Statistics of Coherent Structures Collisions and their Dynamics on the Surface of Deep Water

S. Dremov[1], D. Kachulin[1], A. Dyachenko[2]

[1]Novosibirsk State University, Novosibirsk, Russia
[2] Landau Institute for Theoretical Physics RAS, Chernogolovka, Russia

4, May 2020

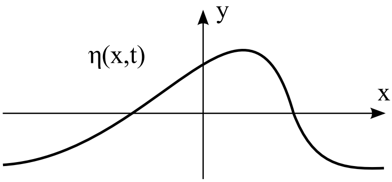
Potential Flow of 2D Ideal Fluid

Potential flow

$$\triangle\phi(x,y,t)=0$$

Boundary conditions:

$$\left[\begin{array}{l}\dfrac{\partial\phi}{\partial t}+\dfrac{1}{2}|\nabla\phi|^2+g\eta=0,\\ \dfrac{\partial\eta}{\partial t}+\eta_x\phi_x=\phi_y\end{array}\right|\text{ at } y=\eta(x,t).$$



$$\frac{\partial\phi}{\partial y}=0,\quad y\to-\infty$$

Hamiltonian H is the total energy of fluid $H=T+U$.

$$\begin{aligned} T &= \frac{1}{2}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\eta}(\nabla\phi)^2dy \\ U &= \frac{g}{2}\int\eta^2dx \end{aligned}\qquad \begin{aligned}\frac{\partial\eta}{\partial t} &= \frac{\delta H}{\delta\psi},\\ \frac{\partial\psi}{\partial t} &= -\frac{\delta H}{\delta\eta},\end{aligned}\qquad \psi(x,t)=\phi(x,y,t)|_{y=\eta}$$

The Hamiltonian expansion in a power series of variables η and ψ up to 4-th order:

$$H(\eta,\psi)=\frac{1}{2}\int\{g\eta^2+\psi\hat{k}\psi\}dx-\frac{1}{2}\int\{(\hat{k}\psi)^2-(\psi_x)^2\}\eta dx+\frac{1}{2}\int\{\psi_{xx}\eta^2\hat{k}\psi+\psi\hat{k}(\eta\hat{k}(\eta\hat{k}\psi))\}dx$$

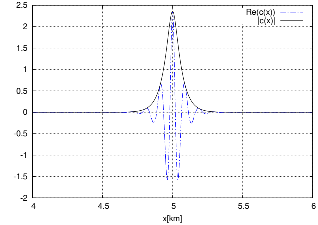
The super compact Zakharov equation (SCZE) and its breather solution

We apply canonical transformation $\eta, \psi \longleftrightarrow c$ to remove cubic nonlinear terms and to drastically simplify fourth-order terms in the Hamiltonian.
The equation of motion takes the following form:

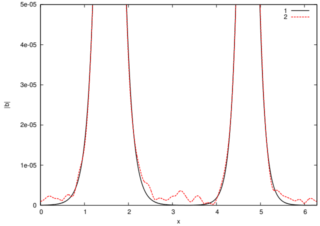
$$\frac{\partial c}{\partial t} + i\hat{\omega} c - i\hat{P}^{+}\frac{\partial}{\partial x}\left(|c|^2 \frac{\partial c}{\partial x}\right) = \hat{P}^{+}\frac{\partial}{\partial x}(\hat{k}|c|^2 c)$$

This equation is called the super compact Zakharov equation (SCZE) and it has a breather solution (can be found numerically by the Petviashvili method):

$$c(x,t) = c_{br}\left(x - V_0 t; \delta\right) e^{ik_0 x - i\omega_{k_0} t - i\delta t},$$



A steep breather. Blue dash-dotted curve shows $Re[c(x)]$; black solid curve indicate the modulus of breather profile $|c(x)|$



The collision of two breathers. Black curve shows $|c(x)|$ at initial moment of time, red curve shows $|c(x)|$ after breather collision.

Nonlinear Schrödinger equation and solitons collision

•The nonlinear Schrödinger equation (NLSE): $$\frac{\partial C}{\partial t}+\frac{i\omega_{k_0}}{8k_0^2}\frac{\partial^2 C}{\partial x^2}+ik_0^2\left[|C|^2C\right]=0.$$

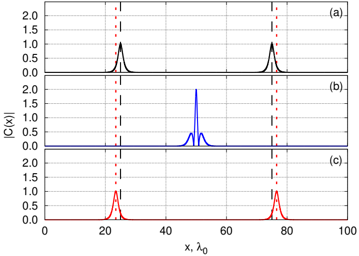
A soliton solution for the NLSE:

$$C_s(x,t)=\frac{\delta}{k_0}\frac{\exp\left[-i\frac{2k_0U}{V_0}x+i\frac{U^2k_0}{V_0}t-i\frac{\delta^2}{2}t\right]}{\cosh\left[\frac{2\delta k_0}{\sqrt{\omega_{k_0}}}(x-Ut)\right]}.$$

• The NLSE is an integrable equation and it has essential peculiarities related to the solitons interaction. Collisions of the NLSE solitons are completely "elastic", that is, there are no energy exchanges between them and their basic parameters, amplitudes and velocities, do not change.

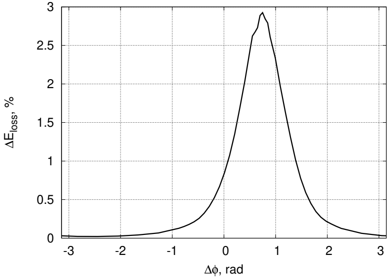
The comparison of soliton collisions in the DZe and the NLSE

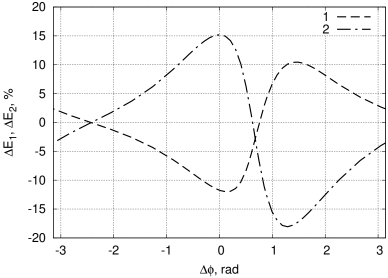
D. Kachulin and A. Gelash, On the phase dependence of the soliton collisions in the Dyachenko–Zakharov envelope equation *Nonlinear Processes in Geophysics*, 25(3), 553-563(2018).



- Numerical simulations were performed in the periodic domain $x \in [0, 100\lambda_0]$ in the reference frame moving with the group velocity $V_0 = \frac{1}{2}\sqrt{\frac{g\lambda_0}{2\pi}}$;
- At the initial time $t = 0$ the distance between the solitons was $50\lambda_0$.
- Solitons have close unidirectional velocities $V_1 = V_0 + U_0$ and $V_2 = V_0 - U_0$.
  - The experiments were carried out with different initial phases of the " left " soliton to study the influence of solitons relative phase on the dynamics of their interaction.
  - At the time $t = \frac{25\lambda_0}{U_0}$ the solitons collided; The calculations were carried out up to the time $t = \frac{50\lambda_0}{U_0}$ when the distance between the solitons became $\approx 50\lambda_0$ again.

Total energy losses and energy interchange





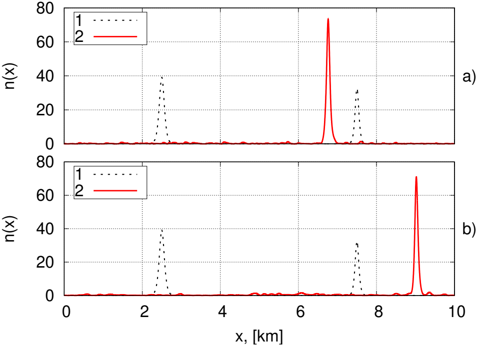
The total energy losses ΔE_{loss} (in percent) of DZe solitons after their collision depending on the relative phase $\Delta\phi$. The wave steepness of the solitons $\mu \approx 0.2$.

The individual energy change (in percent) of the DZe solitons after their collision depending on the relative phase $\Delta\phi$. The dashed curve 1 shows dependence of energy change for the first soliton, while the dash–dotted curve 2 corresponds to the dependence of energy change for the second soliton.

Long-time dynamics of pairwise breathers interactions

One of the breathers having a higher velocity also has a larger number of particles:

$$N = \int n(x)dx = \int \hat{k}^{-1} c(x) c^*(x) dx$$



The initial (black dashed curve) and the final (red solid curve) number of particles densities $n(x)$ for two different numerical experiments.

The experiments showed that in all 32 cases with different initial phases the only one soliton remains at the end, but two main interaction scenarios were discovered.

The first scenario of breather pairwise interaction

We label the soliton that initially had a larger N by index 1, while the second one by index 2.

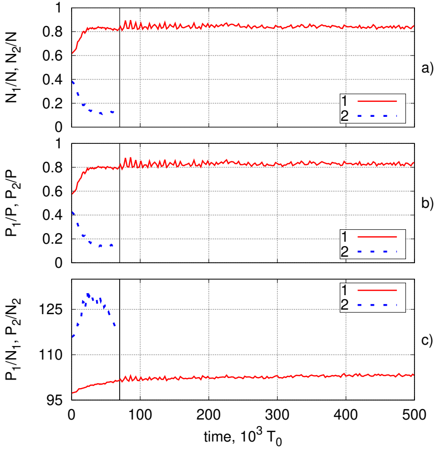
Here number of particles:

$$N = \int \frac{c(x)c^*(x)}{\hat{k}} dx$$

momentum:

$$P = \int c(x)c^*(x) dx$$

In the first scenario N_1 increases, while N_2 decreases after each pairwise interaction until the absorbtion of soliton 2 shown by a vertical black line. After that, N_1 reaches a constant and N_2 vanishes. A similar behaviour for the momentum of P_1 and P_2. The value of $\frac{P_i}{N_i}$ has a dimension of characteristic wave number.



Time dependence for number of particles (a), momentum (b) and wave number (c) in the first scenario.

Estimates for the number of particles, momentum and energy of the SCZE soliton

Following [A. I. Dyachenko, et al., Zh. Eksp. Teor. Fiz, **96**, 2026 (1989)] one can obtain relations for changing the number of particles of each breather after their collision using the conservation laws for the number of particles, momentum and energy.
The expressions for three integrals of motion – the number of particles, the total momentum and the total energy – have the following form:

$$N = \frac{1}{k_0}\int |C|^2 dx, \qquad P = \int |C|^2 dx + \frac{i}{2k_0}\int \left[C\frac{\partial C^*}{\partial x} - C^*\frac{\partial C}{\partial x}\right] dx$$

$$E = \frac{\omega_{k_0}}{k_0}\int |C|^2 dx + i\frac{\omega_{k_0}}{4k_0^2}\int \left[C\frac{\partial C^*}{\partial x} - C^*\frac{\partial C}{\partial x}\right] dx - \frac{\omega_{k_0}}{8k_0^3}\int \left|\frac{\partial C}{\partial x}\right|^2 dx + \frac{k_0}{2}\int |C|^4 dx$$

For a single soliton full momentum P and total energy E can be determined by number of particles $N = \frac{\sqrt{\omega_{k_0}}}{k_0^4}\delta$ and soliton velocity U:

$$P = k_0 N - \frac{4Uk_0^2}{\omega_{k_0}}N, \qquad E = \omega_{k_0}N - 2Uk_0N - \frac{2U^2k_0^2}{\omega_{k_0}}N + \frac{k_0^8}{6\omega_{k_0}}N^3.$$

Expressions for the number of particles changes

The balance expressions for the interaction process of two solitons (N_1, U_1) and (N_2, U_2) with radiation losses δN take the form:

$$
\begin{aligned}
&N_1 + N_2 = \tilde{N}_1 + \tilde{N}_2 + \delta N, \\
&U_1 N_1 + U_2 N_2 = \tilde{U}_1 \tilde{N}_1 + \tilde{U}_2 \tilde{N}_2 - \frac{V_0 k_w}{2k_0} \delta N, \\
&U_1^2 N_1 - \frac{k_0^6}{12} N_1^3 + U_2^2 N_2 - \frac{k_0^6}{12} N_2^3 = \tilde{U}_1{}^2 \tilde{N}_1 - \frac{k_0^6}{12} \tilde{N}_1{}^3 + \tilde{U}_2{}^2 \tilde{N}_2 - \frac{k_0^6}{12} \tilde{N}_2{}^3 + \frac{V_0^2 k_w^2}{4k_0^2} \delta N
\end{aligned}
$$

One can obtain the following expressions for the number of particles changes $\Delta N_1 = \tilde{N}_1 - N_1$ and $\Delta N_2 = \tilde{N}_2 - N_1$ of soliton 1 and 2 correspondingly:

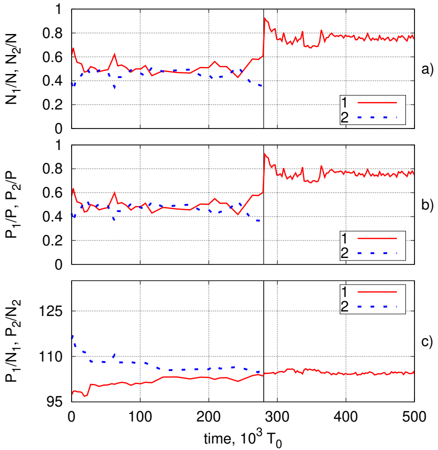
$$
\Delta N_1 = \frac{\left(\frac{k_0^6}{4} N_2^2 + \left(U_2 + \frac{V_0 k_w}{2k_0} \right)^2 \right) \delta N + 2N_1 \Delta U_1 (U_1 - U_2)}{\frac{k_0^6}{4} (N_1^2 - N_2^2) - (U_1 - U_2)^2},
$$

$$
\Delta N_2 = -\frac{\left(\frac{k_0^6}{4} N_1^2 + \left(U_1 + \frac{V_0 k_w}{2k_0} \right)^2 \right) \delta N - 2N_2 \Delta U_2 (U_1 - U_2)}{\frac{k_0^6}{4} (N_1^2 - N_2^2) + (U_1 - U_2)^2}.
$$

Here $\Delta U_1 = \tilde{U}_1 - U_1$ and $\Delta U_2 = \tilde{U}_2 - U_1$ are the velocities changes of soliton 1 and 2 correspondingly.

The second scenario of breather pairwise interaction

Another scenario is less common and is as follows. When the breathers velocities are very close to each other they bind for a while into one periodically oscillating structure resembling the NLSE bi-soliton solution. This scenario can no longer be described by abovementioned expressions for ΔN_i due to the complex and intense interaction of breathers. Interacting in this way for some time they significantly exchange their number of particles and the result remains the same – one of them is completely absorbed.



Time dependence for number of particles (a), momentum (b) and wave number (c) for the second scenario.

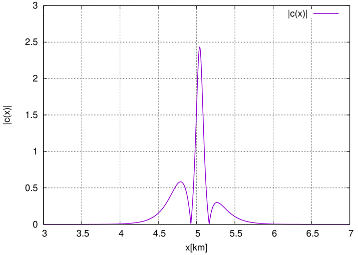
Bound soliton and exact bi-soliton solution of the NLSE

The exact bound bi-soliton solution of the NLSE is well-known:

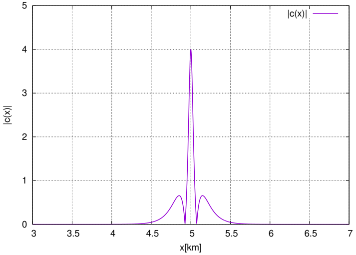
$$C_{bs}(x,t) = \frac{2\frac{C_1-C_2}{C_1+C_2}\left[C_1\cosh\left[C_2\eta_2\right]e^{-\frac{1}{2}ik_0^2C_1^2t} - C_2\cosh\left[C_1\eta_1\right]e^{-\frac{1}{2}ik_0^2C_2^2t}\right]}{\left(\frac{C_1-C_2}{C_1+C_2}\right)^2\cosh\left[C_1\eta_1+C_2\eta_2\right]+\cosh\left[C_1\eta_1-C_2\eta_2\right]-\frac{4C_1C_2}{(C_1+C_2)^2}\cos\left[\left(C_1^2-C_2^2\right)\frac{k_0^2}{2}t\right]}$$

Here, $\eta_1 = \frac{2k_0^2}{\sqrt{\omega_{k_0}}}(x-x_1)$ and $\eta_2 = \frac{2k_0^2}{\sqrt{\omega_{k_0}}}(x-x_2)$.
This solution is periodic in time with period $T = \frac{4\pi}{k_0^2(C_1^2-C_2^2)}$.

The examples of bound soliton solutions



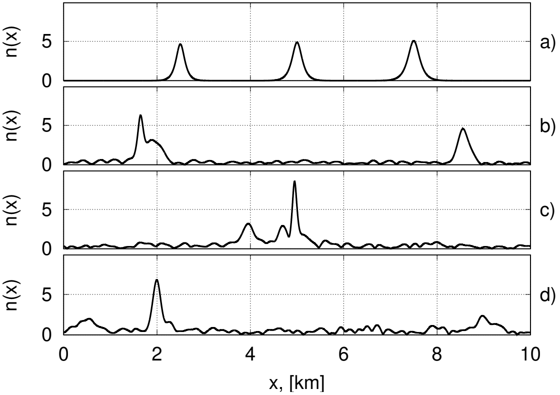
The BS solution of the NLSE for $C_1 = 1.75$, $C_2 = 0.75$ and $x_1 - x_2 = 50$.



The BS solution of the NLSE for $C_1 = 3$, $C_2 = 1$ and $x_1 = x_2$.

Long-time interaction of three breathers

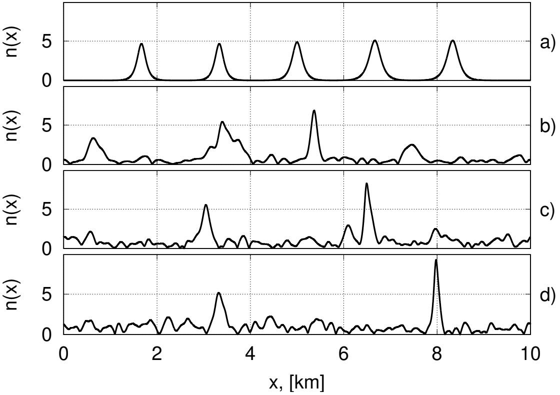
An increasing number of solitons in the domain will not lead to any drastic changes. They still interact according to two previously discovered scenarios, and still, the only one remains.



The evolution of three interacting breathers. The ordinate shows the particles number density $n(x)$. The top panel (a) presents the initial state ($t = 0$). Panels (b) and (c) corresponds to the states with $t = 5.48 \cdot 10^5$s and $t = 1.5 \cdot 10^6$s. The lower panel (d) presents the state where the only one soliton remained.

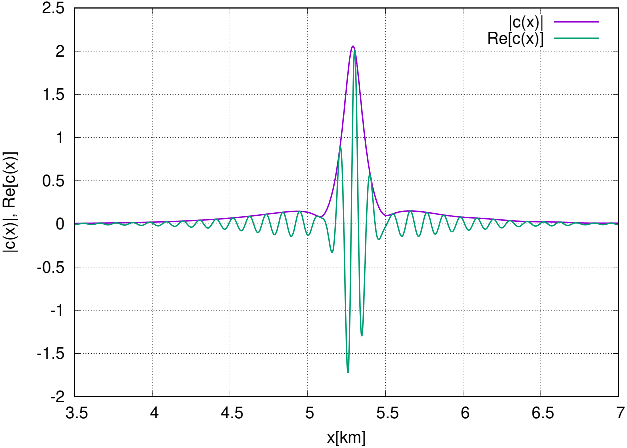
Long-time interaction of five breathers

To increase the number of solitons causes difficulties in calculations. Since one soliton is always taking the particle number from all the others, this will inevitably lead to the wave breaking and the calculation end.



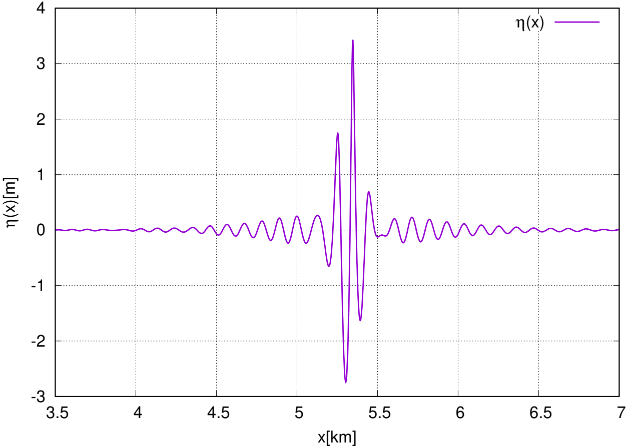
The evolution of five interacting breathers. The ordinate shows the particles number density $n(x)$. The top panel (a) presents the initial state ($t = 0$). The lower panel (d) presents the state of pre-breaking wave.

Bound soliton (bi-breather) solution of the supercompact Zakharov equation



The modulus of $c(x)$ and the real part of $c(x)$ for bound soliton (bi-breather) solution of the supercompact Zakharov equation

Bound soliton (bi-breather) solution of the exact nonlinear equation



Surface profile $\eta(x)$ for the bound soliton (bi-breather) solution of the exact nonlinear equations.

Conclusion

- Long-time dynamics of soliton gas in the SCZE for unidirectional deep water waves was studied. It was shown that after multiple collisions of breathers only one soliton remains regardless of the initial phase. Thus, solitons relative phases do not affect the end result, but play a role in the collision dynamics.
- Despite one outcome, two different scenarios of soliton interaction dynamics were observed. In the first scenario the "strong" soliton initially having a larger number of particles increases after each interaction, while the "weak" soliton decreases. In the second scenario, when the breathers velocities become very close due to collisions, the formation of a bound soliton or "soliton molecule" was observed. The bound structure is similar to the well-known exact bound bi-soliton solution of the NLSE. After a certain number of time periods this bound pair of solitons turns into one large soliton.
- A new solution of the SCZE and the exact equations was found and it is similar to bi-soliton solution of the NLSE.

The work was supported by the Russian Science Foundation (Grant #18-71-00079)