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Session G1.1 – Recent Developments in Geodetic Theory

Laplacian Structure Mirroring Surface Topography in Determining the Gravity Potential by Successive Approximations

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1. Laplacian and Transformation of Coordinates:

1.1. Rectangular to Curvilinear Coordinates

Consider Euclidean space E^3 and a one-to-one mapping F of the domain $\Omega \subset E^3$ given by the equations

$$y_i = y_i(x_1, x_2, x_3), \quad i = 1, 2, 3 \quad (1.1)$$

x_i are rectangular Cartesian coordinates, the functions y_i are assumed twice continuously differentiable and

we suppose that the Jacobian $\left| \frac{\partial y_i}{\partial x_j} \right| \neq 0$ in the domain Ω .

The transformation (1.1) is then reversible, i.e.

$$x_i = x_i(y_1, y_2, y_3), \quad i = 1, 2, 3 \quad \text{for } y \in F(\Omega)$$

and y_i are the curvilinear coordinates of the point $x \equiv (x_1, x_2, x_3)$

Obviously, $\left| \frac{\partial x_i}{\partial y_j} \right| = \left| \frac{\partial y_i}{\partial x_j} \right|^{-1}$ and $x_i(\mathbf{y}) \equiv x_i(y_1, y_2, y_3)$ have continuous 2nd derivatives in the domain $F(\Omega)$.

Following the conventions of the tensor calculus, we can write the components

$$g_{ij}(\mathbf{y}) = \frac{\partial x_k}{\partial y_i} \frac{\partial x_k}{\partial y_j} = \frac{\partial \mathbf{x}}{\partial y_i} \frac{\partial \mathbf{x}}{\partial y_j}$$

of the *fundamental metric tensor* in the coordinates y_i .

Supposing that the determinant

$$g = |g_{ij}| \neq 0 \text{ in } F(\Omega), \text{ we further have } g^{ij} = \frac{G^{ij}}{g}$$

for the components of the *associated (conjugate) metric tensor* where G^{ij} is the cofactor of g_{ij} in the determinant g . Recall that g^{ij} is a contravariant tensor.

In addition we put

$$[ij, k] = \frac{1}{2} \left(\frac{\partial g_{jk}}{\partial y_i} + \frac{\partial g_{ik}}{\partial y_j} - \frac{\partial g_{ij}}{\partial y_k} \right) \quad \text{and} \quad \Gamma_{ij}^k = g^{kl} [ij, l]$$

which are the **Christoffel symbols** of the **1st and the 2nd kind**, respectively.

We are now ready to write the **Laplacian** of a scalar function T in terms of the curvilinear coordinates y_i . Following Sokolnikoff (1971), we have

$$\Delta_y T = g^{ij} \left[\frac{\partial^2 T}{\partial y_i \partial y_j} - \Gamma_{ij}^k \frac{\partial T}{\partial y_k} \right] \quad (1.2)$$

(Obviously, ΔT is the 2nd covariant derivative of T , or the 1st covariant derivative of the gradient of T , contracted with the associated metric tensor g^{ij} to form an invariant.)

1.2. Curvilinear to New Curvilinear Coordinates

We consider a *one-to-one mapping* of the domain $F(\Omega)$ given by

$$y_i = z_i + \mu_i(z_1, z_2, z_3), \quad i = 1, 2, 3$$

z_i are *new curvilinear coordinates* of the original point $x(y)$.

We will suppose that the functions μ_i are twice continuously differentiable and that the *Jacobian*

$$D = \left| \frac{\partial y_i}{\partial z_j} \right| = \left| \delta_{ij} + \frac{\partial \mu_i}{\partial z_j} \right| \neq 0 \quad \text{for any value of the variables}$$

δ_{ij} denotes the ordinary *two-index Kronecker delta*.

Using the *contravariant transformation law*, we obtain

$$\bar{g}^{ij} = \frac{\partial z_i}{\partial y_m} \frac{\partial z_j}{\partial y_n} g^{nm}$$

for the components of the *associated metric tensor* expressed in terms of the *curvilinear coordinates* z_i .

In contrast the transformation of the **Christoffel symbols** Γ_{ij}^k **is more complicated, in particular it does not have the tensor character. Following Sokolnikoff (1971), we can write**

$$\bar{\Gamma}_{ij}^k = \frac{\partial z_k}{\partial y_l} \frac{\partial y_p}{\partial z_i} \frac{\partial y_q}{\partial z_j} \Gamma_{pq}^l + \frac{\partial^2 y_p}{\partial z_i \partial z_j} \frac{\partial z_k}{\partial y_p}$$

for the Christoffel symbols of the 2nd kind in the coordinates z_i .

Finally, for the Laplacian of T expressed in terms of the new curvilinear coordinates z_i we have

$$\Delta_z T = \bar{g}^{ij} \left[\frac{\partial^2 T}{\partial z_i \partial z_j} - \bar{\Gamma}_{ij}^k \frac{\partial T}{\partial z_k} \right]$$

which is an analogue of (1.2), i.e. of

$$\Delta_y T = g^{ij} \left[\frac{\partial^2 T}{\partial y_i \partial y_j} - \Gamma_{ij}^k \frac{\partial T}{\partial y_k} \right]$$

We are now ready to use the above results and perform all the summations . We arrive at

$$\Delta_z T = g^{mn} \left[\frac{\partial z_i}{\partial y_m} \frac{\partial z_j}{\partial y_n} \frac{\partial^2 T}{\partial z_i \partial z_j} - \frac{\partial z_k}{\partial y_l} \Gamma_{mn}^l \frac{\partial T}{\partial z_k} - \frac{\partial z_i}{\partial y_m} \frac{\partial z_j}{\partial y_n} \frac{\partial^2 y_p}{\partial z_i \partial z_j} \frac{\partial z_k}{\partial y_p} \frac{\partial T}{\partial z_k} \right]$$

where an elementary calculation yields

$$\frac{\partial^2 y_p}{\partial y_m \partial y_n} = \frac{\partial^2 y_p}{\partial z_i \partial z_j} \frac{\partial z_i}{\partial y_m} \frac{\partial z_j}{\partial y_n} + \frac{\partial y_p}{\partial z_i} \frac{\partial^2 z_i}{\partial y_m \partial y_n} = 0$$

In consequence

$$\frac{\partial z_i}{\partial y_m} \frac{\partial z_j}{\partial y_n} \frac{\partial^2 y_p}{\partial z_i \partial z_j} \frac{\partial z_k}{\partial y_p} = - \frac{\partial^2 z_k}{\partial y_m \partial y_n}$$

Using this identity , we immediately see that $\Delta_z T$ above turns into

$$\Delta_z T = g^{mn}(\mathbf{y}) \left[\frac{\partial z_i}{\partial y_m} \frac{\partial z_j}{\partial y_n} \frac{\partial^2 T}{\partial z_i \partial z_j} + \left(\frac{\partial^2 z_k}{\partial y_m \partial y_n} - \Gamma_{mn}^l(\mathbf{y}) \frac{\partial z_k}{\partial y_l} \right) \frac{\partial T}{\partial z_k} \right]$$

Recalling in addition that $y_i = z_i + \mu_i(z_1, z_2, z_3)$, $i = 1, 2, 3$, we can also write

$$\Delta_z T = g^{mn}(y) \frac{\partial z_i}{\partial y_m} \frac{\partial z_j}{\partial y_n} \frac{\partial^2 T}{\partial z_i \partial z_j} + \frac{\partial T}{\partial z_k} \Delta_y z_k$$

and it remains to express all the derivatives of z_i with respect to y_m

From $y_i = z_i + \mu_i(z_1, z_2, z_3)$ we easily deduce that

$$\delta_{im} = \frac{\partial z_i}{\partial y_m} + \frac{\partial \mu_i}{\partial z_k} \frac{\partial z_k}{\partial y_m} \quad \text{or} \quad \left(\delta_{ik} + \frac{\partial \mu_i}{\partial z_k} \right) \frac{\partial z_k}{\partial y_m} = \delta_{im} \quad (1.3)$$

Here, according to our supposition the determinant of the system (1.3) is different from zero and the system can be solved for the derivatives

$$\frac{\partial z_k}{\partial y_m}, \quad k = 1, 2, 3. \quad \text{We obtain} \quad \frac{\partial z_k}{\partial y_m} = \left(\delta_{ik} + \frac{\partial \mu_k}{\partial z_i} \right)^{-1} \delta_{im}$$

Similarly, the derivative of (1.3), i.e. of

$$\left(\delta_{ik} + \frac{\partial \mu_i}{\partial z_k} \right) \frac{\partial z_k}{\partial y_m} = \delta_{im}$$

with respect to y_n yields

$$\frac{\partial^2 \mu_i}{\partial z_k \partial z_l} \frac{\partial z_k}{\partial y_m} \frac{\partial z_l}{\partial y_n} + \left(\delta_{ik} + \frac{\partial \mu_i}{\partial z_k} \right) \frac{\partial^2 z_k}{\partial y_m \partial y_n} = 0$$

In consequence

$$\begin{aligned} \frac{\partial^2 z_k}{\partial y_m \partial y_n} &= - \left(\delta_{ik} + \frac{\partial \mu_k}{\partial z_i} \right)^{-1} \frac{\partial^2 \mu_i}{\partial z_p \partial z_q} \frac{\partial z_p}{\partial y_m} \frac{\partial z_q}{\partial y_n} = \\ &= - \left(\delta_{ik} + \frac{\partial \mu_k}{\partial z_i} \right)^{-1} \left(\delta_{rp} + \frac{\partial \mu_p}{\partial z_r} \right)^{-1} \left(\delta_{sq} + \frac{\partial \mu_q}{\partial z_s} \right)^{-1} \delta_{rm} \delta_{sn} \frac{\partial^2 \mu_i}{\partial z_p \partial z_q} \end{aligned} \quad (1.4)$$

At this stage we use the famous **Neumann's series** . First we put

$$M = \left(\frac{\partial \mu_k}{\partial z_i} \right)$$

and will assume that the functions $\mu_i(\mathbf{z}) \equiv \mu_i(z_1, z_2, z_1)$ have the nature of **small perturbations** such that

$$\|M\| < 1$$

where $\|M\|$ is a suitable norm of the matrix M , e.g.

$$\|M\| = \max_i \sum_{j=1}^3 |M_{ij}|$$

Now we express the inverse of the matrix $I + M$, where $I = (\delta_{ij})$ is the unit matrix, by means of **Neumann's series** , i.e.

$$(I + M)^{-1} = I + \sum_{n=1}^{\infty} (-1)^n M^n$$

Finally, confining ourselves to linear and quadratic terms ,

$$(I + M)^{-1} = I - M + M^2 - \dots$$

and (1.3) immediately yields

$$\frac{\partial z_k}{\partial y_m} \doteq \delta_{km} - \frac{\partial \mu_k}{\partial z_m} + \frac{\partial \mu_k}{\partial z_l} \frac{\partial \mu_l}{\partial z_m} - \dots$$

With the same accuracy we obtain

$$\begin{aligned} \frac{\partial z_i}{\partial y_m} \frac{\partial z_j}{\partial y_n} &\doteq \delta_{im} \delta_{jn} - \delta_{im} \frac{\partial \mu_j}{\partial z_n} - \delta_{jn} \frac{\partial \mu_i}{\partial z_m} + \\ &+ \delta_{im} \frac{\partial \mu_j}{\partial z_l} \frac{\partial \mu_l}{\partial z_n} + \delta_{jn} \frac{\partial \mu_i}{\partial z_l} \frac{\partial \mu_l}{\partial z_m} + \frac{\partial \mu_i}{\partial z_m} \frac{\partial \mu_j}{\partial z_n} - \dots \end{aligned}$$

and similarly (1.4) gives

$$\frac{\partial^2 z_k}{\partial y_m \partial y_n} = - \frac{\partial^2 \mu_k}{\partial z_m \partial z_n} + \frac{\partial \mu_k}{\partial z_i} \frac{\partial^2 \mu_i}{\partial z_m \partial z_n} + \frac{\partial \mu_i}{\partial z_m} \frac{\partial^2 \mu_k}{\partial z_i \partial z_n} + \frac{\partial \mu_i}{\partial z_n} \frac{\partial^2 \mu_k}{\partial z_m \partial z_i}$$

where, in addition, we have neglected cubic and higher order products of derivatives of μ_k .

The results above enable us to write

$$\Delta_z T = D^{(0)}T + D^{(1)}T + D^{(2)}T + \dots$$

where

$$D^{(0)}T = g^{mn}(\mathbf{y}) \left[\frac{\partial^2 T}{\partial z_m \partial z_n} - \Gamma_{mn}^k(\mathbf{y}) \frac{\partial T}{\partial z_k} \right], \quad \mathbf{y} = \mathbf{z} + \boldsymbol{\mu}(\mathbf{z})$$

$$D^{(1)}T = -g^{mn}(\mathbf{y}) \left[\frac{\partial \mu_i}{\partial z_m} \frac{\partial^2 T}{\partial z_i \partial z_n} + \frac{\partial \mu_i}{\partial z_n} \frac{\partial^2 T}{\partial z_m \partial z_i} + \left(\frac{\partial^2 \mu_k}{\partial z_m \partial z_n} - \Gamma_{mn}^l(\mathbf{y}) \frac{\partial \mu_k}{\partial z_l} \right) \frac{\partial T}{\partial z_k} \right]$$

$$D^{(2)}T = g^{mn}(\mathbf{y}) \left[\frac{\partial \mu_i}{\partial z_m} \frac{\partial \mu_j}{\partial z_n} \frac{\partial^2 T}{\partial z_i \partial z_j} + \frac{\partial \mu_i}{\partial z_l} \frac{\partial \mu_l}{\partial z_m} \frac{\partial^2 T}{\partial z_i \partial z_n} + \frac{\partial \mu_i}{\partial z_l} \frac{\partial \mu_l}{\partial z_n} \frac{\partial^2 T}{\partial z_m \partial z_i} + \right. \\ \left. + \left(\frac{\partial \mu_k}{\partial z_i} \frac{\partial^2 \mu_i}{\partial z_n \partial z_m} + \frac{\partial \mu_i}{\partial z_m} \frac{\partial^2 \mu_k}{\partial z_i \partial z_n} + \frac{\partial \mu_i}{\partial z_n} \frac{\partial^2 \mu_k}{\partial z_m \partial z_i} - \Gamma_{mn}^l(\mathbf{y}) \frac{\partial \mu_i}{\partial z_l} \frac{\partial \mu_k}{\partial z_i} \right) \frac{\partial T}{\partial z_k} \right]$$

2. Example - Ellipsoidal Coordinates and an Attenuation Function

Consider an **ellipsoid of revolution** with semiaxes a and b , $a \geq b$ and the linear eccentricity $E = \sqrt{a^2 - b^2}$

In this case we can introduce **ellipsoidal coordinates** u, β, λ related to **Cartesian coordinates** x_1, x_2, x_3 by the equations

$$x_1 = \sqrt{u^2 + E^2} \cos\beta \cos\lambda, \quad x_2 = \sqrt{u^2 + E^2} \cos\beta \sin\lambda, \quad x_3 = u \sin\beta.$$

We will suppose that a **function** $h(\beta, \lambda)$ describes the boundary $\partial\Omega$ of our domain Ω with respect to the ellipsoid $u = b$, i.e. $\partial\Omega$ is represented by

$$x_1 = \sqrt{[b + h(\beta, \lambda)]^2 + E^2} \cos\beta \cos\lambda$$

$$x_2 = \sqrt{[b + h(\beta, \lambda)]^2 + E^2} \cos\beta \sin\lambda$$

$$x_3 = [b + h(\beta, \lambda)] \sin\beta$$

We now return to our concept of **modification of curvilinear coordinates**. Generally, we expressed it by $y_i = z_i + \mu_i(z_1, z_2, z_3)$. Nevertheless, in the following example we will consider a more simple (anyway meaningful) case.

Our starting point will be the mapping as above, i.e.

$$x_1 = \sqrt{u^2 + E^2} \cos\beta \cos\lambda$$

$$x_2 = \sqrt{u^2 + E^2} \cos\beta \sin\lambda$$

$$x_3 = u \sin\beta$$

but with

$$u = z + \omega(z)h(\beta, \lambda)$$

where z is a new coordinate and $\omega(z)$ is a twice continuously differentiable attenuation function defined for $z \in [b, \infty)$, such that

$$\omega(z)h(\beta, \lambda) > -b$$

$$\omega(b) = 1, \quad \frac{d\omega}{dz}(b) = 0$$

and

$$\omega(z) = 0 \quad \text{for } z \in [z_{ext}, \infty), \quad \text{where } b + h_{\max} < z_{ext}$$

Stress that the assumption concerning the continuity of ω and its 1st and the 2nd derivatives implies

$$\lim \omega(z) = 0, \quad \lim \frac{d\omega(z)}{dz} = 0, \quad \lim \frac{d^2\omega(z)}{dz^2} = 0 \quad \text{for } z \rightarrow z_{ext}^-$$

Note: z, β, λ form a system of new curvilinear coordinates and for

$$\frac{du}{dz} = 1 + \frac{d\omega}{dz}h > 0$$

the transformation is a **one-to-one mapping between Ω and the outer space Ω_{ell} of our oblate ellipsoid of revolution.**

The construction of the *attenuation function* $\omega(z)$ in the interval $[b, z_{ext})$, i.e. for $b \leq z < z_{ext}$, deserves some attention.

Here we give an example. We put

$$\omega(z) = e^{\left[2 - \frac{2(\Delta z)^2}{(\Delta z)^2 - (z-b)^2}\right]}, \quad \text{where } e \approx 2,71828$$

is the base of natural logarithm and $\Delta z = z_{ext} - b$. By direct computation we can verify that $\omega(b) = 1$ and $\lim_{z \rightarrow z_{ext}^-} \omega(z) = 0$.

For the 1st derivative of $\omega(z)$ we obtain

$$\frac{d\omega(z)}{dz} = - \frac{4(\Delta z)^2 (z-b)}{\left[(\Delta z)^2 - (z-b)^2\right]^2} \omega(z)$$

$$\frac{d\omega(b)}{dz} = 0 \quad \text{and} \quad \lim_{z \rightarrow z_{ext}^-} \frac{d\omega(z)}{dz} = 0$$

Similarly for the 2nd derivative of $\omega(z)$ we have

$$\frac{d^2\omega(z)}{dz^2} = - \frac{4(\Delta z)^2(z-b)}{\left[(\Delta z)^2 - (z-b)^2\right]^2} \frac{d\omega(z)}{dz} -$$

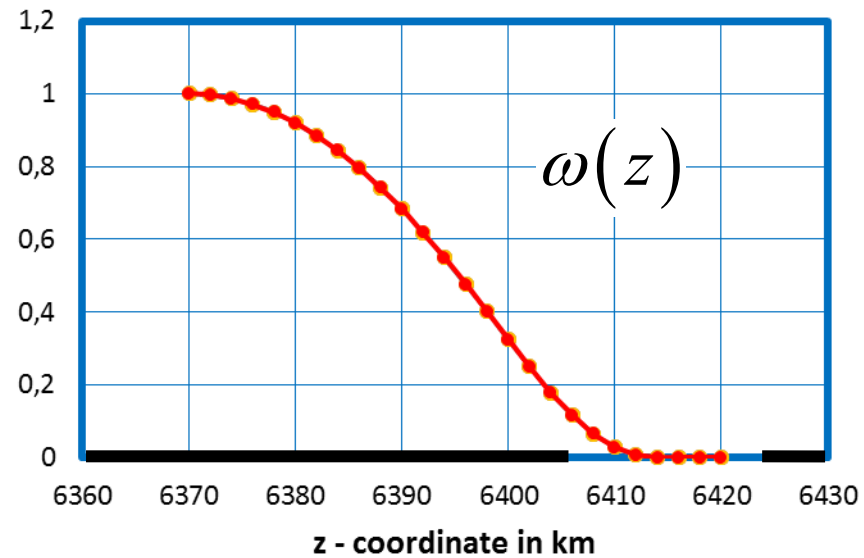
$$- \left\{ \frac{4(\Delta z)^2}{\left[(\Delta z)^2 - (z-b)^2\right]^2} + \frac{16(\Delta z)^2(z-b)^2}{\left[(\Delta z)^2 - (z-b)^2\right]^3} \right\} \omega(z)$$

and

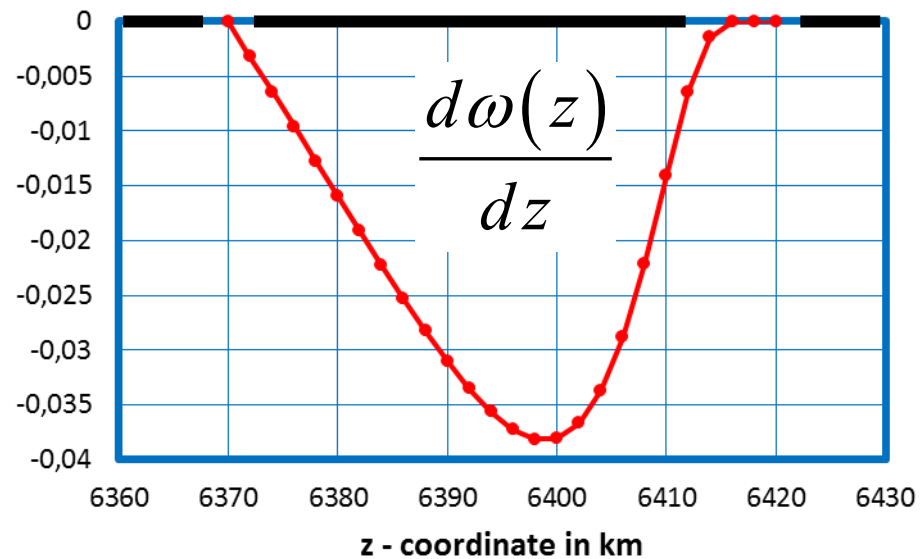
$$\lim_{z \rightarrow z_{\text{ext}}^-} \frac{d^2\omega(z)}{dz^2} = 0$$

The attenuation function and its derivatives are *demonstrated on the following slide.*

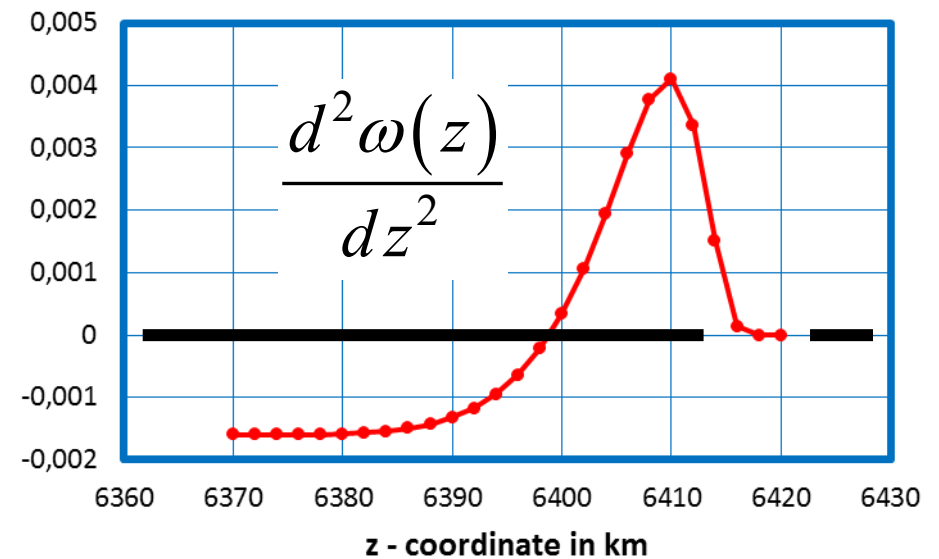
Attenuation Function



1st Derivative of the Attenuation Function



2nd Derivative of the Attenuation Function



3. Laplacian and Topography-Dependent Coefficients

Recall now that

$$x_1 = \sqrt{u^2 + E^2} \cos\beta \cos\lambda$$

$$x_2 = \sqrt{u^2 + E^2} \cos\beta \sin\lambda$$

$$x_3 = u \sin\beta$$

with

$$u = z + \omega(z)h(\beta, \lambda)$$

In the *sense of the notation used in Section 1.2* **this means** that

$$y_1 = u, \quad y_2 = \beta, \quad y_3 = \lambda$$

and the *new (modified) coordinates* are

$$z_1 = z, \quad z_2 = \beta, \quad z_3 = \lambda$$

Hence from $y_i = z_i + \mu_i(z_1, z_2, z_3)$, $i = 1, 2, 3$, we see that the small perturbations μ_i , $i = 1, 2, 3$ are

$$\mu_1 = \omega(z)h(\beta, \lambda), \quad \mu_2 = 0, \quad \mu_3 = 0$$

Going now back to the computations of the derivatives

$$\frac{\partial z_k}{\partial y_m}, \quad k = 1, 2, 3 \quad \text{from the system} \quad \left(\delta_{ik} + \frac{\partial \mu_i}{\partial z_k} \right) \frac{\partial z_k}{\partial y_m} = \delta_{im}$$

we see that the matrix

$$\left(\delta_{ik} + \frac{\partial \mu_i}{\partial z_k} \right) = \begin{vmatrix} 1 + \frac{d\omega}{dz} & \frac{\partial h}{\partial \beta} & \frac{\partial h}{\partial \lambda} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

can be easily inverted.

Thus the **successive approximation based on Neumann's series** even need not be used for the computation of the derivatives

$$\frac{\partial z_k}{\partial y_m} = \left(\delta_{ik} + \frac{\partial \mu_k}{\partial z_i} \right)^{-1} \delta_{im}$$

This opens the road to the structure Laplace's operator applied on T . **In terms of the curvilinear coordinates** z_i (i.e. in z, β, λ) we obtain

$$\Delta T = \frac{z^2 + E^2 \sin^2 \beta}{(z + \omega h)^2 + E^2 \sin^2 \beta} \left[\Delta_{ell} T - \delta(T, h) \right]$$

where

$$\Delta_{ell} T = \frac{1}{z^2 + E^2 \sin^2 \beta} \left[(z^2 + E^2) \frac{\partial^2 T}{\partial z^2} + 2z \frac{\partial T}{\partial z} + \frac{\partial^2 T}{\partial \beta^2} - \frac{\sin \beta}{\cos \beta} \frac{\partial T}{\partial \beta} + \frac{z^2 + E^2 \sin^2 \beta}{(z^2 + E^2) \cos^2 \beta} \frac{\partial^2 T}{\partial \lambda^2} \right]$$

$$\delta(T, h) = A_1 \frac{\partial T}{\partial z} + A_2 \frac{\partial^2 T}{\partial z^2} + A_3 \frac{1}{\sqrt{z^2 + E^2 \sin^2 \beta}} \frac{\partial^2 T}{\partial z \partial \beta} + A_4 \frac{\sqrt{\alpha}}{\sqrt{z^2 + E^2} \cos \beta} \frac{\partial^2 T}{\partial z \partial \lambda}$$

and A_i are *topography dependent coefficients* given by

$$\begin{aligned} A_1 = & \left(1 + \frac{d\omega}{dz}h\right)^{-1} \left[2\left(\frac{d\omega}{dz} - \frac{\omega}{z}\right) \frac{zh}{z^2 + E^2 \sin^2 \beta} + \omega \Delta_E h \right] - \\ & - 2\left(1 + \frac{d\omega}{dz}h\right)^{-2} \omega \frac{d\omega}{dz} |\mathbf{grad}_E h|^2 + \left(1 + \frac{d\omega}{dz}h\right)^{-3} \left[\frac{(z + \omega h)^2 + E^2}{z^2 + E^2 \sin^2 \beta} + \omega^2 |\mathbf{grad}_E h|^2 \right] \frac{d^2 \omega}{dz^2} h, \end{aligned}$$

$$\begin{aligned} A_2 = & \left(1 + \frac{d\omega}{dz}h\right)^{-2} \left\{ 2\left(\frac{d\omega}{dz} - \frac{\omega z}{z^2 + E^2}\right) h + \left[\left(\frac{d\omega}{dz}\right)^2 - \frac{\omega^2}{z^2 + E^2} \right] h^2 \right\} \frac{z^2 + E^2}{z^2 + E^2 \sin^2 \beta} - \\ & - \left(1 + \frac{d\omega}{dz}h\right)^{-2} \omega^2 |\mathbf{grad}_E h|^2, \end{aligned}$$

$$A_3 = \left(1 + \frac{d\omega}{dz}h\right)^{-1} \frac{2\omega}{\sqrt{z^2 + E^2 \sin^2 \beta}} \frac{\partial h}{\partial \beta}, \quad A_4 = \left(1 + \frac{d\omega}{dz}h\right)^{-1} \frac{2\omega \sqrt{\alpha}}{\sqrt{z^2 + E^2 \cos \beta}} \frac{\partial h}{\partial \lambda}$$

where $\alpha = w^2(z + \omega h, \beta)$ **with** $w(u, \beta) = \sqrt{\frac{u^2 + E^2 \sin^2 \beta}{u^2 + E^2}}$

while

$$|\mathbf{grad}_E h|^2 = \frac{1}{z^2 + E^2 \sin^2 \beta} \left[\left(\frac{\partial h}{\partial \beta} \right)^2 + \frac{z^2 + E^2 \sin^2 \beta}{(z^2 + E^2) \cos^2 \beta} \left(\frac{\partial h}{\partial \lambda} \right)^2 \right]$$

and

$$\Delta_E h = \frac{1}{z^2 + E^2 \sin^2 \beta} \left[\frac{\partial^2 h}{\partial \beta^2} - \frac{\sin \beta}{\cos \beta} \frac{\partial h}{\partial \beta} + \frac{z^2 + E^2 \sin^2 \beta}{(z^2 + E^2) \cos^2 \beta} \frac{\partial^2 h}{\partial \lambda^2} \right]$$

being the 1st and the 2nd Beltrami differential operators.

Note also that in the computation of ΔT we neglected

$$w^2(z + \omega h, \beta) - w^2(z, \beta) \leq \frac{E^2}{z^2} \left[2\omega \frac{h}{z} + \left(\omega \frac{h}{z} \right)^2 \right] \cos^2 \beta$$

Nevertheless, this omission is small and can be accepted.

4. Application of the Results - Linear GBVP

The disturbing potential T is a harmonic function in the original solution domain Ω . In the space of the curvilinear coordinates z, β, λ , therefore, $\Delta T = 0$ for $z > b$. This yields

$$\Delta_{ell} T = \delta(T, h) \quad \text{for} \quad z > b$$

where

$$\delta(T, h) = A_1 \frac{\partial T}{\partial z} + A_2 \frac{\partial^2 T}{\partial z^2} + A_3 \frac{1}{\sqrt{z^2 + E^2 \sin^2 \beta}} \frac{\partial^2 T}{\partial z \partial \beta} + A_4 \frac{\sqrt{\alpha}}{\sqrt{z^2 + E^2 \cos \beta}} \frac{\partial^2 T}{\partial z \partial \lambda}$$

In addition one can show that in the coordinates z, β, λ the linear gravimetric boundary value problem attains the form

$$\Delta_{ell} T = f \quad \text{in} \quad \Omega_{ell}$$

$$\frac{\partial T}{\partial n} = -\sqrt{1 + \varepsilon} \delta g \quad \text{on} \quad \partial \Omega_{ell}$$

where $f = \delta(T, h)$ and ε may be omitted.

Neglecting the fact that $f = \delta(T, h)$ depends on T , we can represent the solution of the problem *formally* by means of a classical apparatus of mathematical physics. Indeed, we may construct the respective Green's function of the second kind $N(x, y)$, which solves Neumann's boundary value problem for the domain Ω_{ell} . Then (formally)

$$T(y) = \frac{1}{4\pi} \int_{\partial\Omega_{ell}} \delta g(x) N(x, y) d_x S - \frac{1}{4\pi} \int_{\Omega_{ell}} f(x) N(x, y) d_x V .$$

5. Iteration Process

Nevertheless, Neumann's function $N(x, y)$ can also be used to solve the transformed LGBVP, where $f = \delta(T, h)$ depends on T . In this case the integral formula above represents an integro-differential equation for T .

For clarity we can put

$$F(\mathbf{y}) = \frac{1}{4\pi} \int_{\partial\Omega_{ell}} \delta g(\mathbf{x}) N(\mathbf{x}, \mathbf{y}) d_x S$$

$$KT(\mathbf{y}) = -\frac{1}{4\pi} \int_{\Omega_{ell}} \delta [T(\mathbf{x}), h(\mathbf{x})] N(\mathbf{x}, \mathbf{y}) d_x V$$

where $F(\mathbf{y})$ is a harmonic function and $KT(\mathbf{y})$ is an integro-differential operator applied on T , such that

$$\Delta_{ell} KT = \delta(T, h) \text{ in } \Omega_{ell} \text{ and } \frac{\partial KT}{\partial n} = 0 \text{ on } \partial\Omega_{ell}$$

Under this notation the problem is to find T from

$$T = F + KT$$

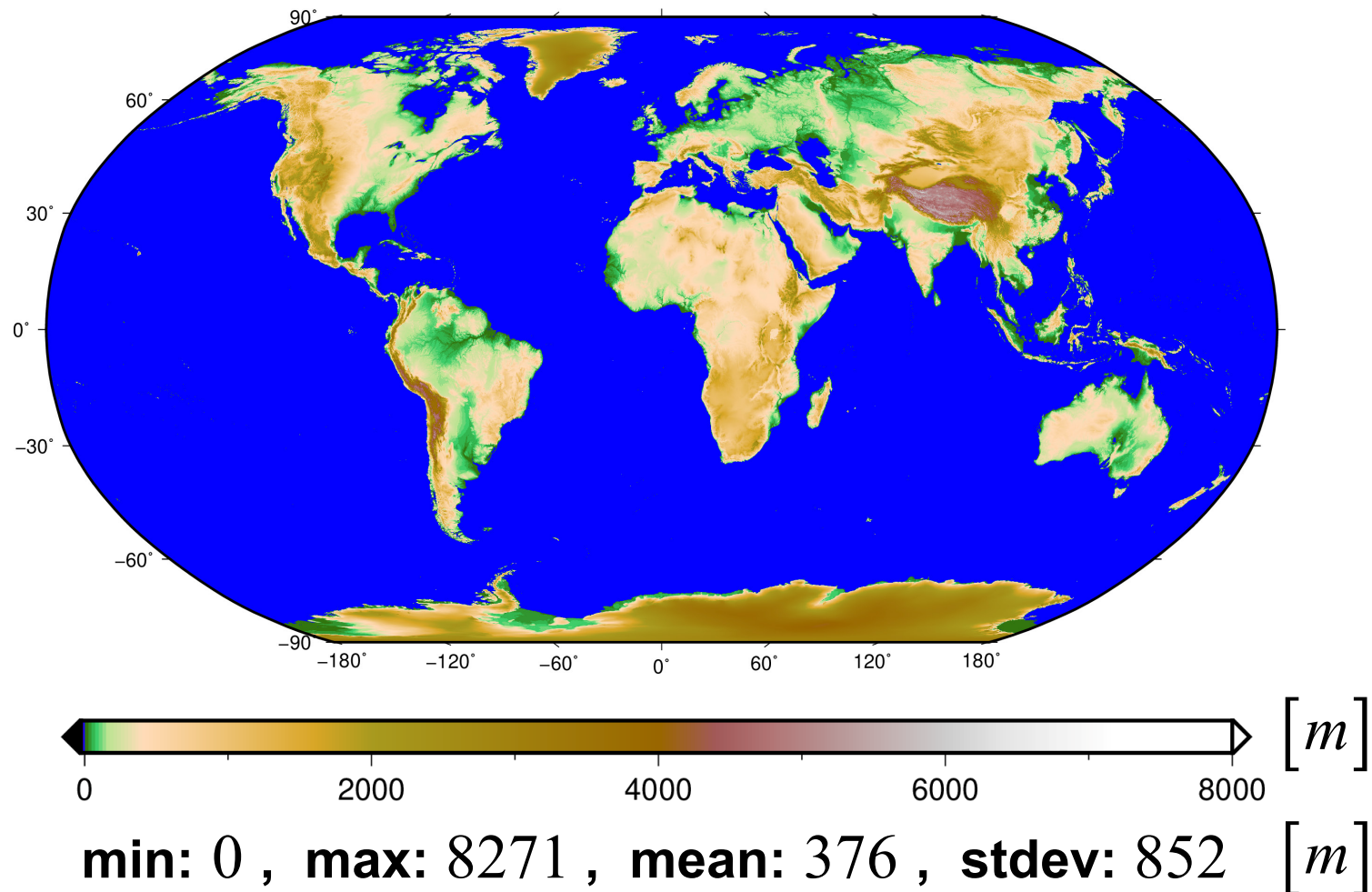
We apply the method of successive approximations, i.e.

$$T = \lim_n T_n, \quad T_n = F + KT_{n-1}$$

where T_0 is the starting approximation, e.g. $T_0 = F$.

6. Example of the Topography Dependent Coefficients

ETOPO1_ice global relief model of Earth's surface



**Topography Dependent Coefficients
for the global relief model of Earth's surface related to a sphere
of radius $R = 6378137$ m in case that $z = R$**

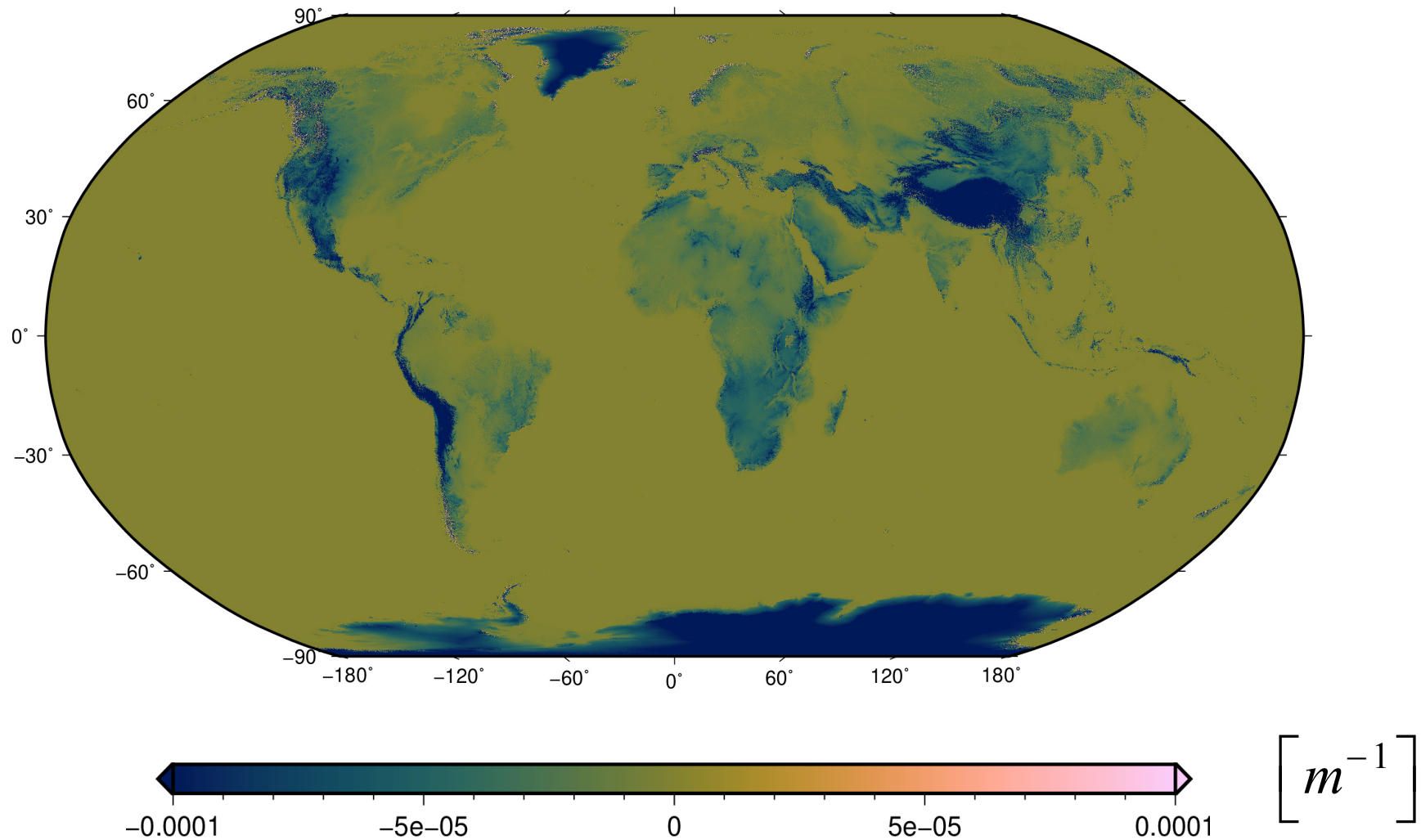
$$A_1 = \frac{1}{R^2} \left[-2h - \frac{\sin \beta}{\cos \beta} \frac{\partial h}{\partial \beta} + \frac{\partial^2 h}{\partial \beta^2} + \frac{1}{\cos^2 \beta} \frac{\partial^2 h}{\partial \lambda^2} \right] +$$

$$+ \frac{1}{R^2} \left\{ (R+h)^2 + \left[\left(\frac{\partial h}{\partial \beta} \right)^2 + \frac{1}{\cos^2 \beta} \left(\frac{\partial h}{\partial \lambda} \right)^2 \right] \right\} \frac{d^2 \omega(R)}{dz^2} h ,$$

$$A_2 = -\frac{2Rh + h^2}{R^2} - \frac{1}{R^2} \left[\left(\frac{\partial h}{\partial \beta} \right)^2 + \frac{1}{\cos^2 \beta} \left(\frac{\partial h}{\partial \lambda} \right)^2 \right] ,$$

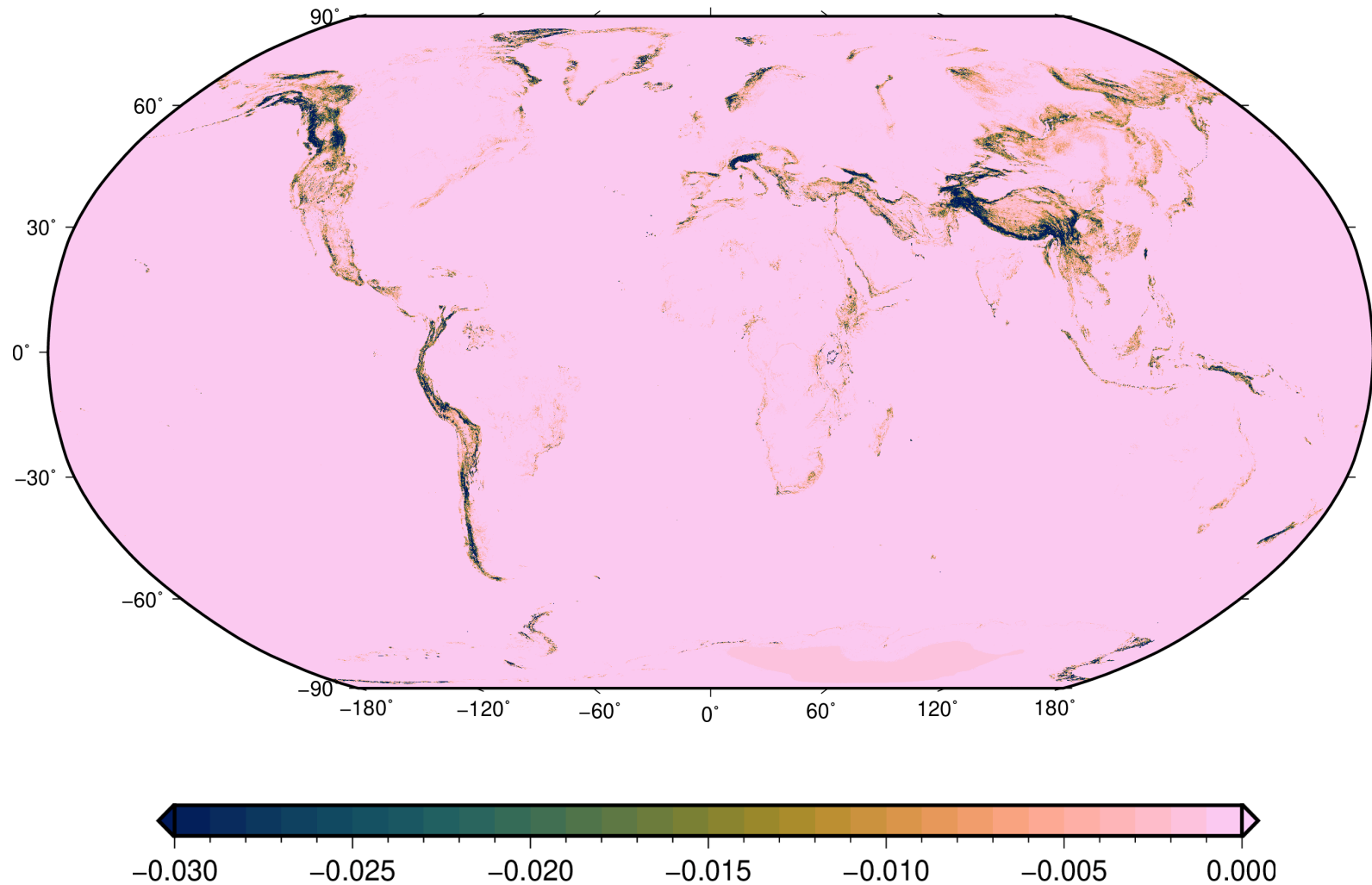
$$A_3 = \frac{2}{R} \frac{\partial h}{\partial \beta} , \quad A_4 = \frac{2}{R \cos \beta} \frac{\partial h}{\partial \lambda}$$

Coefficient A_1



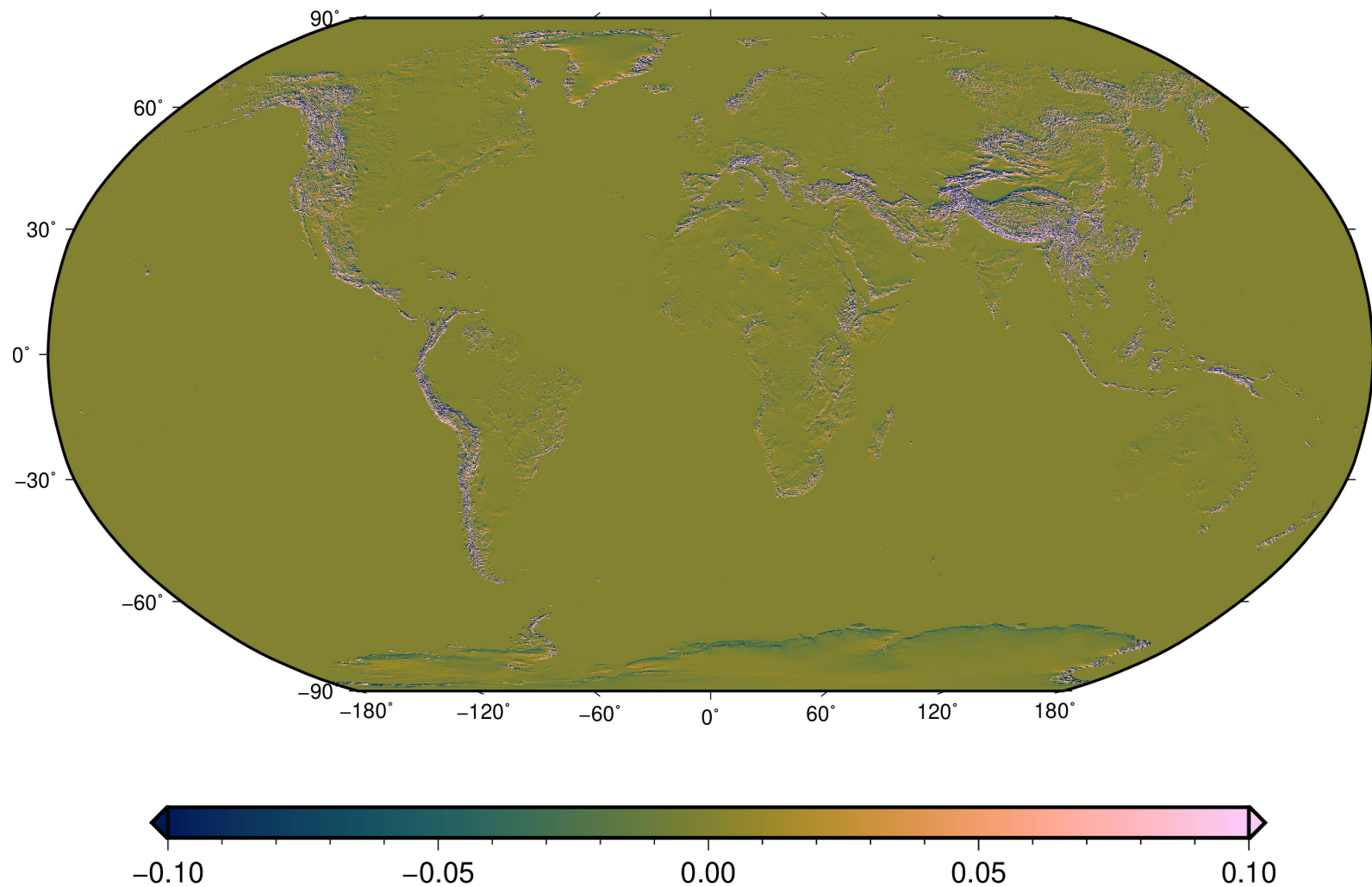
min: -0.011 , max: 0.007 , mean: -0.000015 , stdev: 0.00006

Coefficient A_2



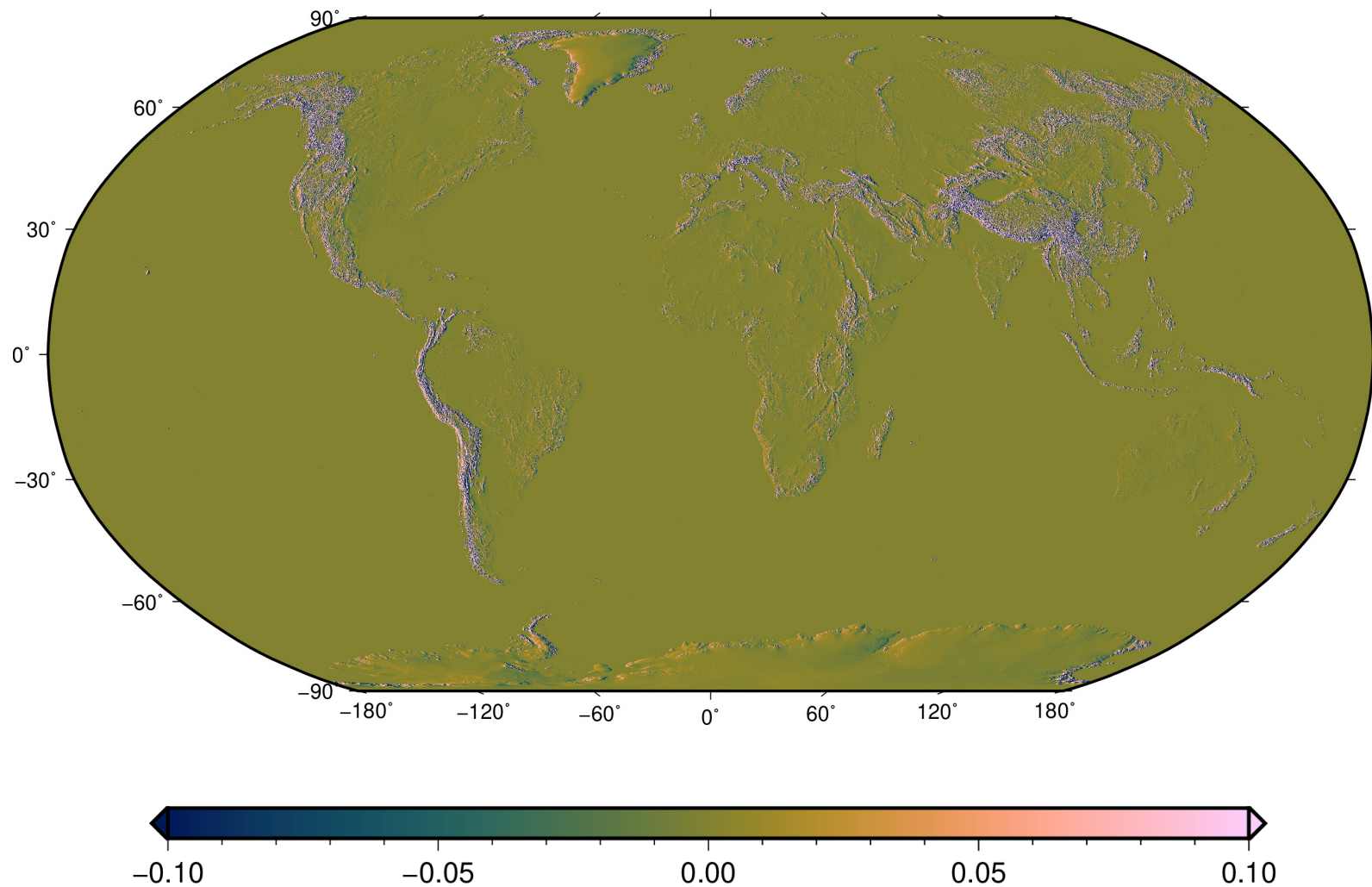
min: -2.86 , max: 0.00 , mean: -0.0009 , stdev: 0.00068

Coefficient A_3



min: -1.663 , max: 1.713 , mean: -0.0003 , stdev: 0.034

Coefficient A_4



min: -2.935 , max: 3.362 , mean: 0.000 , stdev: 0.043

Thank you for your attention !

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