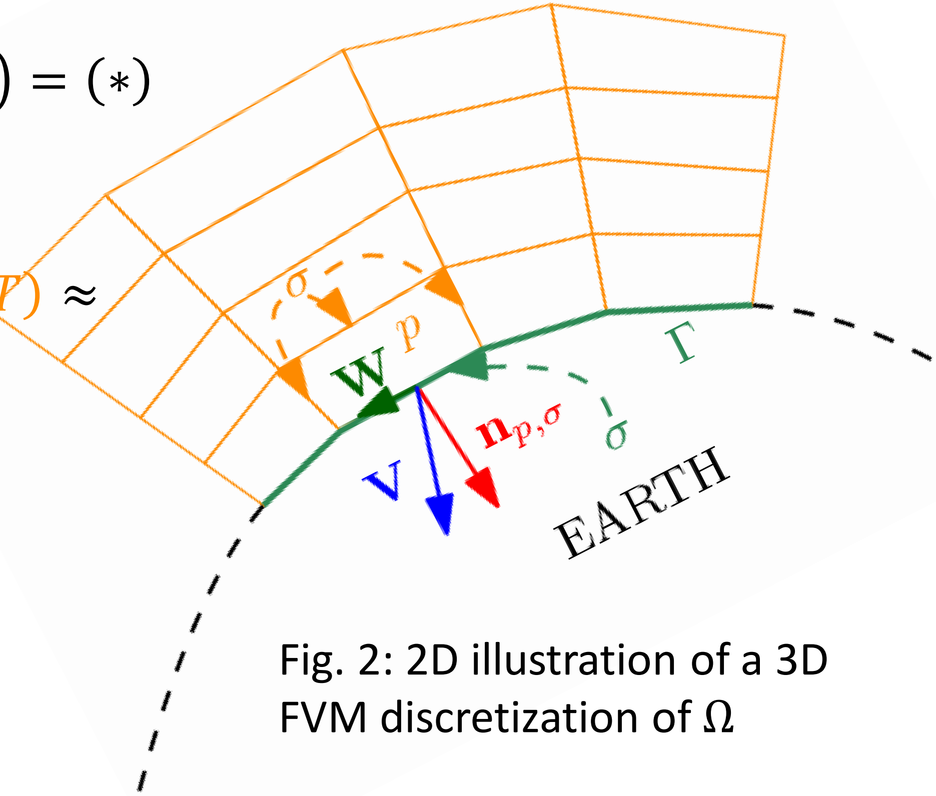


Fig. 2: 2D illustration of a 3D FVM discretization of Ω

2. Generic Finite Volume method (FVM)

$$0 = \iint_{\sigma} \nabla T \cdot \boldsymbol{W} = \iint_{\sigma} \nabla_{\Gamma} \cdot (T\boldsymbol{W}) - T\nabla_{\Gamma} \cdot \boldsymbol{W}$$

$$= \sum_{e \in \mathfrak{E}(\sigma)} \int_{e} T\,\boldsymbol{W} \cdot \boldsymbol{n}_{\sigma,e} - \iint_{\sigma} T\nabla_{\Gamma} \cdot \boldsymbol{W} = (*)$$

- Choice of central scheme
  - Approximate T on the edge e by constant T_e
  - Approximate T on the face σ by constant T_σ

$$(*) = \sum_{e \in \mathfrak{E}(\sigma)} T_e \int_{e} \boldsymbol{W} \cdot \boldsymbol{n}_{\sigma,e} - T_\sigma \iint_{\sigma} \nabla_{\Gamma} \cdot \boldsymbol{W}$$

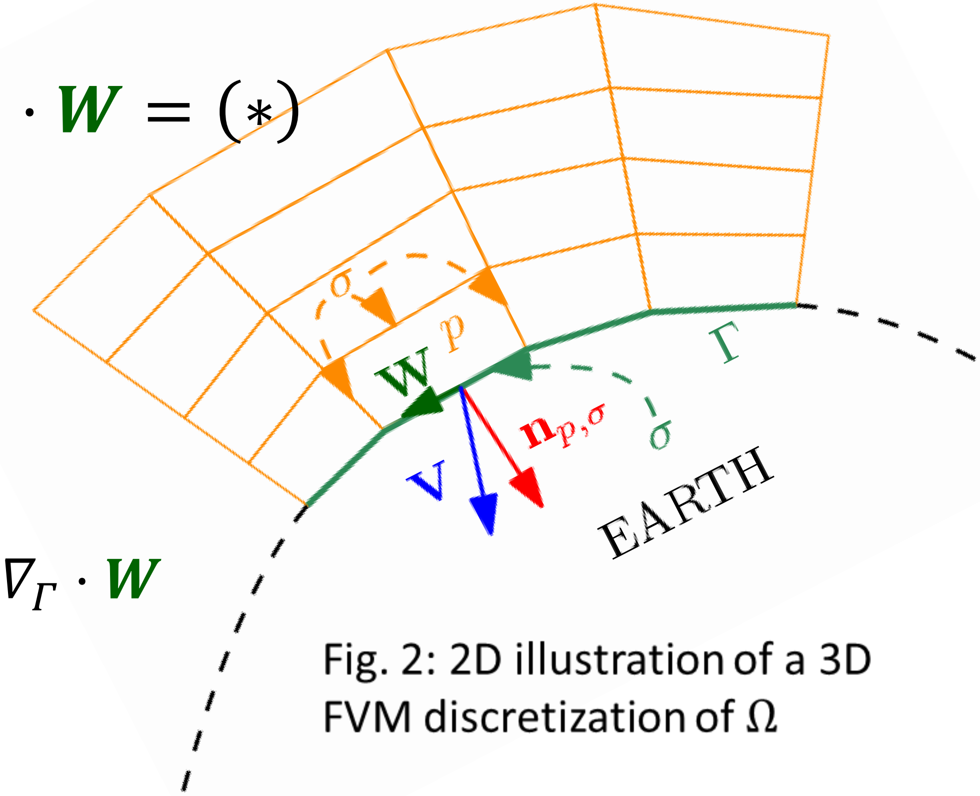


Fig. 2: 2D illustration of a 3D FVM discretization of Ω

2. Generic Finite Volume method (FVM)

- From a numerical analysis [1] we add a small amount of boundary diffusion for a stability purposes
- Resulting scheme

$$\sum_{\sigma\in\mathfrak{S}(p)\setminus\mathfrak{S}_\Gamma} \mathcal{F}^{\Omega}_{p,\sigma}(T) + \sum_{\sigma\in\mathfrak{S}(p)\setminus\mathfrak{S}_\Gamma}\ \sum_{e\in\mathfrak{E}(\sigma)} T_e \int_e \boldsymbol{W}\cdot\boldsymbol{n}_{\sigma,e} - T_\sigma \iint_\sigma \nabla_\Gamma\cdot\boldsymbol{W}$$

$$+ Rh_\Gamma \sum_{\sigma\in\mathfrak{S}(p)\setminus\mathfrak{S}_\Gamma}\ \sum_{e\in\mathfrak{E}(\sigma)} \mathcal{F}^{\Omega}_{p,\sigma}(T) = \sum_{\sigma\in\mathfrak{S}(p)\setminus\mathfrak{S}_\Gamma} \iint_\sigma g$$

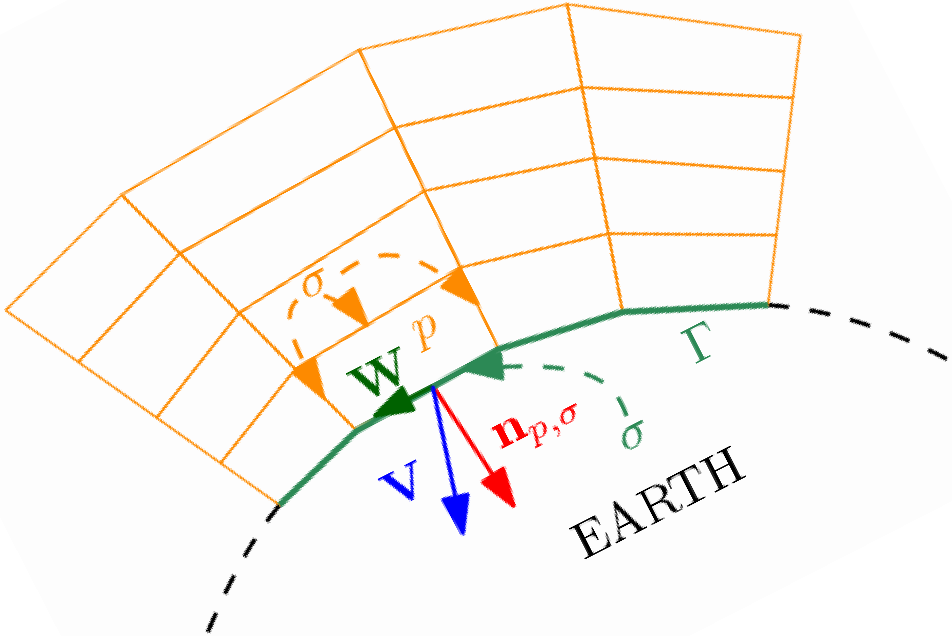


Fig. 2: 2D illustration of a 3D FVM discretization of Ω

3. Choice of fluxes discretization

- Chose some finite volume approximation of inner volume fluxes $\mathcal{F}^{\Omega}_{p,\sigma}(T)$
- Chose some finite volume approximation of boundary fluxes $\mathcal{F}^{\Omega}_{p,\sigma}(T)$
- For our choices see [1]

$$\sum_{\sigma\in\mathfrak{S}(p)\setminus\mathfrak{S}_\Gamma} \mathcal{F}^{\Omega}_{p,\sigma}(T) + \sum_{\sigma\in\mathfrak{S}(p)\setminus\mathfrak{S}_\Gamma} \sum_{e\in\mathfrak{E}(\sigma)} T_e \int_e \boldsymbol{W}\cdot\boldsymbol{n}_{\sigma,e} - T_\sigma \iint_\sigma \nabla_\Gamma\cdot\boldsymbol{W}$$

$$+ R h_\Gamma \sum_{\sigma\in\mathfrak{S}(p)\setminus\mathfrak{S}_\Gamma} \sum_{e\in\mathfrak{E}(\sigma)} \mathcal{F}^{\Omega}_{p,\sigma}(T) = \sum_{\sigma\in\mathfrak{S}(p)\setminus\mathfrak{S}_\Gamma} \iint_\sigma g$$

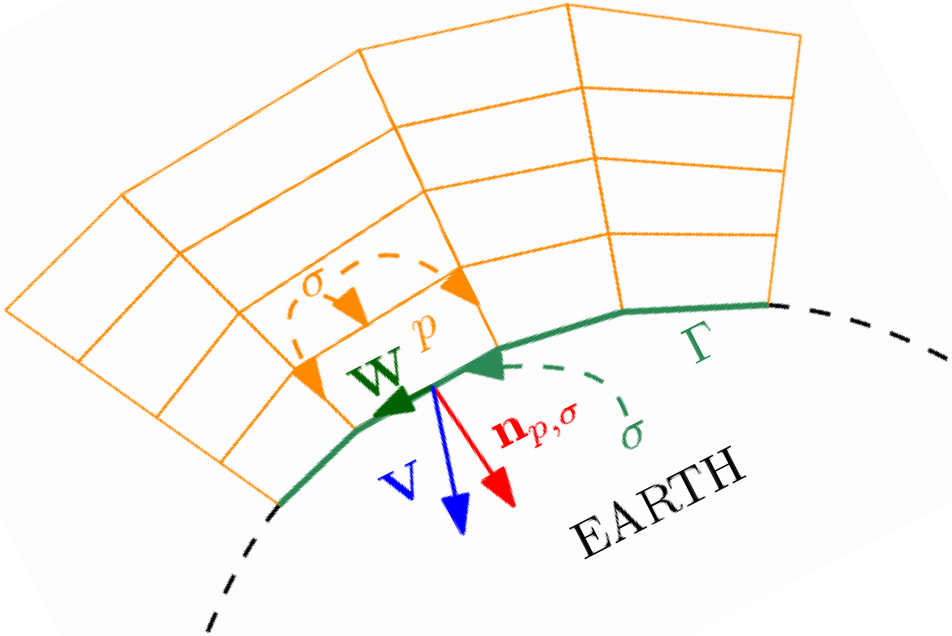
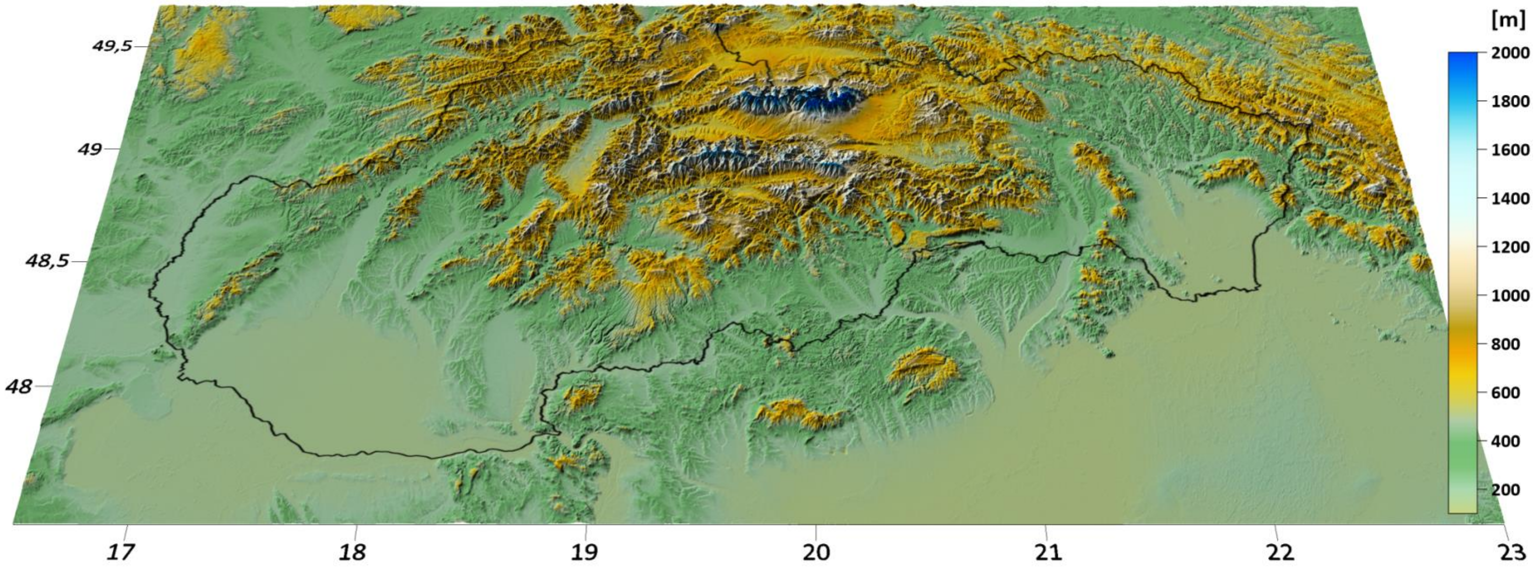


Fig. 2: 2D illustration of a 3D FVM discretization of Ω

4. Numerical results for local gravity field modelling in the area of Slovakia

Fig. 3: Topography in the area of Slovakia



	Boundaries	Resolution	#points
Latitude direction	16.5° - 23°	0.002° (200 m)	3251
Longitude direction	47.5° - 49.7°	0.002° (200 m)	1101
Radial direction	Topography – 230 km	250 m – 1 km	127

4. Numerical results for local gravity field modelling in the area of Slovakia

Boundary conditions:

- Bottom boundary condition (the gravity disturbances) generated
  - inside Slovakia using the CBA2G software [2]
  - Outside Slovakia interpolated from the GGMPlus database [3]
- Upper boundary condition (disturbing potential) generated from the GO_CONS_GCF_2_DIR_R5 geopotential model up to d/o 300 [4]
- Side boundaries condition (disturbing potential) generated from the EIGEN-6C4 geopotential model up to d/o 2160 [5]

4. Numerical results for local gravity field modelling in the area of Slovakia

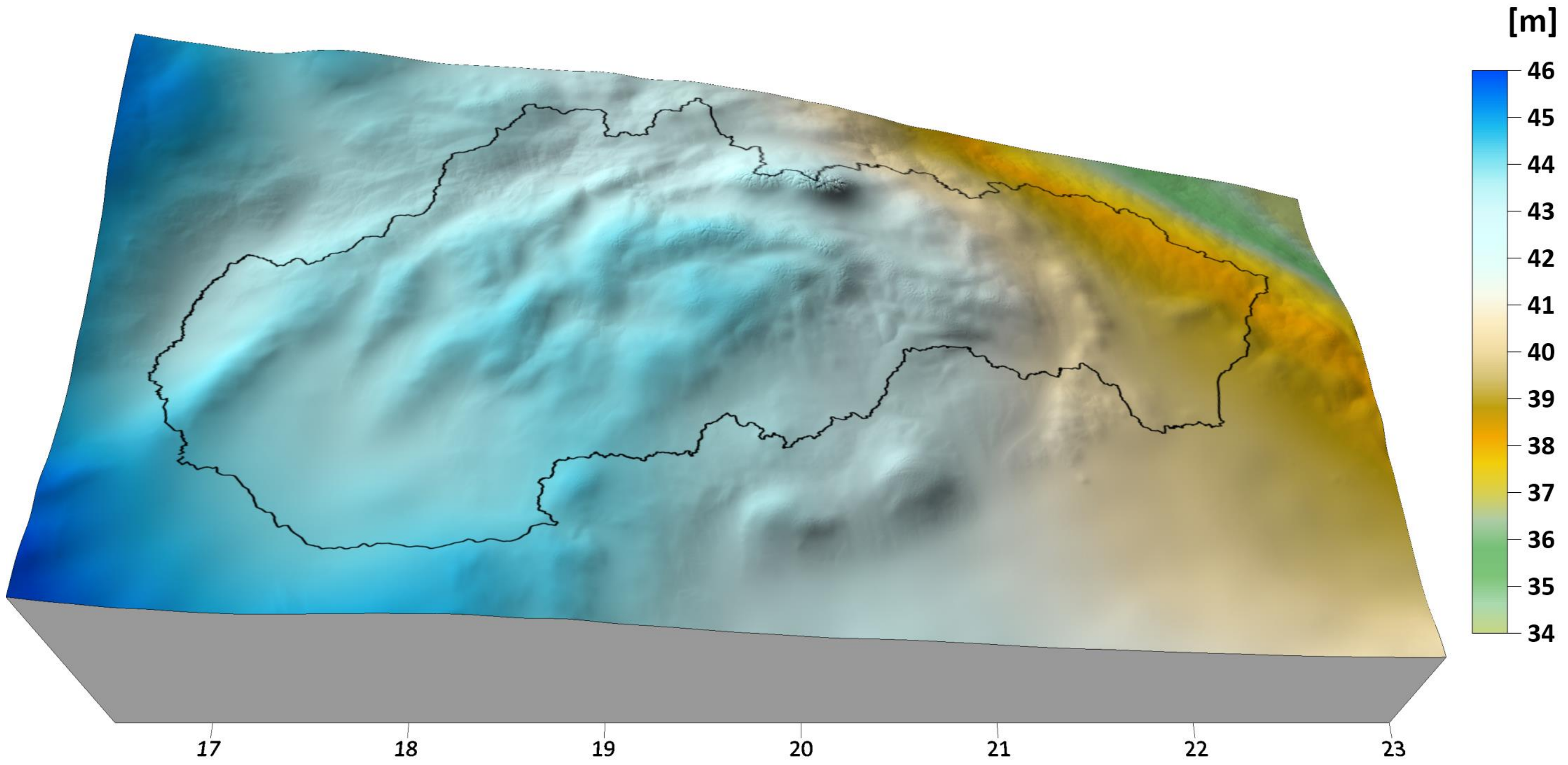


Fig. 5: Local quasigeoid model in the area of Slovakia obtained from the FVM solution

4. Numerical results for local gravity field modelling in the area of Slovakia

Statistics of the GNSS/Levelling test:

Characteristic	For all points	Without outliers
Number of points	404	395
Minimum	0.131 m	0.147 m
Maximum	0.352 m	0.352 m
Range	0.221 m	0.205 m
Mean	0.231 m	0.231 m
Median	0.230 m	0.230 m
Standard deviation	0.028 m	0.026 m

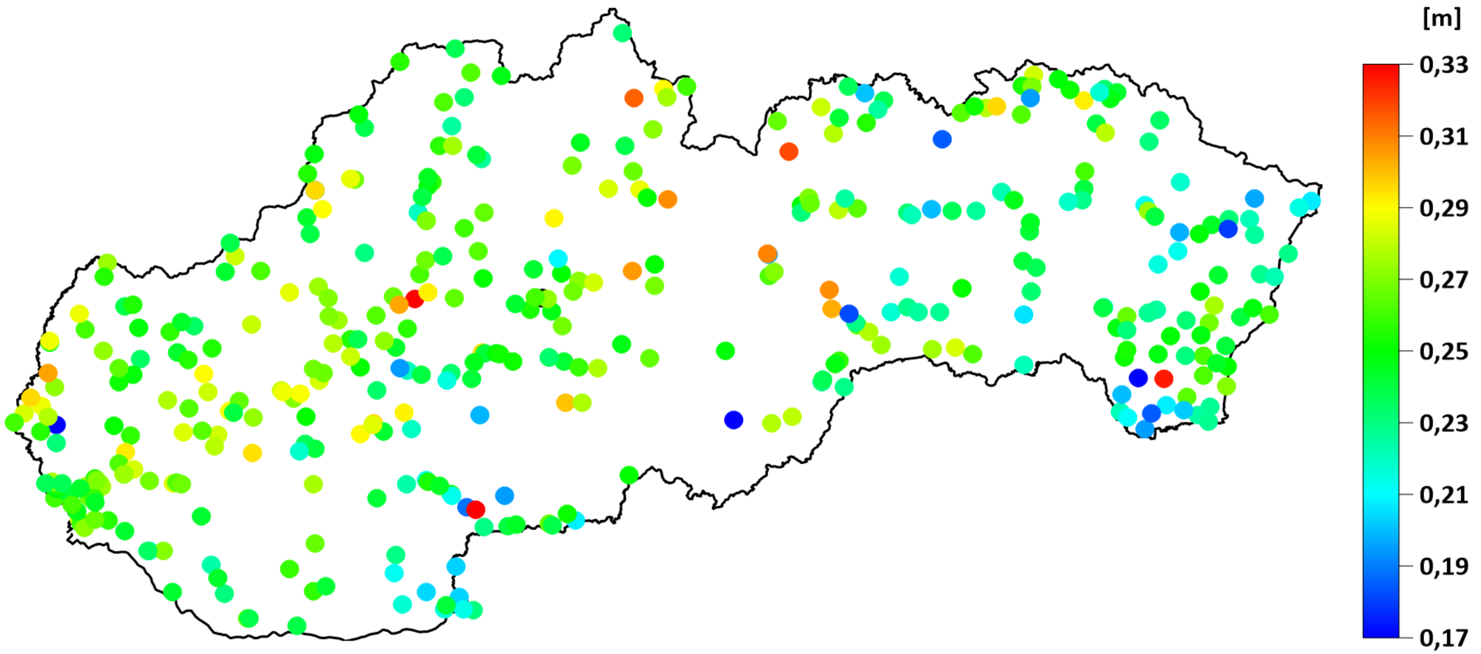


Fig. 6: GNSS/levelling test of the local quasigeoid model in Slovakia at 404 benchmarks

References

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- [5] Förste Ch, Bruinsma SL, Abrikosov O, Lemoine JM, Marty JCh, Flechtner F, Balmino G, Barthelmes F, Biancale R (2014) EIGEN-6C4 - The latest combined global gravity field model including GOCE data up to degree and order 2190 of GFZ Potsdam and GRGS Toulouse. GFZ Data Services. http://doi.org/10.5880/icgem.2015.1