Experimental Study of Dispersion and Modulational Instability of Surface Gravity Waves on Constant Vorticity Currents

James N. Steer[1,5]†, Alistair G. L. Borthwick[1], Dimitris Stagonas[2], Eugeny Buldakov[3] and Ton S. van den Bremer[4]

[1]School of Engineering, University of Edinburgh, Edinburgh EH9 3FB, UK

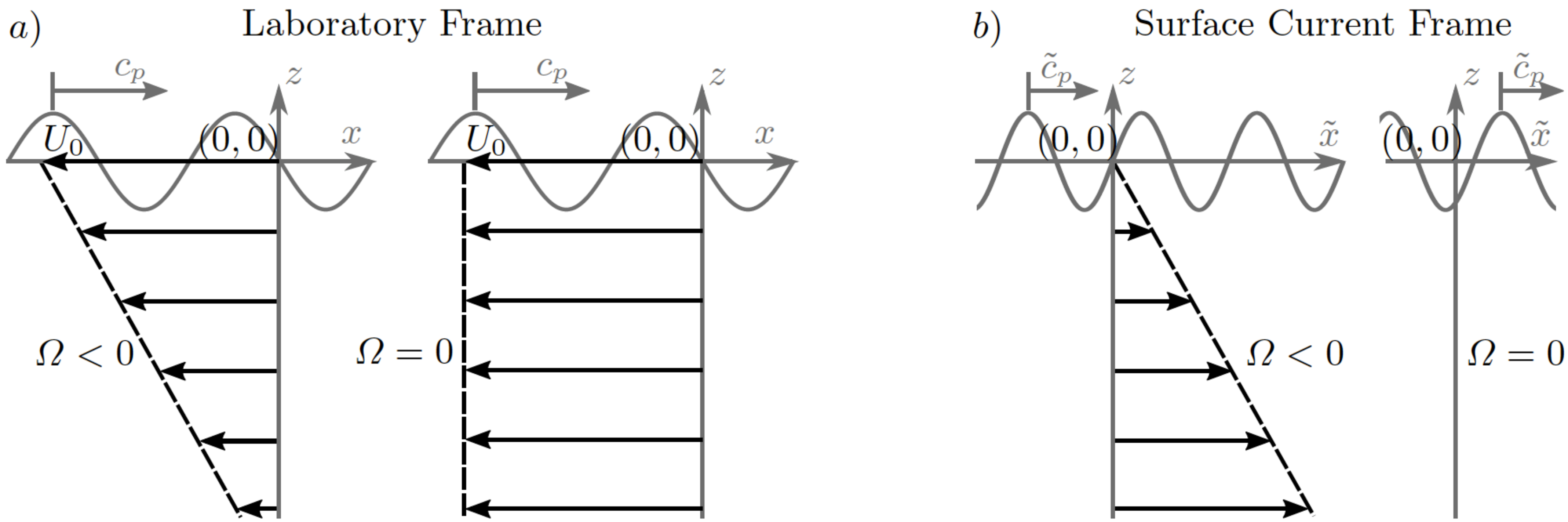
[2]School of Water, Energy and Environment, Cranfield University, Cranfield MK43 0AL, UK

[3]Department of Civil, Environmental and Geomatic Engineering, University College London, Chadwick Building, London, WC1 6BT

[4]Department of Engineering Science, University of Oxford, Oxford OX1 3PJ, UK

[5]Wind and Marine Energy Systems Center for Doctoral Training

Experiments for negatively sheared current



Linear background current: $U = U_0 + \Omega z.$

Tilde denotes surface current reference frame: $\omega = \tilde{\omega} + U_0 k,$

Waves have potential: $\boldsymbol{u} = U(z)\hat{\boldsymbol{i}} + \boldsymbol{\nabla}\phi,$

Governing equations and boundary conditions

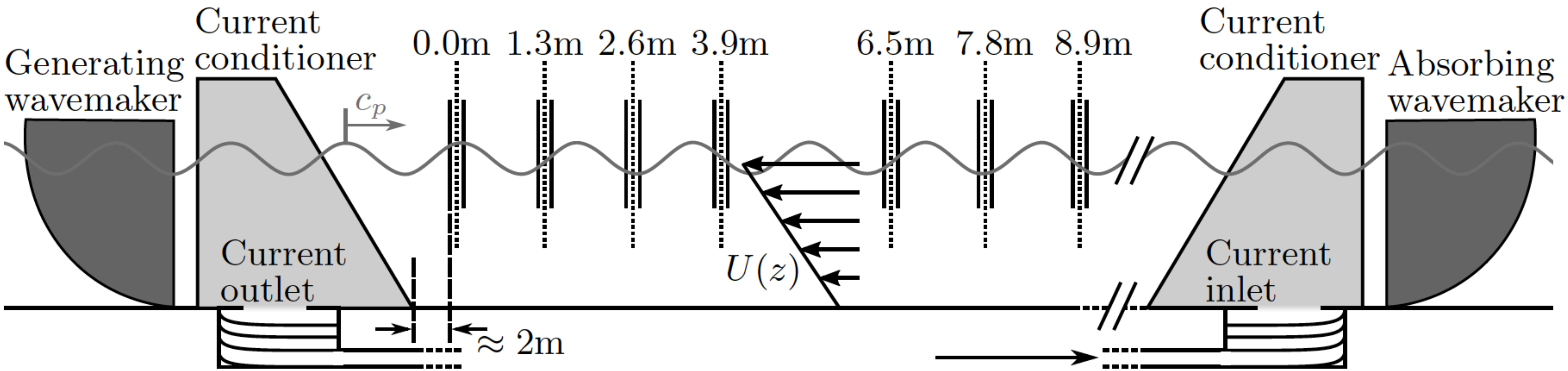
Laplace: $\nabla^2 \phi = 0 \qquad -d < z < \eta(x,t)$

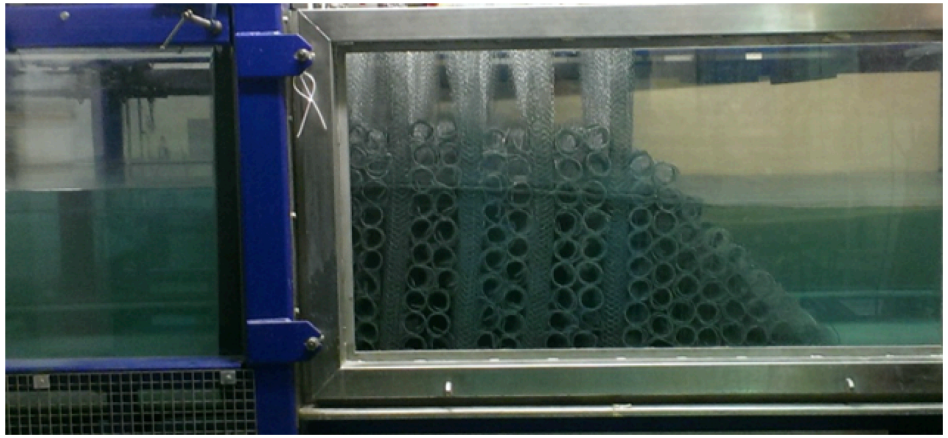
Kinematic free surface boundary condition: $\eta_t + (\Phi_x + \Omega\eta)\eta_x - \Phi_z = 0 \qquad z = \eta(x,t),$

Dynamic free surface boundary condition: $\Phi_t + \frac{1}{2}\Phi_x^2 + \frac{1}{2}\Phi_z^2 + \Omega\eta\Phi_x + g\eta - \Omega\Psi = 0 \qquad z = \eta(x,t),$

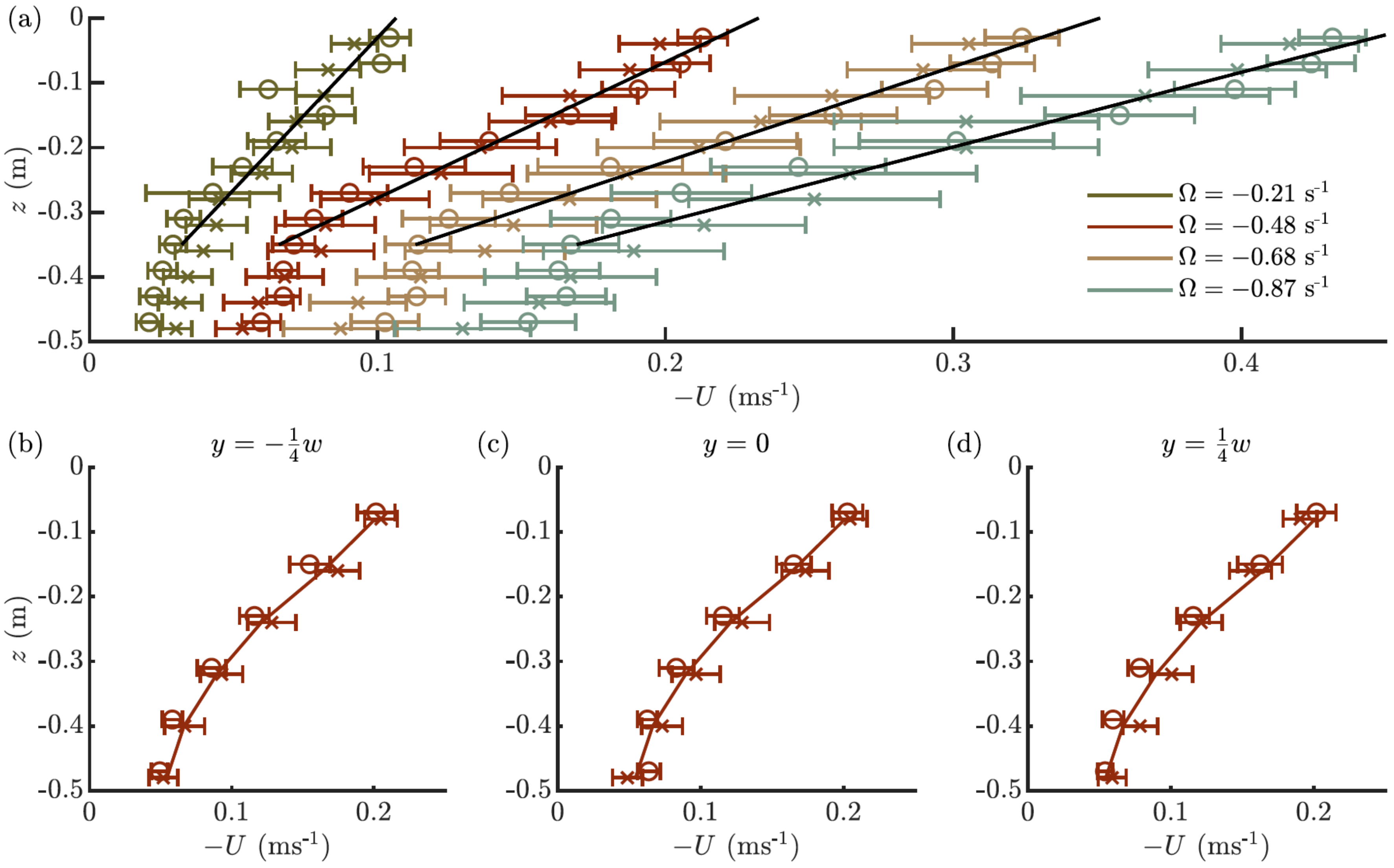
Free surface values: $\Psi \equiv \psi(z = \eta(x,t)) \qquad \Phi \equiv \phi(z = \eta(x,t))$

Laboratory experiments (UCL)



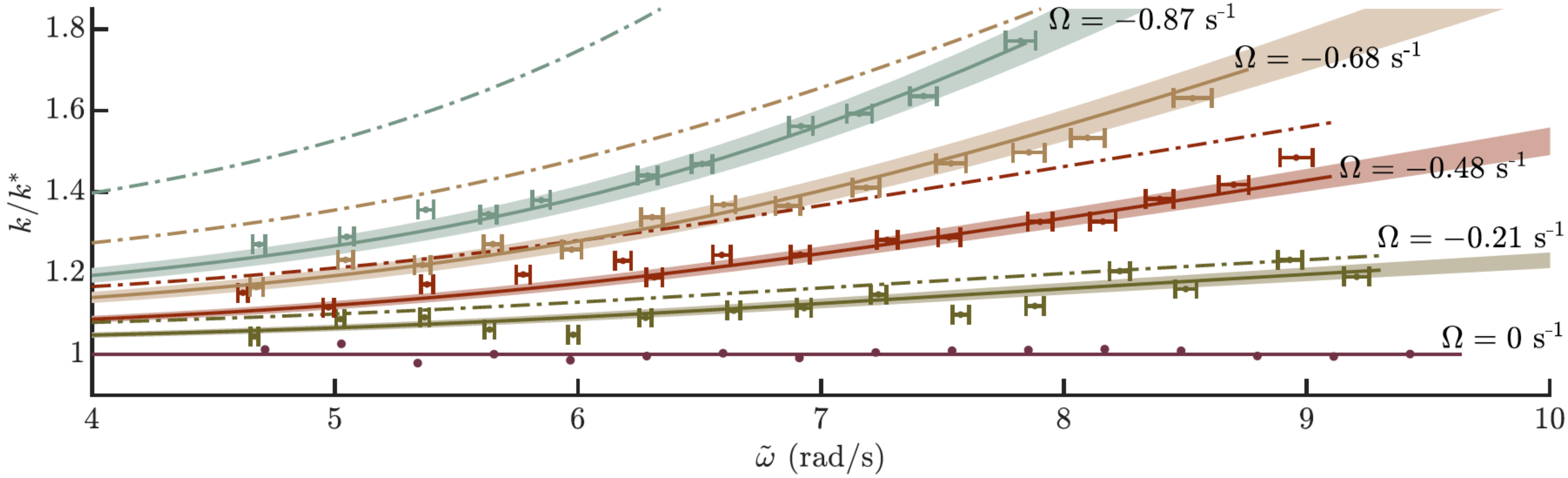


Velocity profiles



Ω (s^{-1})	U_0 (m s^{-1})
0	0
-0.21 ± 0.01	-0.11 ± 0.01
-0.48 ± 0.01	-0.22 ± 0.01
-0.68 ± 0.02	-0.33 ± 0.01
-0.87 ± 0.04	-0.44 ± 0.01

Linear dispersion relationship: $\tilde{\omega}_0^2 + (\tilde{\omega}_0 \Omega - g k_0) \tanh k_0 d = 0.$



Vor-NLSE

Scaled space and time: $\xi = \epsilon(\tilde{x} - \tilde{c}_g t)$ $\tau = \epsilon^2 t$,

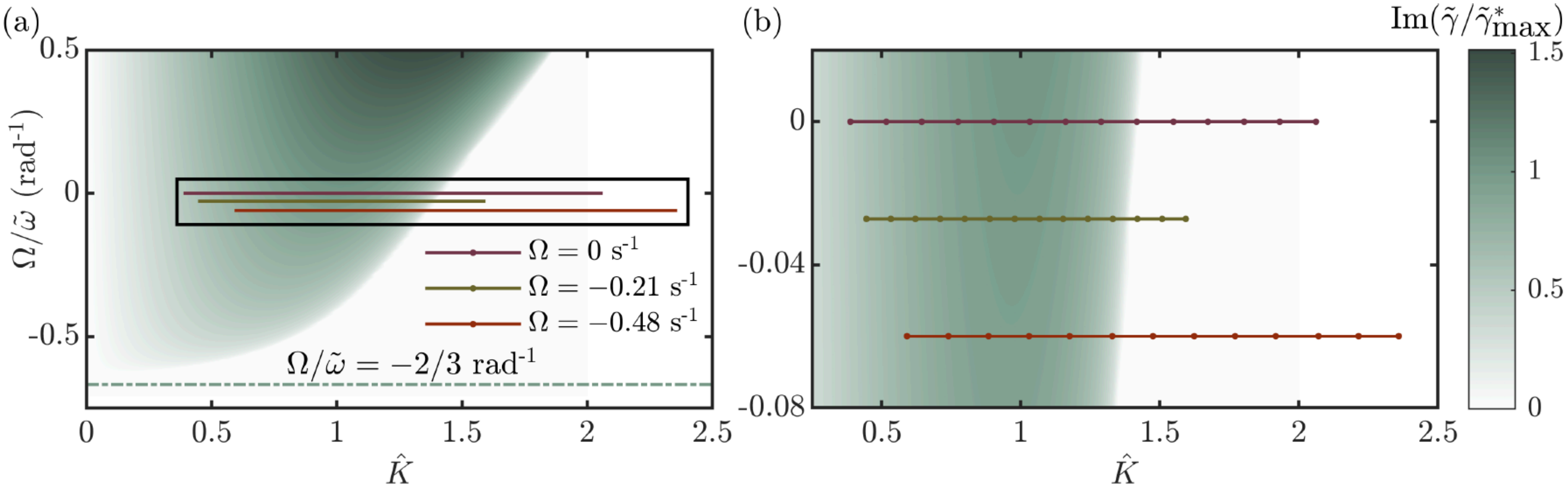
NLSE: $iA_\tau + LA_{\xi\xi} - M|A|^2 A = 0.$

Coefficients: $L = -\dfrac{\tilde{\omega}_0 (1+\bar{\Omega})^2}{k_0^2 (2+\bar{\Omega})^3}$ and $M = \dfrac{\tilde{\omega}_0 k_0^2}{8(1+\bar{\Omega})} \left(4 + 10\bar{\Omega} + 8\bar{\Omega}^2 + 3\bar{\Omega}^3\right),$

$\bar{\Omega} = \Omega / \tilde{\omega}_0$

From envelope to free surface: $\eta^{(1)} = \mathrm{Re}\left[\epsilon A(\xi, \tau) e^{i(k_0 \tilde{x} - \tilde{\omega}_0 t)}\right]$

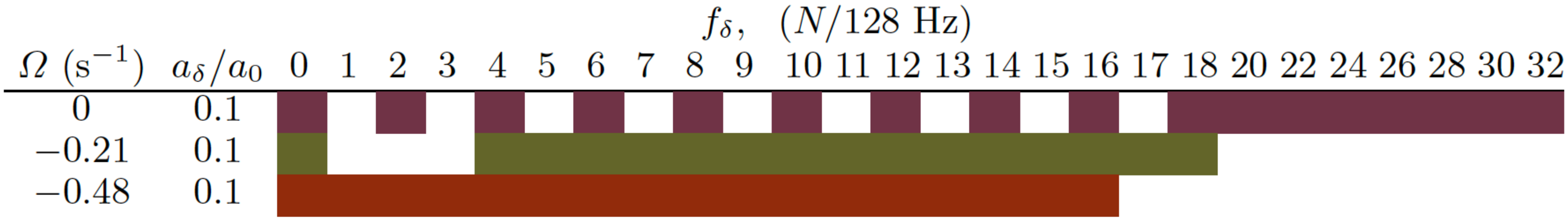
Linear stability analysis



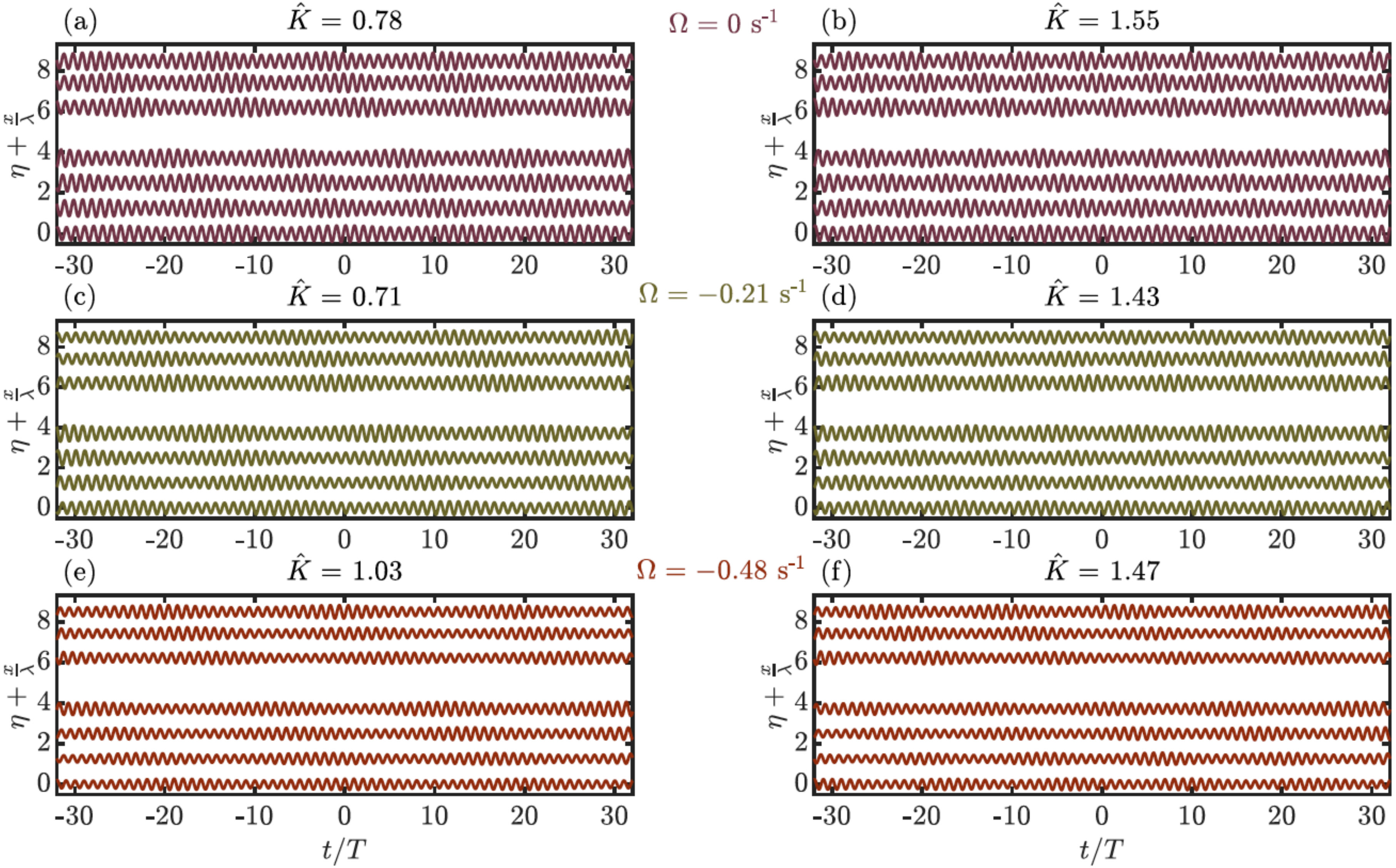
$$A = [a_0 + \delta(\tau, \xi)]e^{-iMa_0^2\tau} \qquad \tilde{\gamma} = \pm\sqrt{K^2L(K^2L + 2Ma_0^2)}.$$

Matrix of experiments

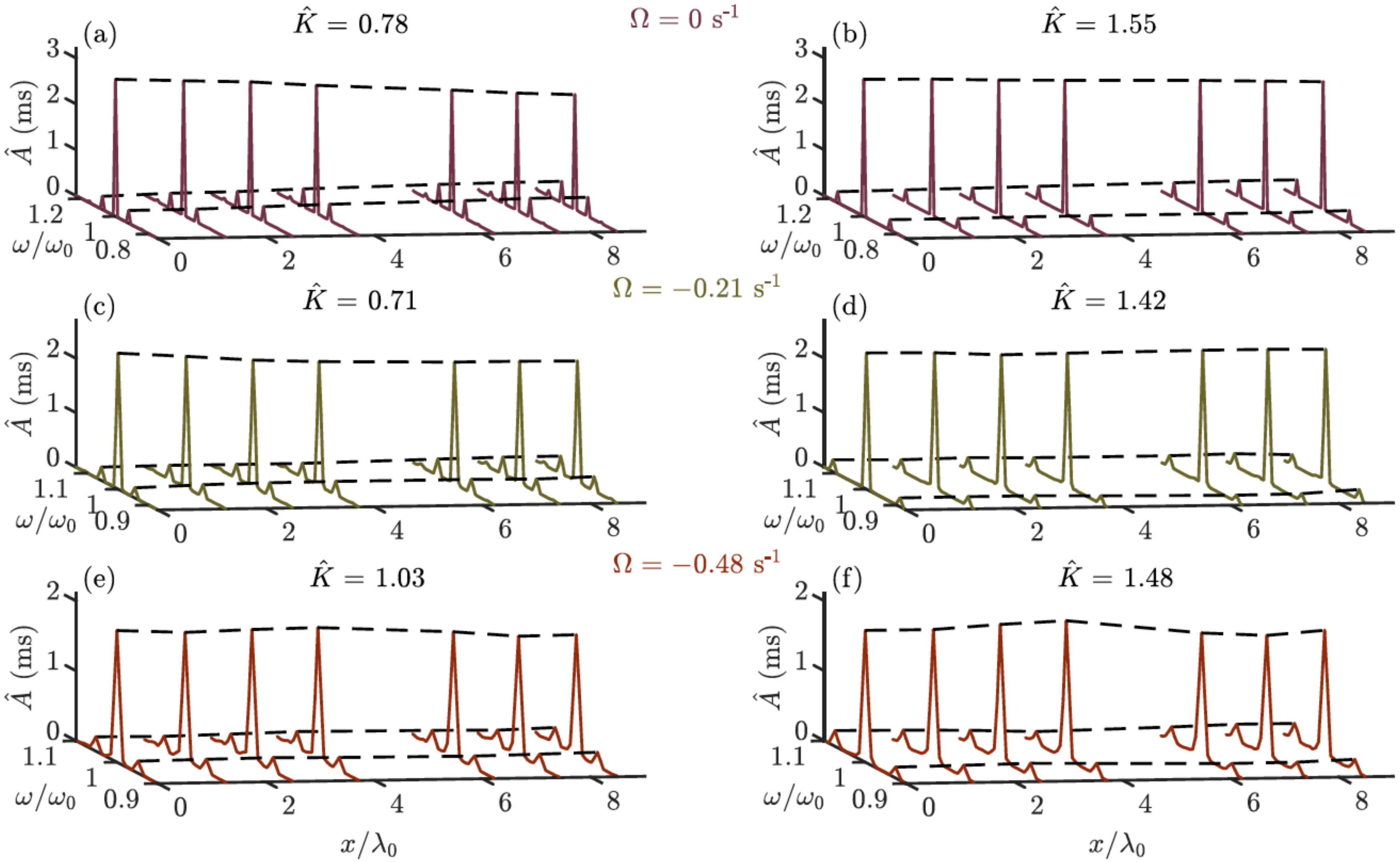
Ω (s^{-1})	ω ($\mathrm{rad\,s}^{-1}$)	ka_0
0	7.62	0.15
−0.21	7.17	0.12
−0.48	6.63	0.10



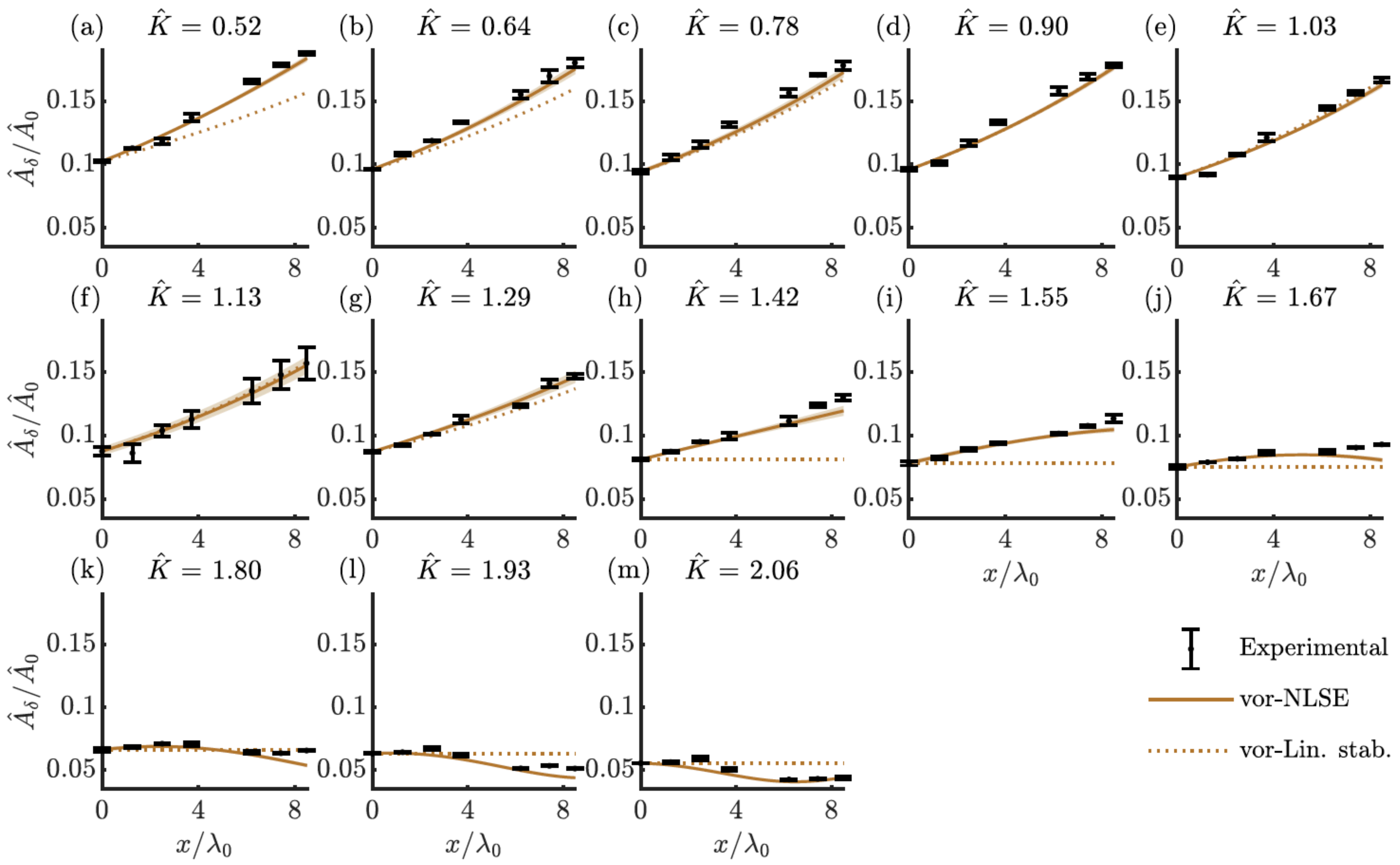
Example time series



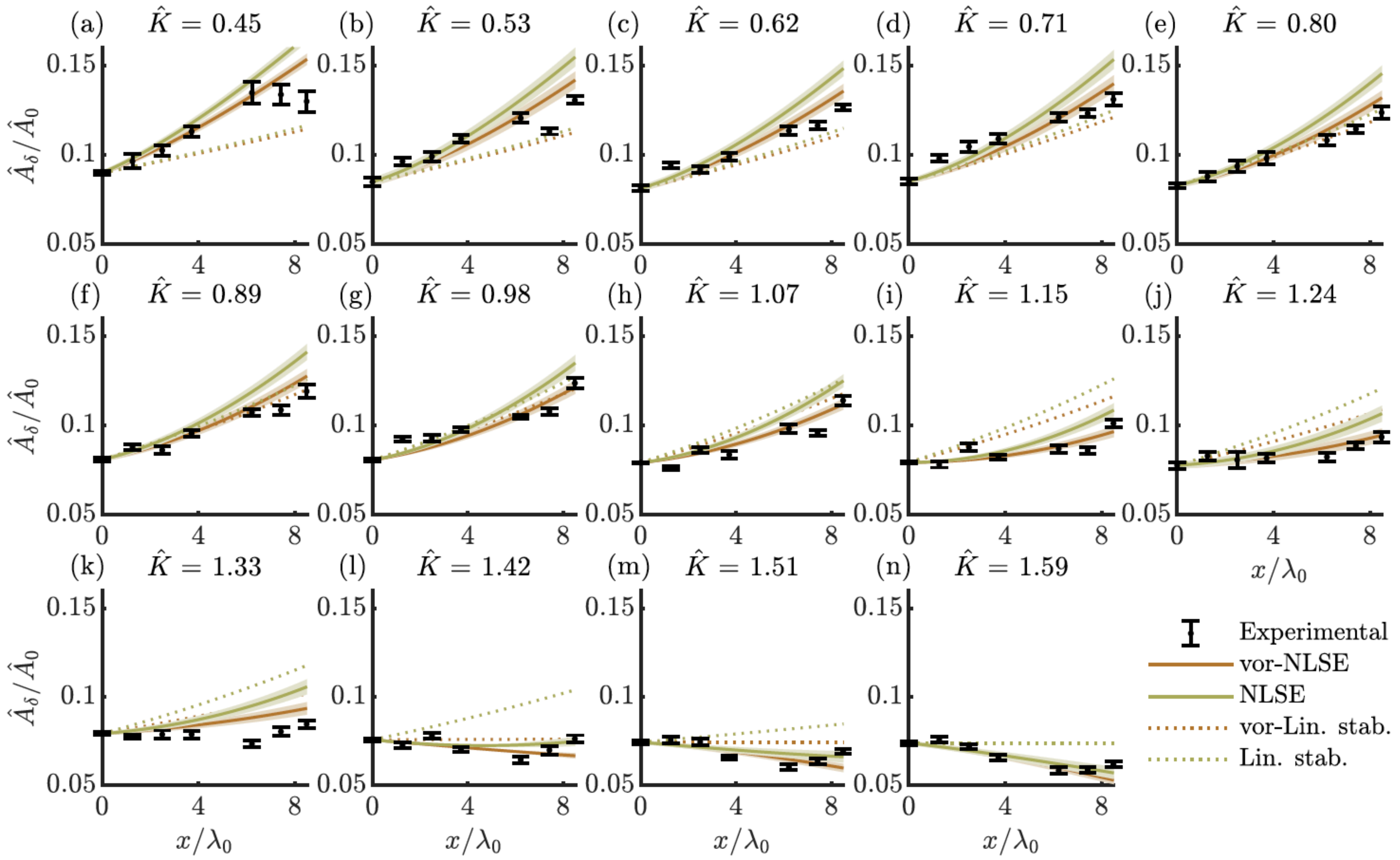
Example time spectra



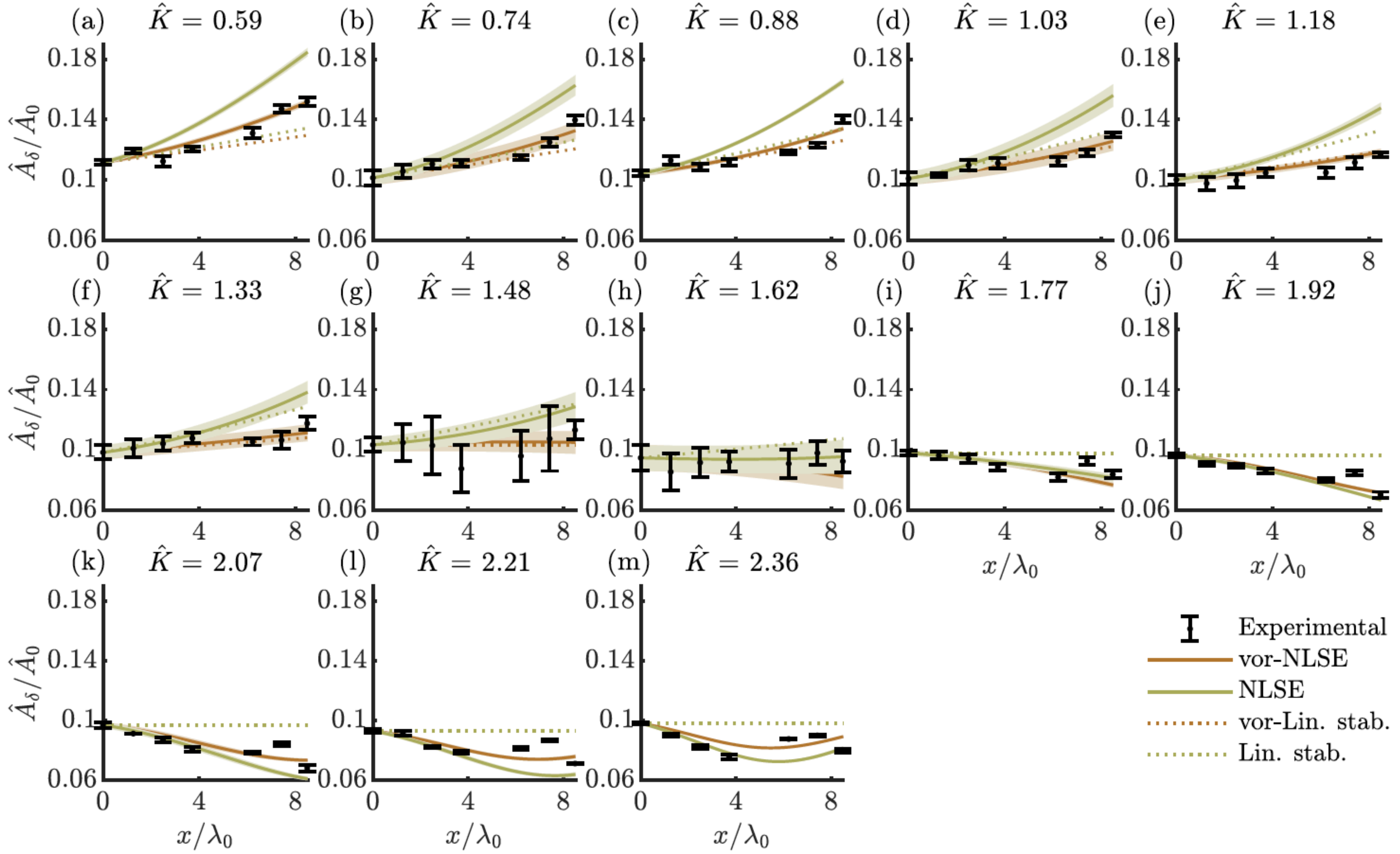
Combined upper and lower sideband: $\boldsymbol{\Omega = 0}$



Combined upper and lower sideband: $\Omega = -0.21$ 1/s



Combined upper and lower sideband: $\Omega = -0.48$ 1/s



Maximum amplification

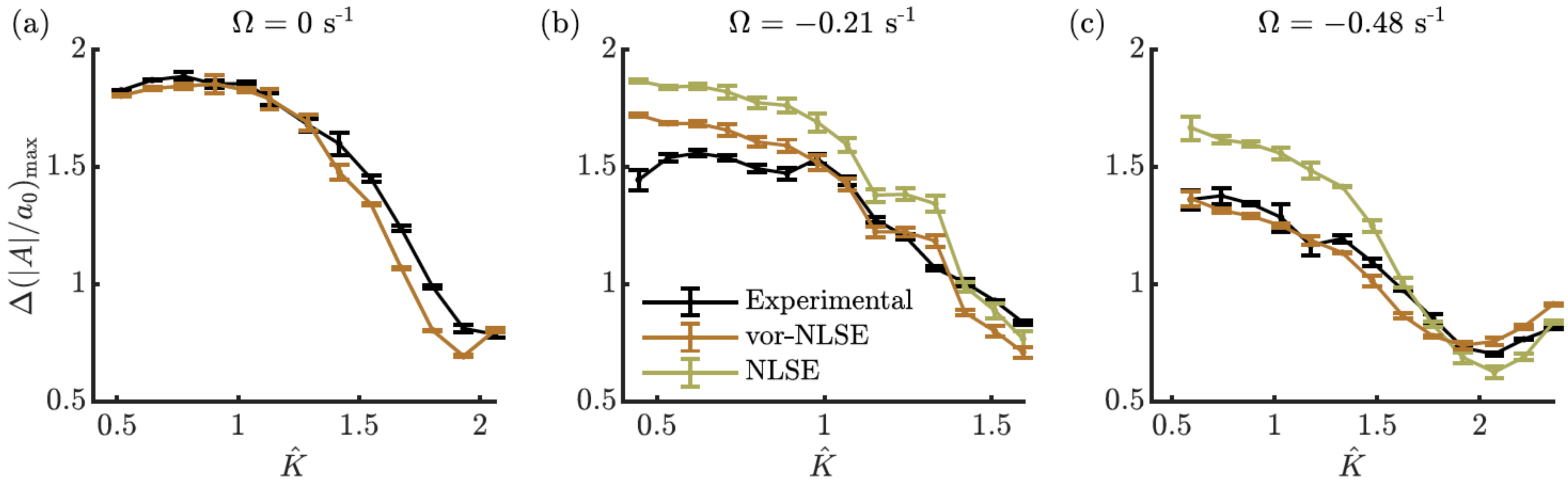


FIGURE 11. Maximum amplification factors, denoting the ratio between the maximum envelope amplitudes at the first and final gauges, as a function of the normalised sideband wavenumber parameter $\hat{K} = K/\left(a_0\sqrt{-M^*/L^*}\right)$ and for the three shear rates.

Conclusions

- Can robustly observed shear-modified linear dispersion relationship (for negative shear / opposing currents).
- Negative shear stabilizes the modulational instability: vor-NLSE better than NLSE.

Steer, J.N, A.G.L. Borthwick, D. Stagonas, E. Buldakov and T.S. van den Bremer (2020) Experimental study of dispersion and modulational instability of surface gravity waves on constant vorticity currents. Journal of Fluid Mechanics, **884**, A40.