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Session G1.1 – Recent Developments in Geodetic Theory

Differential Geometry and Curvatures of Equipotential Surfaces in the Realization of the World Height System

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1. Introduction

The calculus of *Exterior Differential Forms* was created by an ingenious French mathematician *Elie Cartan* (*1869-1961*) on the turn of the last century.

Beginning with twenties *Cartan's method* started to penetrate into diverse domains of mathematics, first of all in differential geometry, in the theory of *J. Pfaff's systems* and differential equations, in the theory of Lie's groups and in the integral calculus.

The stimulus of this paper comes from the work devoted by professor *Zbyněk Nádeník* to differential geometry and its applications in geodesy. It is, therefore, natural to start with the introductory pages of his paper in *Studia geophysica et geodaetica*, 15(1971), pp. 1-6, that follow:

LES FORMES DIFFÉRENTIELLES EXTÉRIEURES DANS LA GÉODÉSIE I: COURBURE DE GAUSS

ZBYNĚK NÁDENÍK

*École Polytechnique, Prague**)

Резюме: Посредством методов Картана 1) дан краткий вывод Гауссовой кривизны в общих параметрах поверхности, 2) выражена Гауссова кривизна как функция длины радиус-вектора точки поверхности и 3) модифицировано доказательство Гильберта однозначной определенности шаровой поверхности.

1. INTRODUCTION

Le calcul des formes différentielles extérieures a été créé par l'ingénieux mathématicien français Elie Cartan (1869—1961) au changement de siècle. À partir des années vingt, les méthodes de Cartan ont commencé à pénétrer dans les divers domaines des mathématiques, avant tout dans la géométrie différentielle, dans la théorie des systèmes de J. Pfaff et des équations différentielles, dans la théorie des groupes de S. Lie et dans le calcul intégral. De la grande série des livres qui traitent le calcul de Cartan et ses applications dans ces théories, nous introduisons seulement le suivant:

E. Cartan: Les systèmes différentiels extérieurs et leurs applications géométriques, Paris 1945 (traduction russe, Moscou 1962).

Actuellement, il y a plusieurs travaux qui font usage des méthodes de Cartan aussi dans la physique. Nous rappelons seulement les livres:

H. Flanders: Differential Forms with Applications to the Physical Sciences, New York—London 1963;

G. A. Deschamps: Exterior Differential Forms (dans Mathematics Applied to Physics, sous la rédaction de E. Robine), Berlin—Heidelberg—New York—Paris 1970,

qui s'occupent en détail des applications du calcul de Cartan au point de vue de la physique théorique. Parce qu'on peut tenir la géodésie pour un domaine scientifique entre la physique et les mathématiques, dont spécialement la géométrie différentielle et la théorie du potentiel ont un rôle important, il semble qu'il soit utile d'essayer les applications convenables des méthodes de Cartan dans la géodésie.

Nous récapitulons seulement les opérations fondamentales élémentaires: Soient ω, ω^*, Ω, Ω^* les formes de Pfaff (c'est-à-dire les formes différentielles linéaires) et f une fonction scalaire dans un certain domaine. Pour le produit extérieur de ω et Ω, on utilise la notation $[\omega\Omega]$ ou $\omega \wedge \Omega$; nous préférons la première avec les parenthèses. Pour la différentielle extérieure de ω, il y a beaucoup de notations; nous employerons $\mathrm{d}\omega$. Dans ces deux cas, il s'agit de la notation du livre sus-mentionné de Cartan. Le produit extérieur et la différentielle extérieure sont les formes différentielles quadratiques qui satisfont aux règles

$$[\omega\Omega] = -[\Omega\omega]\,, \quad [\omega + \omega^*, \Omega] = [\omega\Omega] + [\omega^*\Omega]\,, \quad [f\omega\Omega] = f[\omega\Omega]\,;$$

$$\mathrm{d}(\omega + \omega^*) = \mathrm{d}\omega + \mathrm{d}\omega^*\,, \quad \mathrm{d}(f\omega) = [\mathrm{d}f, \omega] + f\,\mathrm{d}\omega\,, \quad \mathrm{d}\omega = 0 \Leftrightarrow \omega = \mathrm{d}f\,;$$

$\mathrm{d}f$ désignant la différentielle ordinaire de la fonction f. La dernière relation est un cas très spécial du théorème de Poincaré. Rappelons encore la plus simple forme du lemme de Cartan: Si ω, ω^* sont les formes indépendantes — c'est-à-dire $[\omega\omega^*] \neq 0$, la relation $[\omega\Omega] + [\omega^*\Omega^*] = 0$ signifie que les formes Ω, Ω^* sont des combinaisons linéaires des formes ω, ω^* à coefficients symétriques. Nous nous servirons pour la plupart seulement de ces règles les plus élémentaires.

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2. CONSIDÉRATIONS PRÉLIMINAIRES

Attachons à chaque point $\boldsymbol{x}$[1]) d'un domaine D de l'espace euclidien à trois dimensions E_3 un trièdre trirectangle formé par les trois vecteurs unitaires $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{t}_3$ d'origine $\boldsymbol{x}$. Puis, dans la symbolique conventionelle du calcul de Cartan, on a[2])

$$(2,1) \qquad \mathrm{d}\boldsymbol{x} = \sum_{i=1}^{3} \omega_i \mathbf{t}_i\,, \quad \mathrm{d}\mathbf{t}_i = \sum_{j=1}^{3} \omega_{ij}\mathbf{t}_j\,; \quad \omega_{ij} = -\omega_{ji} \quad (i, j = 1, 2, 3)\,;$$

les ω sont les formes de Pfaff assujéties aux équations de structure de E_3

$$(2,2) \qquad \mathrm{d}\omega_1 = [\omega_2\omega_{21}] + [\omega_3\omega_{31}]\,, \quad \mathrm{d}\omega_2 = [\omega_1\omega_{12}] + [\omega_3\omega_{32}]\,,$$

$$(2,3) \qquad \mathrm{d}\omega_3 = [\omega_1\omega_{13}] + [\omega_2\omega_{23}]\,,$$

$$(2,4) \qquad \mathrm{d}\omega_{12} = [\omega_{13}\omega_{32}]\,, \quad \mathrm{d}\omega_{13} = [\omega_{12}\omega_{23}]\,, \quad \mathrm{d}\omega_{23} = [\omega_{21}\omega_{13}]\,.$$

Prenons en considération dans D une surface S dont le paramétrage $\boldsymbol{x} = \boldsymbol{x}(u, v)$ est — dans un certain domaine des paramètres u, v — de la classe C^3 et satisfait à la condition $\boldsymbol{x}_u \times \boldsymbol{x}_v \neq \boldsymbol{0}$[3]). Soit $\boldsymbol{N} \equiv \mathbf{t}_3$ le vecteur normal à la surface S au point $\boldsymbol{x}$; puis les vecteurs $\mathbf{t}_1$, $\mathbf{t}_2$ se trouvent dans le plan tangent de S en $\boldsymbol{x}$ et, d'après (2,1)—(2,3), nous aurons, dans la description détaillée,

$$(2,5) \qquad \mathrm{d}\boldsymbol{x} = \omega_1\mathbf{t}_1 + \omega_2\mathbf{t}_2\,; \quad [\omega_1\omega_2] \neq 0\,;$$

$$(2,6) \qquad \begin{aligned} \mathrm{d}\mathbf{t}_1 &= \phantom{\omega_{21}\mathbf{t}_1 +{}} \omega_{12}\mathbf{t}_2 + \omega_{13}\boldsymbol{N}\,,\\ \mathrm{d}\mathbf{t}_2 &= \omega_{21}\mathbf{t}_1 \phantom{{}+ \omega_{12}\mathbf{t}_2} + \omega_{23}\boldsymbol{N}\,,\\ \mathrm{d}\boldsymbol{N} &= \omega_{31}\mathbf{t}_1 + \omega_{32}\mathbf{t}_2\,; \end{aligned}$$

$$(2,7) \qquad \mathrm{d}\omega_1 = [\omega_2\omega_{21}]\,, \quad \mathrm{d}\omega_2 = [\omega_1\omega_{12}]\,;$$

$$(2,8) \qquad [\omega_1\omega_{13}] + [\omega_2\omega_{23}] = 0\,;$$

les équations (2,4) ne subissent aucun changement. Tous les ω peuvent être considérés comme les formes de Pfaff seulement en u, v (c' est-à-dire, les expressions de la forme $(.)\,\mathrm{d}u + (..)\,\mathrm{d}v$).

La courbure de Gauss K de la surface S étant le quotient de l'élément d'aire $[\omega_{31}\omega_{32}]$ de l'image sphérique de S et de l'élément d'aire $[\omega_1\omega_2]$ de S, on trouve pour K d'après la première équation (2,4)

$$(2,9) \qquad K = -\mathrm{d}\omega_{12} : [\omega_1\omega_2]\,.$$

En posant

$$(2,10) \qquad \omega_{12} = g_1\omega_1 + g_2\omega_2\,,$$

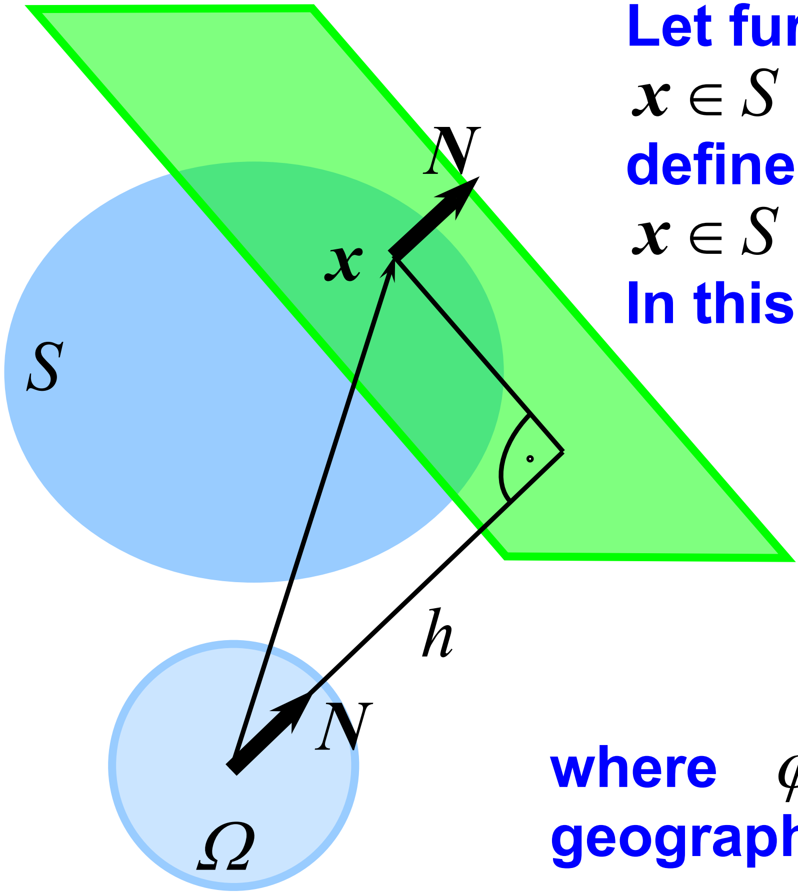
[1]) Il est superflu de faire ici une distinction entre le point et son rayon vecteur.

[2]) Au lieu de ω_i, ω_{ij} on écrit souvent ω^i, ω_i^j et puis, on adopte, quant au signe de sommation, la convention d'Einstein; cf. le livre sus-mentionné de Cartan.

[3]) Les indices désignent les dérivées partielles. Les vecteurs tangents en question sont donc linéairement indépendants.

2. Support Function

In this section we start from Holota P. and Nádeník Z.: *Les formes différentieles extérieures dans la géodésie II: Courbure moyenne*. Studia geophysica et geodaetica, 15 (1971), pp. 106-112. We suppose that S is a ***convex surface***, so that $[\omega_{31}\omega_{32}] \neq 0$.



Let further the normal vector $\boldsymbol{N}$ in the point $\boldsymbol{x} \in S$ be oriented to the half space that is defined by the ***tangent plane*** in the point $\boldsymbol{x} \in S$ and does not contain the surface S. In this case

$$h = \boldsymbol{x} \cdot \boldsymbol{N}$$

is a ***support function*** of S defined on the sphere Ω with parameterization

$$\boldsymbol{N} = (\cos\varphi\cos\lambda, \cos\varphi\sin\lambda, \sin\varphi)$$

where $\varphi \in \langle -\frac{1}{2}\pi, \frac{1}{2}\pi \rangle$ and $\lambda \in \langle 0, 2\pi \rangle$ are geographical coordinates.

3. Dual Representation of the Surface S

The *unit tangential vectors*

$$N_\varphi = \frac{\partial N}{\partial \varphi} = (-\sin\varphi\cos\lambda, -\sin\varphi\sin\lambda, \cos\varphi)$$

$$\frac{N_\lambda}{\cos\varphi} = \frac{1}{\cos\varphi}\frac{\partial N}{\partial \lambda} = (-\sin\lambda, -\cos\lambda, 0)$$

of parametric lines on the unit sphere Ω are tangential vectors of the surface S too and one can easily show that

$$N_\varphi \cdot (N_\lambda / \cos\varphi) = 0\,, \qquad N_\varphi \cdot N = 0\,, \qquad (N_\lambda / \cos\varphi)\cdot N = 0$$

Thus they form a *trirectangular trihedron* and one can deduce that

$$x(\varphi,\lambda) \;=\; h\cdot N \;+\; h_\varphi \cdot N_\varphi + \frac{h_\lambda}{\cos\varphi}\cdot\frac{N_\lambda}{\cos\varphi}$$

i.e. the surface S is *represented* by the *support function* h with respect to the *moving trihedron* given by N, N_φ, $N_\lambda / \cos\varphi$.

- **Now in general, putting**

$$\boldsymbol{x} \;=\; \tau_1 \boldsymbol{t}_1 + \tau_2 \boldsymbol{t}_2 + \; h\boldsymbol{N}$$

we determine parameters τ_1 and τ_2 . Using *Cartan's lemma*, we have

$$\omega_{31} = a\omega_1 + b\omega_2 \quad \text{and} \quad \omega_{32} = b\omega_1 + c\omega_2$$

where

$$a + c = \frac{1}{R_1} + \frac{1}{R_2} \quad \text{and} \quad ac - b^2 = \frac{1}{R_1} \cdot \frac{1}{R_1}$$

is the *mean* and *Gauss curvature* of our surface S.

- **Subsequently, the calculation of $d\boldsymbol{x}$ and the comparison with**

$$d\boldsymbol{x} \;=\; \omega_1 \boldsymbol{t}_1 + \; \omega_2 \boldsymbol{t}_2$$

immediately yield

$$\omega_1 = d\tau_1 + \tau_2\omega_{21} + h\omega_{31} \;, \qquad \omega_2 = d\tau_2 + \tau_1\omega_{12} + h\omega_{32}$$

$$dh = \tau_1\omega_{31} + \tau_2\omega_{32}$$

where $\tau_1 = h_1$, $\tau_2 = h_2$ are derivatives with respect to ω_{31} and ω_{32}.

Hence, from the results above we obtain

$$R_1 + R_2 = \frac{[\omega_{31}\omega_2] + [\omega_1\omega_{32}]}{[\omega_{31}\omega_{32}]} = 2h + \Delta_2 h$$

$$R_1 R_2 = \frac{[\omega_1\omega_2]}{[\omega_{31}\omega_{32}]} = h^2 + h\,\Delta_2 h + \nabla_2 h$$

where

$$\Delta_2(h) = \frac{[dh_1 + h_2\omega_{21}, \omega_{32}] + [\omega_{31}, dh_2 + h_1\omega_{12}]}{[\omega_{31}\omega_{32}]} = \frac{[d(h_1\omega_{32} - h_2\omega_{31})]}{[\omega_{31}\omega_{32}]}$$

is *Beltrami's differential* operator of *2nd order* with respect to Ω (i.e. the sphere) and *recall that* in coordinates φ and λ it reads

$$\Delta_2 h = \frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial h}{\partial\varphi}\right) + \frac{1}{\cos^2\varphi}\frac{\partial^2 h}{\partial\varphi^2}$$

Similarly, $$\nabla_2(h) = \frac{[dh_1 + h_2\omega_{21}, dh_2 + h_1\omega_{12}]}{[\omega_{31}\omega_{32}]}$$

is *also a differential operator of 2nd order* with respect to Ω.

4. Weingarten's Formula

The first formula above, i.e.

$$R_1 + R_2 = 2h + \Delta_2 h$$

has been derived by *Weingarten in 1884* and was also a point of departure for *Hurwitz's* classical proof of *Christoffel's* theorem, who wrote:

„In Folge dieses Satzes kann die Gestalt der Fläche E mit jeder beliebigen Genauigkeit bestimmt werden, wenn man im Stande ist, für eine genügende Anzahl von Punkten derselben, über deren gegenseitige Lage keine anderweitigen Angaben erforderlich sind, 1. die sphärische Koordinaten ihres wahren Zeniths, und 2. die Summe der in ihnen stattfindenden Hauptkrümmungshalbmesser zu ermitteln."

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see E.B. Christoffel: Über die Bestimmung der Gestalt einer krummen Oberfläche durch geodätische Messungen auf derselben. Journal für die reine und angewandte Mathematik 64(1864), 193-209.

5. Christoffel's Theorem - Proof

To see that the theorem really holds, suppose that h can be developed into a convergent series

$$h = h_0 + h_1 + h_2 + \cdots$$

of surface spherical harmonics h_i, $i = 0, 1, 2, \ldots$. Thus

$$R_1 + R_2 = 2h + \Delta_2 h \ = \ -\sum_{n=0}^{\infty} (n-1)(n+2)\, h_n$$

In parallel suppose that

$$R_1 + R_2 = \sum_{n=0}^{\infty} Q_n$$

where Q_n are the respective surface spherical harmonics. Hence

$$h_n = -\frac{1}{(n-1)(n+2)} Q_n \quad \text{for} \quad i = 0, 2, 3, \ldots$$

Moreover, one can easily show that $Q_1 \equiv 0$ identically, so that h is determined apart from h_1, but this depends just from the position of the origin of coordinates.

6. Generalizatized Weingaten Formula

Weingarten's formula **can be also generalized in the following sense:**

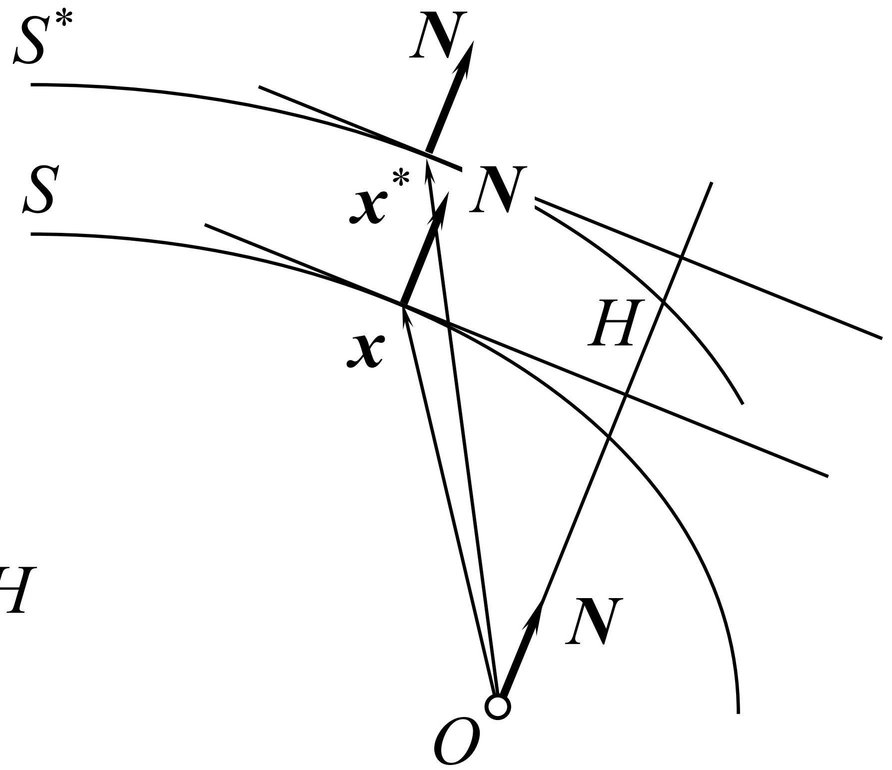
Together with S consider another surface S^* that has its spherical image on Ω, similarly as the surface S. Let h^* be the support function of S^*.

Then, putting

$$H = h^* - h$$

we easily obtain

$$R_1^* + R_2^* - (R_1 + R_2) \;=\; 2H + \Delta_2 H$$



7. Interpretation in terms of Classical Physical Geodesy

In case that S is the *reference ellipsoid* and S^* the *geoid* the generalized *Weingarten formula*, i.e.

$$R_1^* + R_2^* - (R_1 + R_2) = 2H + \Delta_2 H$$

has an interesting interpretation. *In order to see it*, recall that in physical geodesy (within some approximation) we usually put

$$H = \frac{T}{G}$$ where G is a *mean value of gravity over the Earth*

and T is the value of the *disturbing potential*

$$T(r,\varphi,\lambda) = \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} T_n(\varphi,\lambda)$$

for $r = R$, where R is a *mean radius of the Earth*.

In the next step recall also that T meets *Laplace's equation* $\Delta T = 0$, i.e.

$$r^2 \frac{\partial^2 T}{\partial r^2} + 2r \frac{\partial T}{\partial r} + \frac{1}{\cos\varphi} \frac{\partial}{\partial \varphi}\left(\cos\varphi \frac{\partial T}{\partial \varphi}\right) + \frac{1}{\cos^2\varphi} \frac{\partial^2 T}{\partial \varphi^2} = 0$$

which means that

$$r^2 \frac{\partial^2 T}{\partial r^2} + 2r \frac{\partial T}{\partial r} + \Delta_2 T = 0$$

where

$$\Delta_2 T = \frac{1}{\cos\varphi} \frac{\partial}{\partial \varphi}\left(\cos\varphi \frac{\partial T}{\partial \varphi}\right) + \frac{1}{\cos^2\varphi} \frac{\partial^2 T}{\partial \varphi^2}$$

is *Beltrami's differential* operator applied on T. Hence

$$\Delta_2 \left[T(R,\varphi,\lambda)\right] = -R^2 \left.\frac{\partial^2 T}{\partial r^2}\right|_{r=R} - 2R \left.\frac{\partial T}{\partial r}\right|_{r=R}$$

and obviously

$$\Delta_2 H = \left[\frac{1}{G} T(R,\varphi,\lambda)\right] = -\frac{1}{G}\left(R^2 \left.\frac{\partial^2 T}{\partial r^2}\right|_{r=R} + 2R \left.\frac{\partial T}{\partial r}\right|_{r=R}\right)$$

Thus

$$R_1^* + R_2^* - (R_1 + R_2) = 2H + \Delta_2 H = \frac{1}{G}\left(2T - 2r\frac{\partial T}{\partial r} - r^2\frac{\partial^2 T}{\partial r^2}\right)\bigg|_{r=R}$$

Moreover, taking into account that

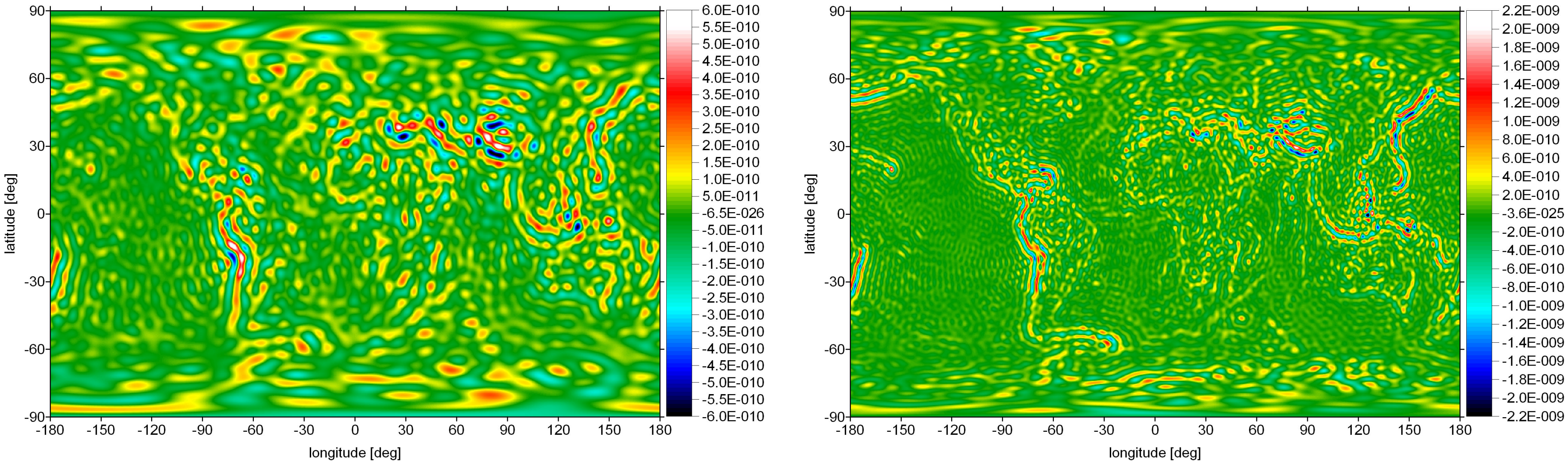
$$T(r,\varphi,\lambda) = \sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^{n+1} T_n(\varphi,\lambda)$$

***we can interpret the above formula* (i.e. *Weingarten generalized formula*) in terms of a *series development*, indeed:**

$$R_1^* + R_2^* - (R_1 + R_2) = -\frac{1}{G}\sum_{n=0}^{\infty}(n-1)(n+2)T_n$$

The results above enables to compute also the anomaly ΔJ of the *Mean Curvature*. In the figures below one can see that with increasing resolution the anomalies ΔJ of the *Mean Curvature* grow.

Figure 1. ***Anomaly*** ΔJ ***of the Mean Curvature*** **of the level surface** $W = W_0$ **with respect to** ***Reference Ellipsoid*** **(*GRS80*).** W **(*EGM2008*) represented by a** ***spherical harmonic development*** **up to degree** $n = 60$ **(*left*) ; up to degree** $n = 120$ **(*right*).**

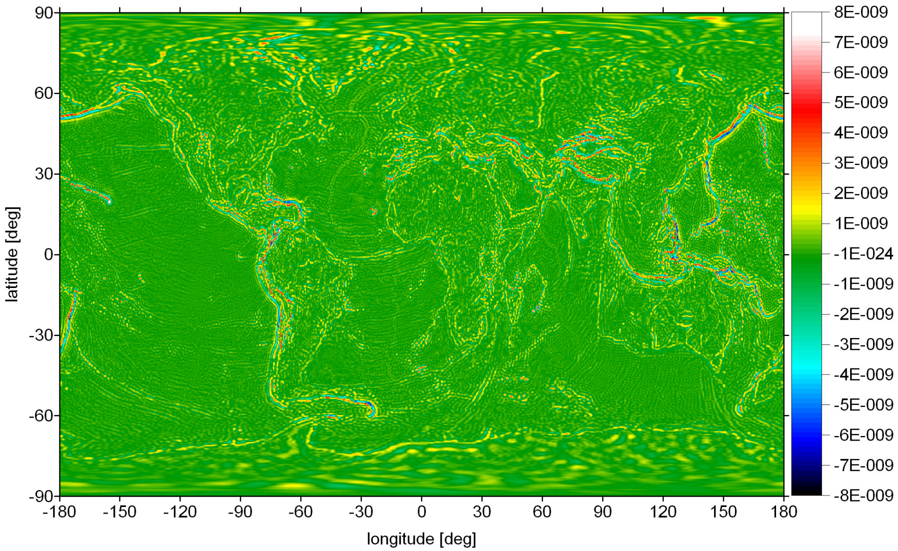


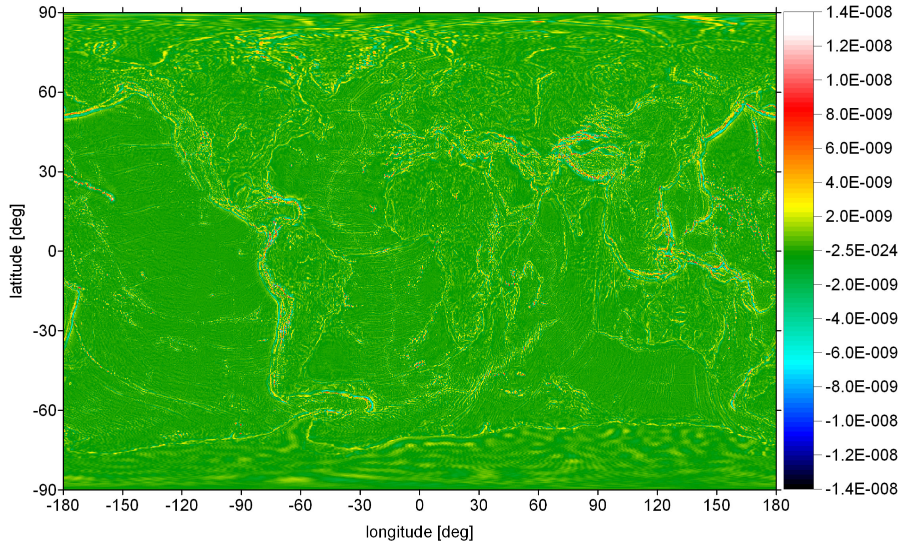
$$\Delta J \in \langle -6 \cdot 10^{-10} m, +6 \cdot 10^{-10} m \rangle \qquad \Delta J \in \langle -2.2 \cdot 10^{-9} m, +2.2 \cdot 10^{-9} m \rangle$$

which means that (approximately)

$$R_i^* - R_i \in \langle -6 \cdot 10^{3} m, +6 \cdot 10^{3} m \rangle \qquad R_i^* - R_i \in \langle -2.2 \cdot 10^{4} m, +2.2 \cdot 10^{4} m \rangle$$

Figure 2. ***Anomaly*** ΔJ ***of the Mean Curvature*** **of the level surface** $W = W_0$ **with respect to** ***Reference Ellipsoid*** **(*GRS80*).** W **(*EGM2008*) represented by a** ***spherical harmonic development*** **up to degree** $n = 360$ **(*left*) ; up to degree** $n = 720$ **(*right*).**



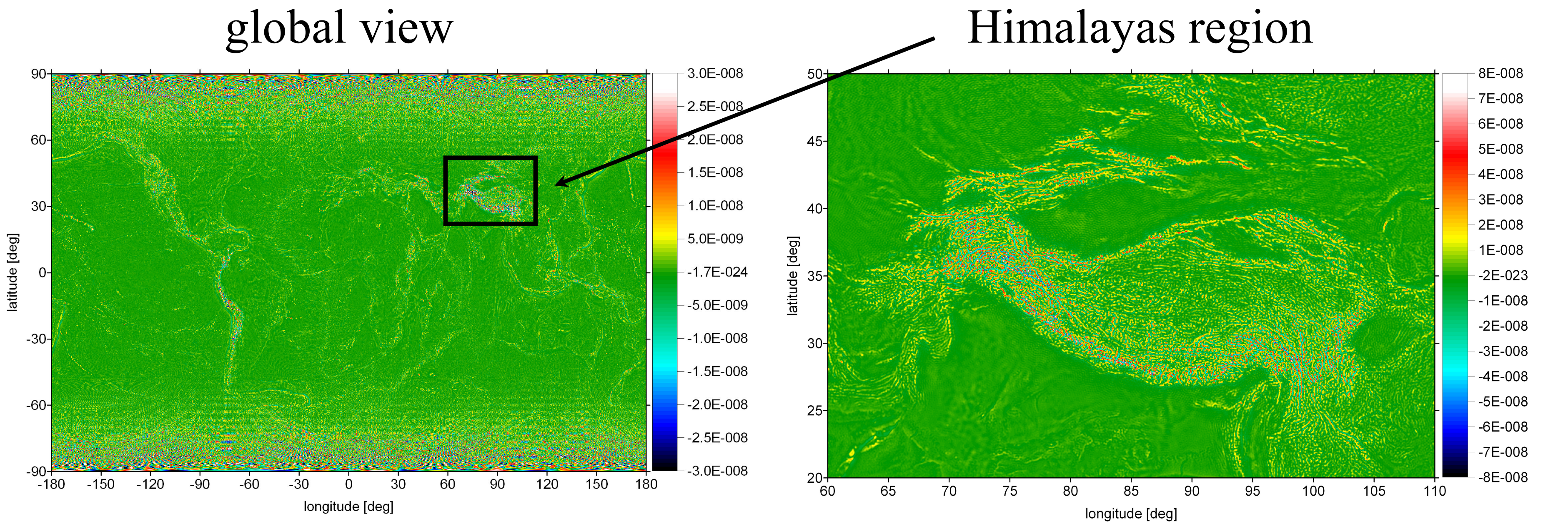


$$\Delta J \in \langle -8\cdot 10^{-9} m, +8\cdot 10^{-9} m\rangle \qquad \Delta J \in \langle -1.4\cdot 10^{-8} m, +1.4\cdot 10^{-8} m\rangle$$

which means that (approximately)

$$R_i^* - R_i \in \langle -8\cdot 10^{4} m, +8\cdot 10^{4} m\rangle \qquad R_i^* - R_i \in \langle -1.4\cdot 10^{5} m, +1.4\cdot 10^{5} m\rangle$$

Figure 3. ***Anomaly* ΔJ *of the Mean Curvature* of the level surface $W = W_0$ with respect to *Reference Ellipsoid* (*GRS80*). W (*EGM2008*) represented by a *spherical harmonic development* up to degree $n = 2160$.**

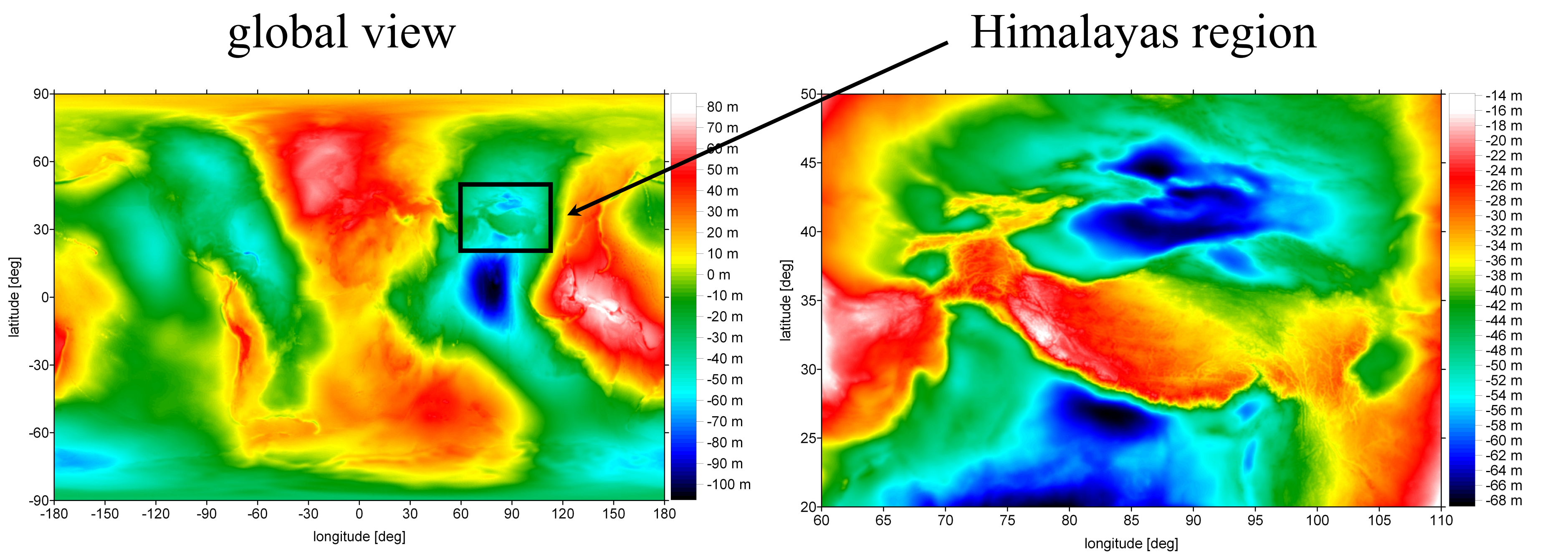


$$\Delta J \in \langle -6 \cdot 10^{-8} m, +6 \cdot 10^{-8} m \rangle$$

which means that (approximately)

$$R_i^* - R_i \in \langle -6 \cdot 10^{5} m, +6 \cdot 10^{5} m \rangle$$

Figure 4 (*for comparison*). **Same regions as in Figure 3, but characterized by *geoid undulations* for $n = 2160$.**



8. Final Remark:
Mean Sea Surface and Geoid

Weingarten's (generalized) formula

$$R_1^* + R_2^* - (R_1 + R_2) \;=\; 2H + \Delta_2 H$$

can also be applied in case that S is the reference ellipsoid as above and that for a potential W we are looking for a level surface $W = W_0 + \Delta W = const.$ that approximates the ***Means Sea Surface*** S^*. Indeed, we put

$$H = \frac{T - \Delta W}{G} \quad \text{where} \quad T(r,\varphi,\lambda) = \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} T_n(\varphi,\lambda)$$

and in analogy to our results above we arrive at

$$\Delta W = \frac{G}{2}\left[R_1^* + R_2^* - (R_1 + R_2)\right] - \frac{1}{2}\left(2T - 2r\frac{\partial T}{\partial r} - r^2\frac{\partial^2 T}{\partial r^2}\right)\Bigg|_{r=R}$$

or

$$\Delta W = \frac{G}{2}\left[R_1^* + R_2^* - (R_1 + R_2)\right] + \frac{1}{2}\sum_{n=0}^{\infty}(n-1)(n+2)T_n$$

The use of this result is also the goal of our future work focused on the determination of the constant ΔW. Motivation also comes from the expected fine-scale ocean surface topography observations with the Surface Water and Ocean Topography (SWOT) Mission planned for launch in late 2021.

Thank you for your attention !

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