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Magneto-inertial waves and planetary rotation

by J. Rekier, S. A. Triana, and V. Dehant

Royal Observatory of Belgium

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Motivation

- This work is part of a long-term effort undertaken at ROB to develop an improved model of the Earth's rotation.
- Current models are based on Sasao et al. (1981). Rotation is represented by a set of coupled Liouville equations expressing the conservation of the Angular Momentum (AM) of the planet and each of its independent layers.
- Sasao et al. make the following assumptions:
  - The flow inside the liquid core has a uniform vorticity (Poincaré flow)
  - The flattening of the Earth is small (first order computation)
  - Forces on the core flow are represented by additional torques at the CMB/ICB (viscous, electromagnetic, etc.)

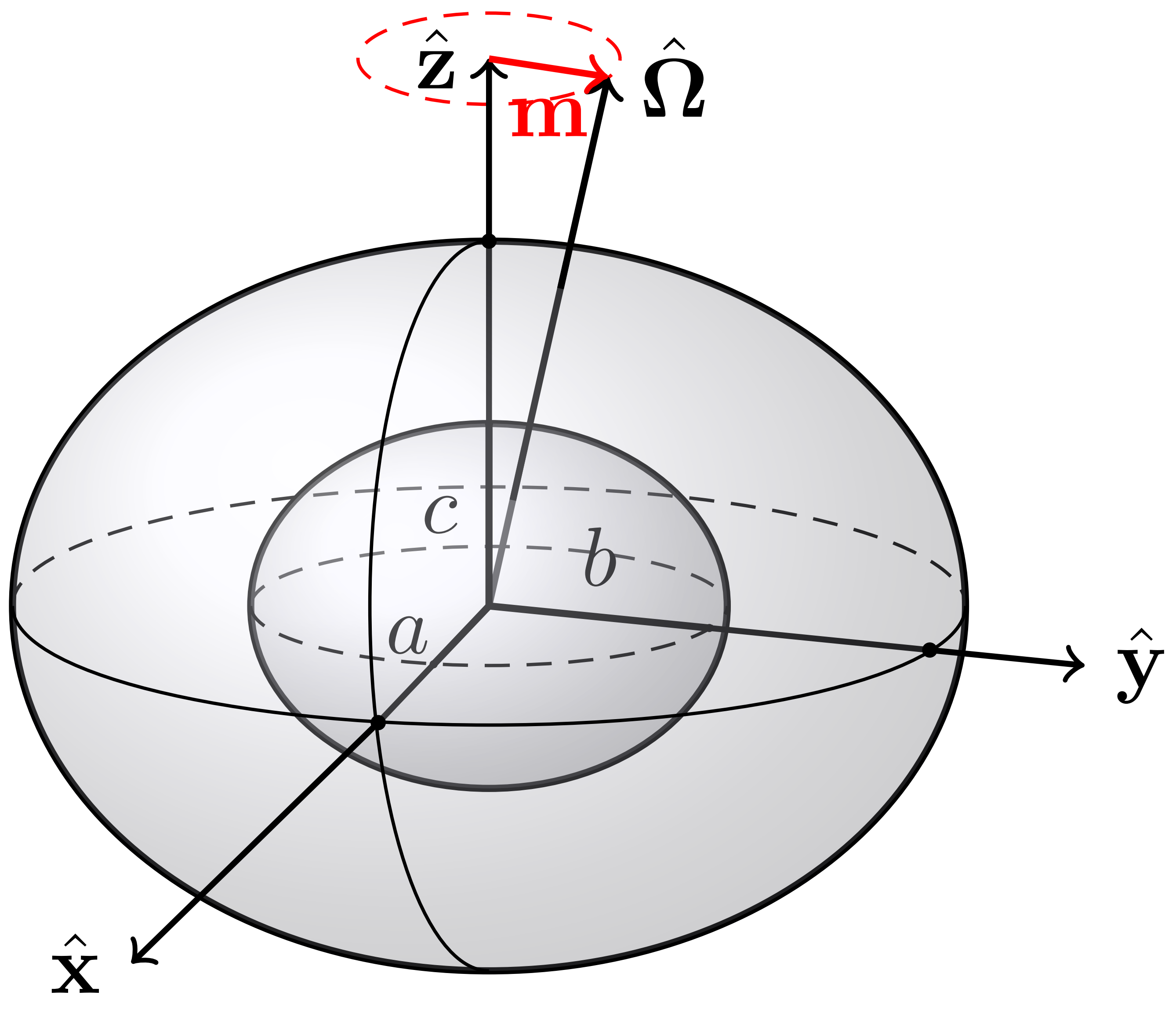
Motivation

- The model based on Sasao et al, though very convenient, cannot, *on its own,* account for some of the phenomena observed in the ***Length Of Day*** (LOD), ***Polar Motion*** (PM) and ***Nutation*** data → **additional elements of formalism are required**.
- E.g. interdecadal (~6yrs) oscillations in LOD, possible modulation in amplitude of the *Free Core Nutation* (FCN), etc.
- *Question:* Can we extend the formalism of Sasao et al. in order to integrate those elements of formalism ?
- Smith & Dahlen (1981) suggested a model equivalent to Sasao et al. which they called the **HLL model** (standing for *Hough, Love & Larmor*). Though less versatile than Sasao et al., this has the merit to make the link with fluid dynamics conceptually simpler.
- In Rekier et al. (2020), we used a similar formalism to clarify the link between the FCN and the ***Inertial Modes*** (IM). More specifically, we showed how the FCN is the equivalent to the simplest IM, the *Spin-Over Mode* (SOM) of a **freely rotating** planet.

Content and model

$$i\omega\mathbf{u} + 2\hat{\mathbf{z}}\times\mathbf{u} + \nabla p + i\omega\mathbf{m}\times\mathbf{r} = \mathrm{Le}^2(\nabla\times\mathbf{B})\times\mathbf{B}$$
$$i\omega\mathbf{B} = \nabla\times(\mathbf{u}\times\mathbf{B})$$

- We extend the work presented in Rekier et al. (2020) to include the effects of the Lorentz force produced by a background magnetic field.
- We start by looking at IM with uniform vorticity: SOM + *Near-Geostrophic Mode* (NGM) (see later).
- Exact analytical solutions allow us to explore the full range of parameters.



With $\hat{\mathbf{\Omega}} \equiv \hat{\mathbf{z}} + \mathbf{m}$

$\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$

$$\mathbf{L} = \mathbf{I}\cdot\hat{\mathbf{\Omega}} + \int_{\mathscr{V}} \rho\mathbf{r}\times\mathbf{u}$$

$$\underbrace{\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}}_{\mathrm{I}} = \underbrace{\begin{pmatrix} A_{\mathrm{m}} & 0 & 0 \\ 0 & B_{\mathrm{m}} & 0 \\ 0 & 0 & C_{\mathrm{m}} \end{pmatrix}}_{\mathrm{I_m}} + \underbrace{\begin{pmatrix} A_{\mathrm{f}} & 0 & 0 \\ 0 & B_{\mathrm{f}} & 0 \\ 0 & 0 & C_{\mathrm{f}} \end{pmatrix}}_{\mathrm{I_f}}.$$

Purely hydrodynamic case
($\mathbf{B} = 0$)

Rekier et al. (2020):

Uniform vorticity flow: $\mathbf{u} = \mathbf{w} \times \mathbf{r} + \nabla\psi$ (Poincaré)

AM conservation: $i\omega\mathbf{L} = -\hat{\mathbf{\Omega}} \times \mathbf{L}$ (Euler-Liouville)

Mantle

$$\begin{pmatrix} \omega & \frac{i(B-C)}{A} & 0 & \frac{\omega(A_f+B_f-C_f)(A_f-B_f+C_f)}{AA_f} & \frac{i(A_f+B_f-C_f)(-A_f+B_f+C_f)}{AB_f} & 0 \\ -\frac{i(A-C)}{B} & \omega & 0 & -\frac{i(A_f+B_f-C_f)(A_f-B_f+C_f)}{A_fB} & \frac{\omega(A_f+B_f-C_f)(-A_f+B_f+C_f)}{BB_f} & 0 \\ 0 & 0 & \omega & 0 & 0 & \frac{\omega(A_f-B_f+C_f)(-A_f+B_f+C_f)}{CC_f} \\ \omega & 0 & 0 & \omega & -\frac{i(A_f-B_f-C_f)}{B_f} & 0 \\ 0 & \omega & 0 & -\frac{i(A_f-B_f+C_f)}{A_f} & \omega & 0 \\ 0 & 0 & \omega & 0 & 0 & \omega \end{pmatrix} \begin{pmatrix} m^x \\ m^y \\ m^z \\ w^x \\ w^y \\ w^z \end{pmatrix} = 0$$

Core

Three solutions →

$$\omega = \omega_{\text{FCN}} \approx -1$$
$$\omega = \omega_{\text{CW}} \approx 0$$
$\omega = 0 \quad \equiv$ **geostrophic flow**

Lorentz force

- The exact shape of the magnetic field inside the core is unknown.
- Once we have chosen the form of $\mathbf{B}_0$, the increment, we obtain $\mathbf{b}$ directly from the (ideal) induction equation:

$$\mathbf{b} = \frac{1}{i\omega}\nabla \times \left(\mathbf{u} \times \mathbf{B}_0\right)$$

- Once $\mathbf{b}$ is known, we immediately obtain the Lorentz force:

$$\mathrm{Le}^2 \left(\nabla \times \mathbf{B}_0 \times \mathbf{b} + \nabla \times \mathbf{b} \times \mathbf{B}_0\right)$$

- A uniform magnetic field $\mathbf{B}_0 = \mathbf{cst}$ produces no Lorentz force. The next simplest form is the *Malkus field*:

$$\nabla \times \mathbf{B}_0 = \mathbf{j} \ \text{ with } \ \mathbf{j} = \mathbf{cst}$$

- Malkus (1967) used $\mathbf{j} = \hat{\mathbf{z}}$, in what follows, we use a general shape for an arbitrary $\mathbf{j}$, that satisfies the boundary condition $\mathbf{B}_0 \cdot \mathbf{n} = 0$, which also implies (from induction equation): $\mathbf{b} \cdot \mathbf{n} = 0$.

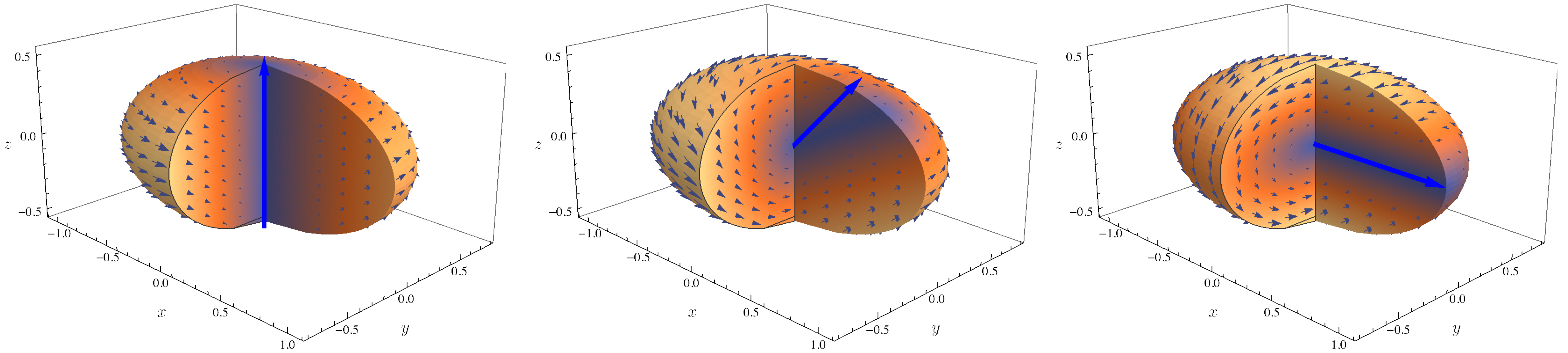
Lorentz force

$$+\mathrm{Le}^2 \begin{pmatrix} \frac{(B_\mathrm{f}-C_\mathrm{f})(-A_\mathrm{f}+B_\mathrm{f}+C_\mathrm{f})\left(B_\mathrm{f}^2 j^{z\,2}(A_\mathrm{f}-B_\mathrm{f}+C_\mathrm{f})+C_\mathrm{f}^2 j^{y\,2}(-A_\mathrm{f}-B_\mathrm{f}+C_\mathrm{f})\right)}{A_\mathrm{f} B_\mathrm{f}^2 C_\mathrm{f}^2 \omega} & \frac{j^x j^y (B_\mathrm{f}-C_\mathrm{f})(A_\mathrm{f}+B_\mathrm{f}-C_\mathrm{f})(-A_\mathrm{f}+B_\mathrm{f}+C_\mathrm{f})}{A_\mathrm{f} B_\mathrm{f}^2 \omega} & \frac{j^x j^z (B_\mathrm{f}-C_\mathrm{f})(A_\mathrm{f}-B_\mathrm{f}-C_\mathrm{f})(A_\mathrm{f}-B_\mathrm{f}+C_\mathrm{f})}{A_\mathrm{f} C_\mathrm{f}^2 \omega} \\ \frac{j^x j^y (A_\mathrm{f}-C_\mathrm{f})(A_\mathrm{f}+B_\mathrm{f}-C_\mathrm{f})(A_\mathrm{f}-B_\mathrm{f}+C_\mathrm{f})}{A_\mathrm{f}^2 B_\mathrm{f} \omega} & -\frac{(A_\mathrm{f}-C_\mathrm{f})(A_\mathrm{f}-B_\mathrm{f}+C_\mathrm{f})\left(A_\mathrm{f}^2 j^{z\,2}(A_\mathrm{f}-B_\mathrm{f}-C_\mathrm{f})+C_\mathrm{f}^2 j^{x\,2}(A_\mathrm{f}+B_\mathrm{f}-C_\mathrm{f})\right)}{A_\mathrm{f}^2 B_\mathrm{f} C_\mathrm{f}^2 \omega} & \frac{j^y j^z (A_\mathrm{f}-C_\mathrm{f})(-A_\mathrm{f}+B_\mathrm{f}-C_\mathrm{f})(-A_\mathrm{f}+B_\mathrm{f}+C_\mathrm{f})}{B_\mathrm{f} C_\mathrm{f}^2 \omega} \\ \frac{j^x j^z (A_\mathrm{f}-B_\mathrm{f})(A_\mathrm{f}+B_\mathrm{f}-C_\mathrm{f})(A_\mathrm{f}-B_\mathrm{f}+C_\mathrm{f})}{A_\mathrm{f}^2 C_\mathrm{f} \omega} & \frac{j^y j^z (A_\mathrm{f}-B_\mathrm{f})(A_\mathrm{f}-B_\mathrm{f}-C_\mathrm{f})(A_\mathrm{f}+B_\mathrm{f}-C_\mathrm{f})}{B_\mathrm{f}^2 C_\mathrm{f} \omega} & -\frac{(A_\mathrm{f}-B_\mathrm{f})(A_\mathrm{f}+B_\mathrm{f}-C_\mathrm{f})\left(A_\mathrm{f}^2 j^{z\,2}(A_\mathrm{f}-B_\mathrm{f}-C_\mathrm{f})+B_\mathrm{f}^2 j^{y\,2}(A_\mathrm{f} j^x-B_\mathrm{f}+A_\mathrm{f})\right)}{A_\mathrm{f}^2 C_\mathrm{f}^2 C_\mathrm{f} \omega} \end{pmatrix} \begin{pmatrix} w^x \\ w^y \\ w^z \end{pmatrix} = 0$$

- Adding the above matrix and solving for $\mathbf{m}$ and $\mathbf{w}$ gives the free rotational modes accounting for the presence of the Lorentz force.
- We start by looking at the steadily rotation solutions for which $\mathbf{m} = \mathbf{0}$. To simplify further, we set:

$$\mathbf{j} = \sin\theta \hat{\mathbf{x}} + \cos\theta \hat{\mathbf{z}}$$

Background magnetic field

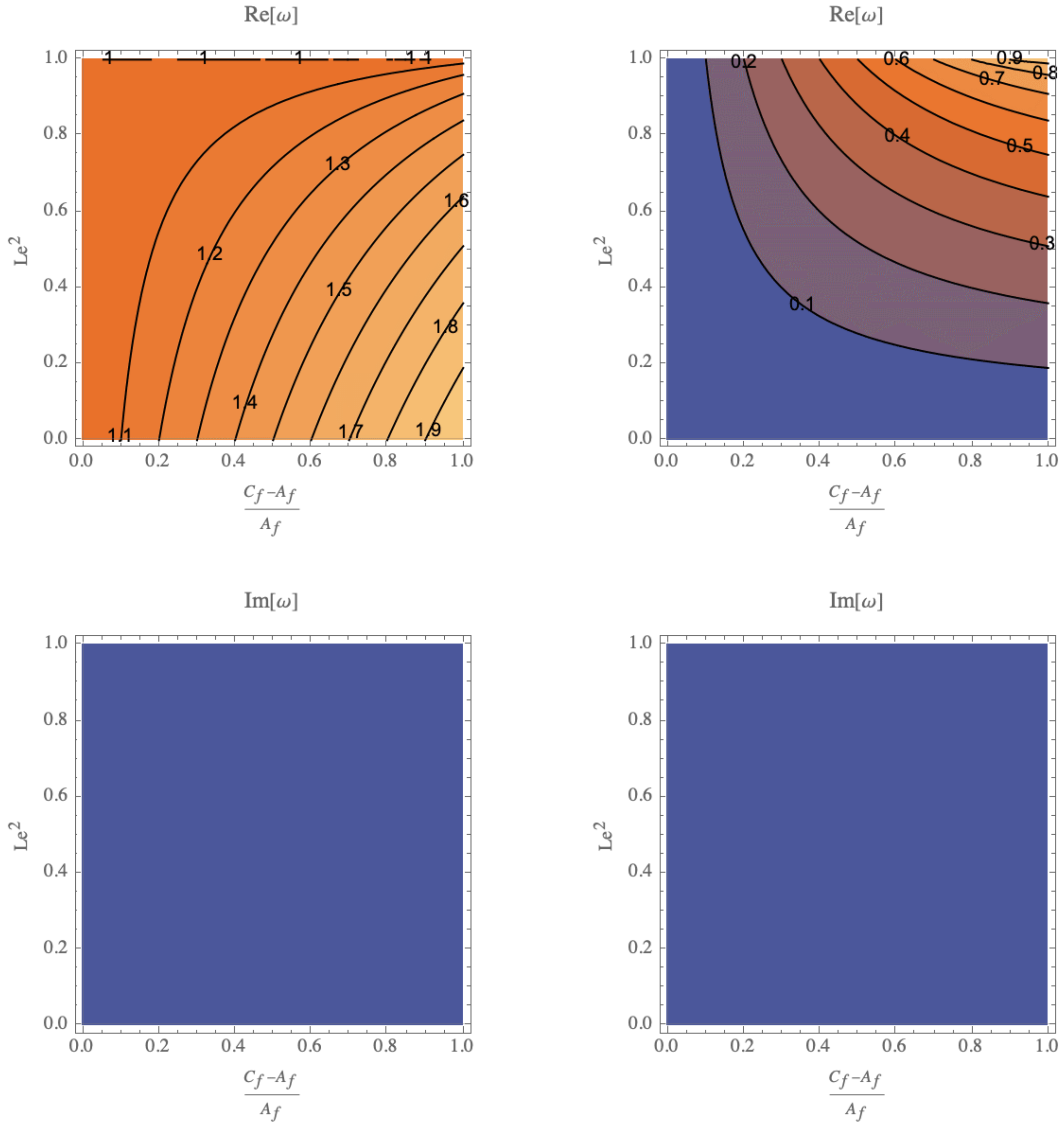


$$\mathbf{j} = \sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{z}}$$

$$\mathbf{B}_0 = -\frac{y\cos(\theta)(-A_{\mathrm{f}} + B_{\mathrm{f}} + C_{\mathrm{f}})}{C_{\mathrm{f}}}\hat{\mathbf{x}} + \frac{(A_{\mathrm{f}} - B_{\mathrm{f}} + C_{\mathrm{f}})(A_{\mathrm{f}}x\cos(\theta) - C_{\mathrm{f}}z\sin(\theta))}{A_{\mathrm{f}}C_{\mathrm{f}}}\hat{\mathbf{y}} + \frac{y\sin(\theta)(A_{\mathrm{f}} + B_{\mathrm{f}} - C_{\mathrm{f}})}{A_{\mathrm{f}}}\hat{\mathbf{z}}$$

Steady rotation

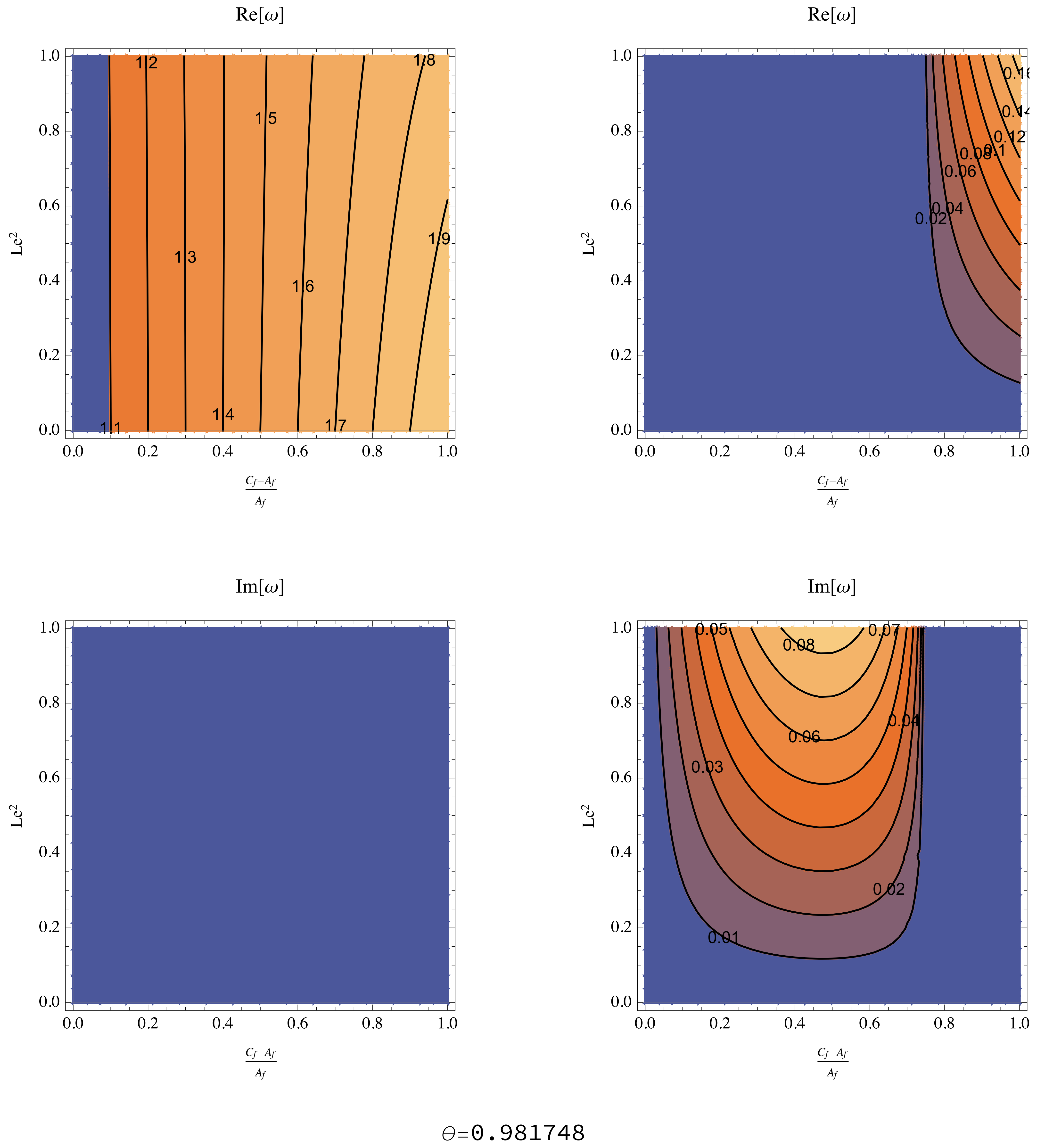
axisymmetric core ($A_\mathrm{f} = B_\mathrm{f}$)



θ=0.

- The magnetic field significantly alters the SOM frequency (left) for $\theta \sim 0$ but this influence diminishes for larger θ
- The SOM frequency remains real $\forall \theta$
- The magnetic field turns the geostrophic flow into a NGM (right)
- The NGM frequency *becomes complex* for $\theta \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

Steady rotation
axisymmetric core ($A_f = B_f$)



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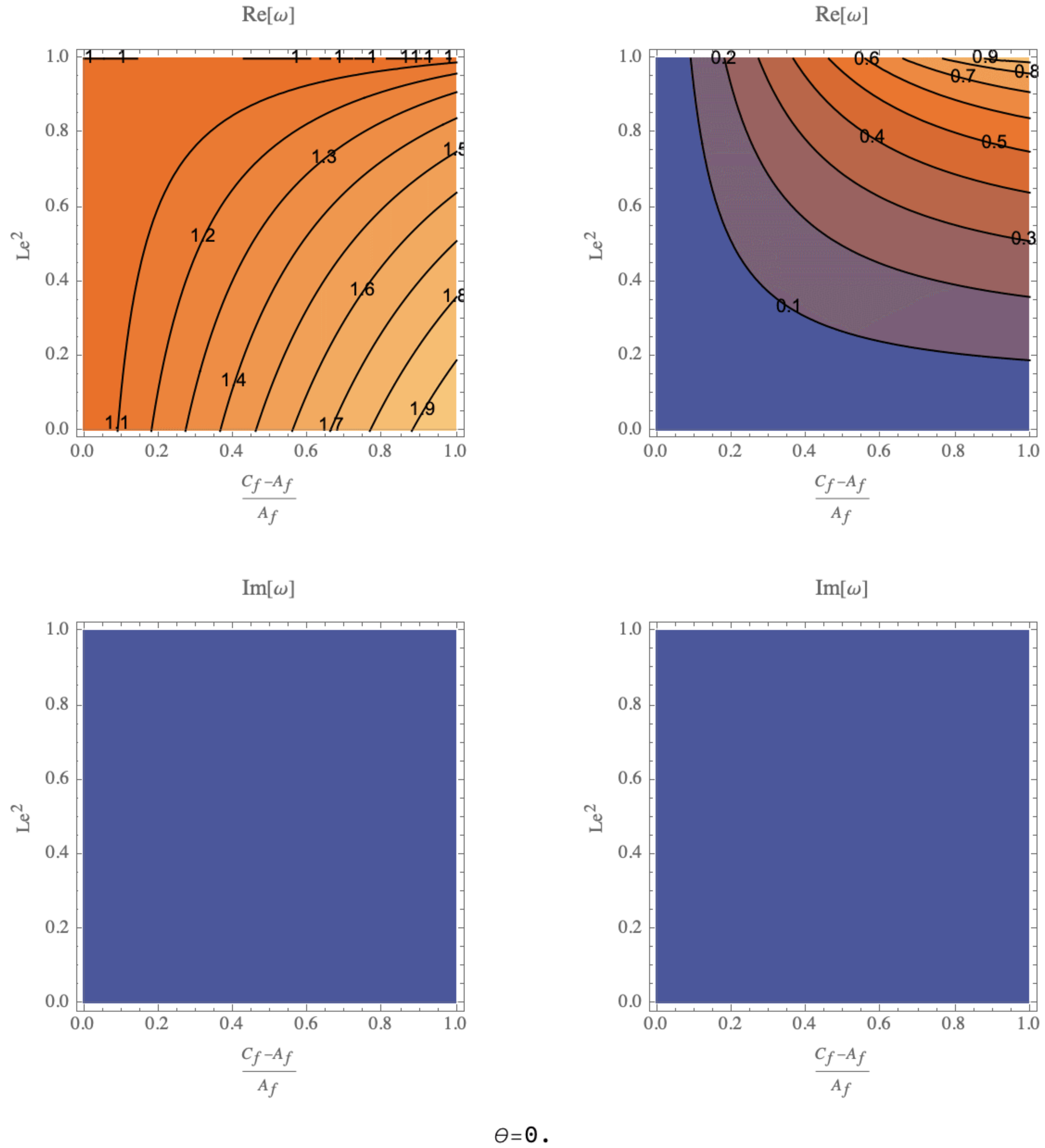
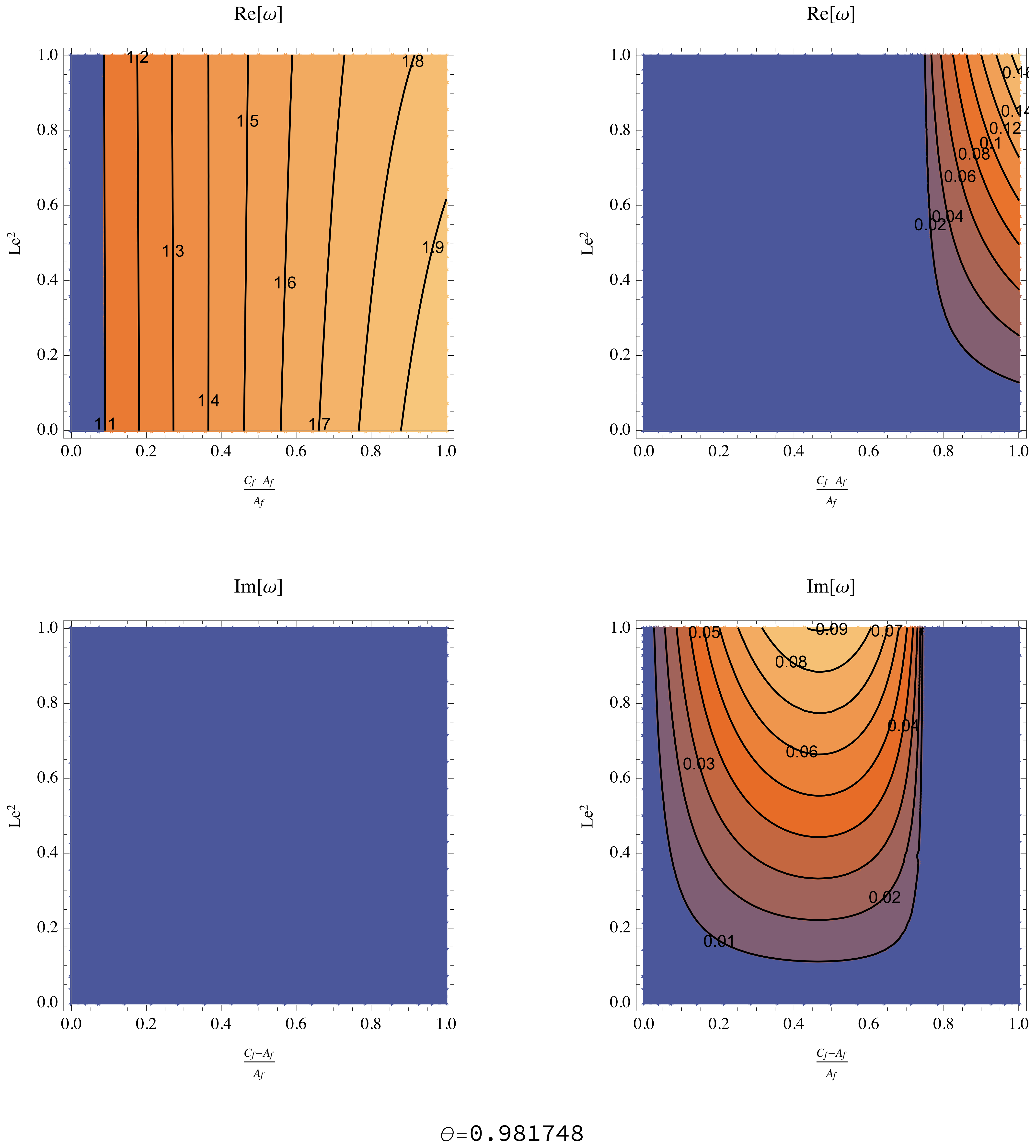
Free rotation ($A_\mathrm{f} = A/10$, "Earth-like") axisymmetric core ($A_\mathrm{f} = B_\mathrm{f}$ and $\mathrm{A} = \mathrm{B} = \mathrm{C}$)



- The magnetic field significantly alters the FCN frequency (left) for $\theta \sim 0$ but this influence diminishes for larger θ
- The FCN frequency remains real $\forall \theta$
- The magnetic field turns the geostrophic flow into a F(ree)NGM (right)
- The FNGM frequency *becomes complex* for $\theta \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

Free rotation ($A_f = A/10$, "Earth-like")
axisymmetric core ($A_f = B_f$ and $A = B = C$)



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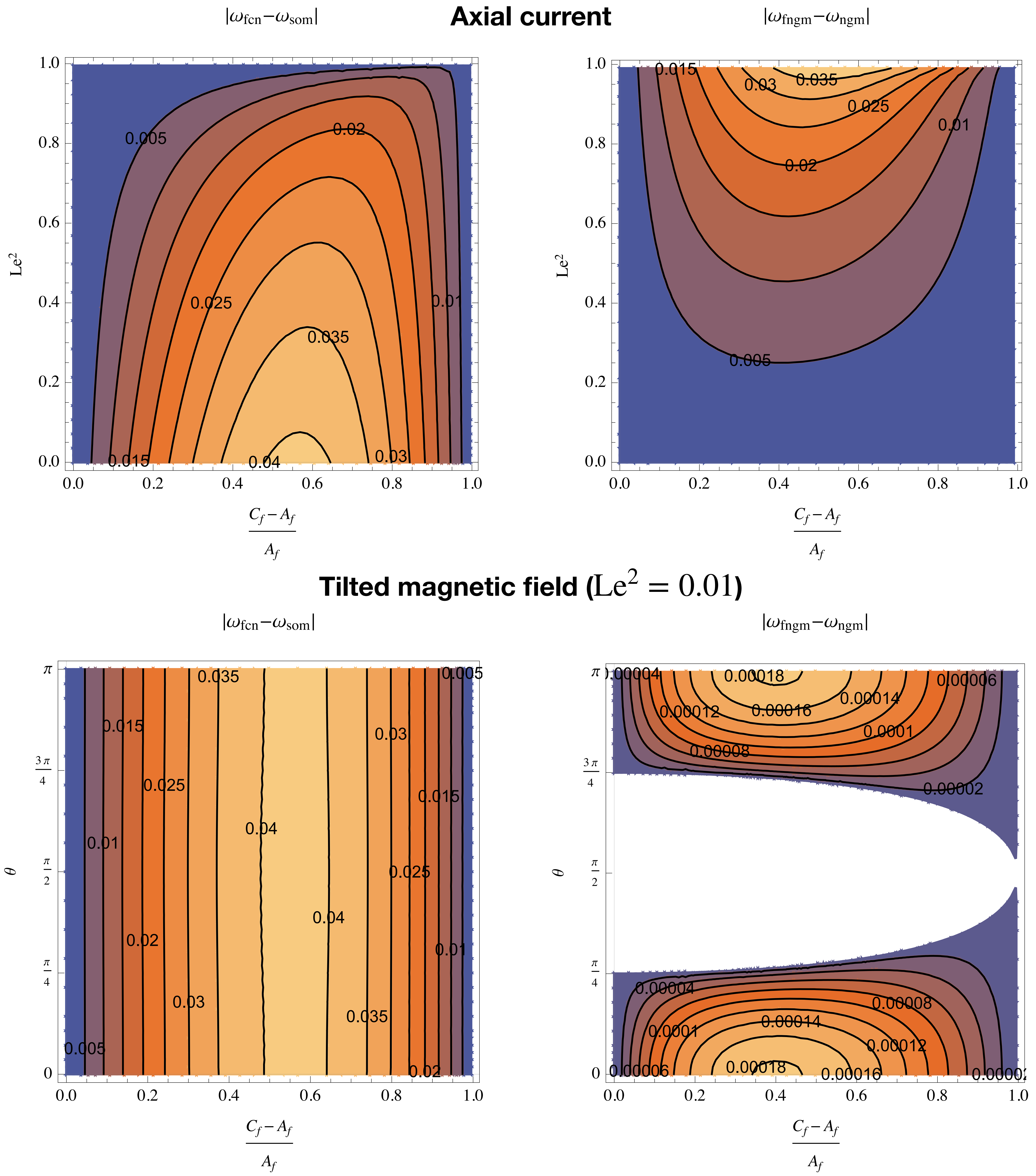
Free rotation
axisymmetric core ($A_\mathrm{f} = B_\mathrm{f}$)

- The change in the frequencies depends on both the magnetic field amplitude and on its orientation
- $|\omega_\mathrm{fcn} - \omega_\mathrm{som}|$ depends very weakly on θ for $\mathrm{Le}^2 \ll 1$ and very strongly for $\mathrm{Le}^2 \geq 1$
- $|\omega_\mathrm{fngm} - \omega_\mathrm{ngm}|$ depends very strongly on θ for all values of Le^2



Free rotation axisymmetric core ($A_{\mathrm{f}} = B_{\mathrm{f}}$)

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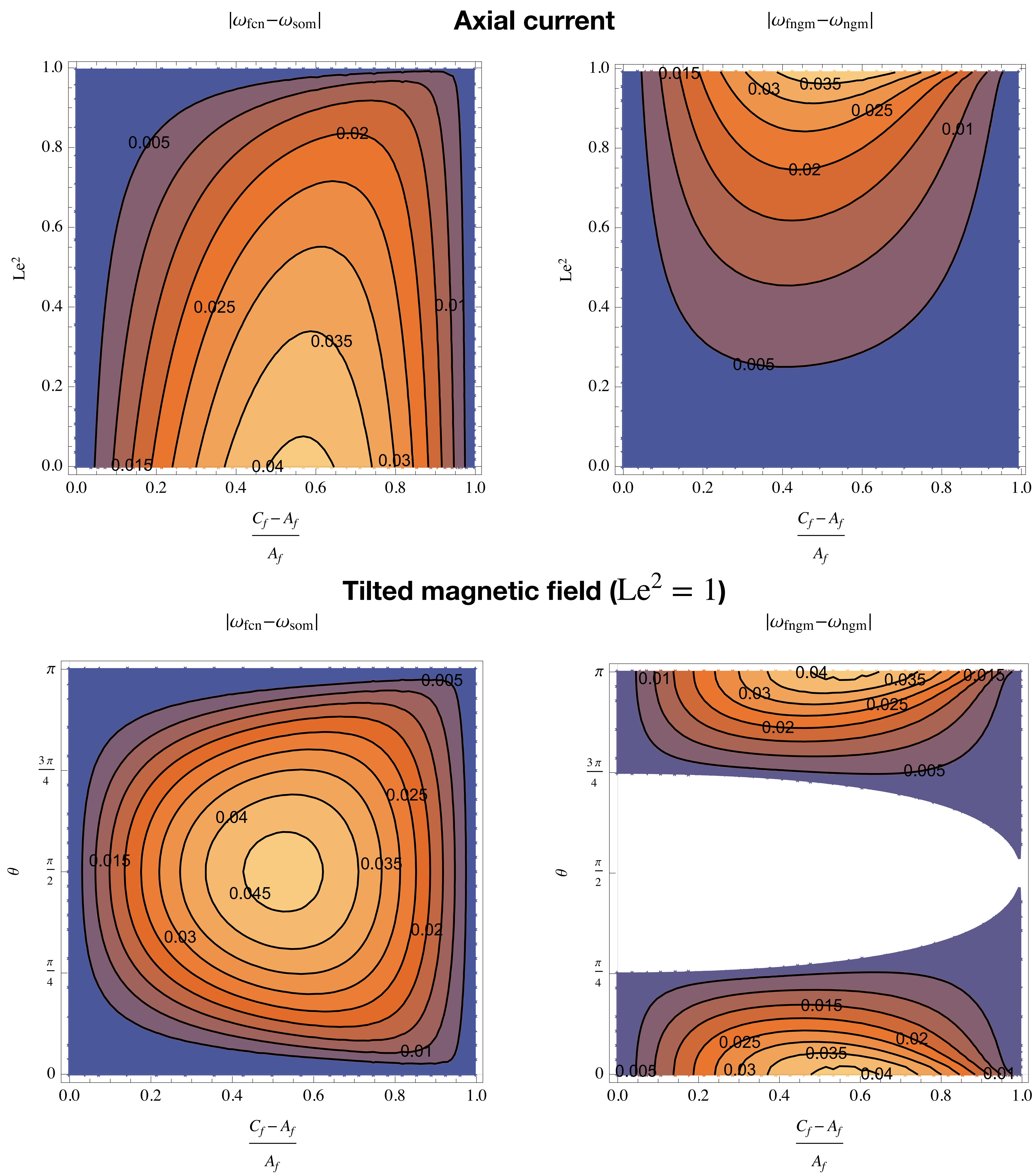

Conclusions

- Perturbations induced by a (Malkus) toroidal field on the uniform vorticity flow inside the core volume have very little effects on the FCN frequency
- They do, however, cause the appearance of a near-geostrophic (long period) mode which, under certain conditions, may become destabilised

What's next ?

- Although a uniform magnetic field exerts no Lorentz force on its own, it can create a Lorentz force by combination with the Malkus field inducing current: $\mathbf{j} \times \mathbf{b}$
- Reintroduce the Electro-Magnetic torque acting on an electrically conducting mantle

  $\rightarrow$ $\exists$ contribution from Inertial modes ?

- Introduce magnetic diffusivity and viscosity (requires numerical resolution)

  $\rightarrow$ on this subject see presentation of Triana et al. (EGU21-12492, this session)