

Solitons in a box-shaped wavefield with noise: perturbation theory and statistics

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Two problems within the perturbation theory

We consider the one-dimensional focusing nonlinear Schrödinger (NLSE) equation NLSE for a complex wavefield $q(t, x)$ in its non-dimensional form:

$$iq_t + \frac{1}{2}q_{xx} + |q|^2q = 0. \quad (1)$$

The model (1) is **integrable** using the Inverse scattering transform (IST) method [Zakharov, Shabat (1972)] and the wave field $q(x)$ at any chosen moment of time can be fully characterized by its **scattering data** (IST spectrum).

There are two problem formulations within the IST perturbation theory:

1. The evolution of q with a small but continuous forcing $f(t, x)$:

$$iq_t + \frac{1}{2}q_{xx} + |q|^2q = f(t, x). \quad (2)$$

2. The effect of perturbed initial conditions and no forcing term, i.e. $f = 0$:

$$q(0, x) \rightarrow q(0, x) + \delta q(x). \quad (3)$$

Here we consider the problem 2!

Introduction: developments on the IST perturbation theory

Solitons and multi-solitons. Classic perturbation ansatz for discrete soliton eigenvalues and continuous spectrum waves reflection coefficient:

- ✓ D.J. Kaup, "A Perturbation expansion for the Zakharov–Shabat Inverse scattering transform", SIAM Journal on Applied Mathematics, 1976.
- ✓ V.I. Karpman and E.M. Maslov, "Perturbation theory for solitons", Soviet Physics JETP, 1977.

Reviews:

- ✓ F. Bass, Y. Kivshar, V. Konotop, and Y. Sinitsyn, "Dynamics of solitons under random perturbations", Physics Reports, 1988.
- ✓ Y. S. Kivshar and B. A. Malomed, "Dynamics of solitons in nearly integrable systems", Reviews of Modern Physics, 1989.

Recent developments:

- ✓ F. J. Garcia-Gomez and V. Aref, "Statistics of the nonlinear discrete spectrum of a noisy pulse", Journal of Lightwave Technology, 2019.

Introduction: soliton appearance/disappearance problem

Analytical approaches:

- ✓ Y. S. Kivshar and B. A. Malomed, "Dynamics of solitons in nearly integrable systems", Reviews of Modern Physics, 1989.

Numerical simulations of elementary soliton formation due to continuous pumping:

- ✓ D.J. Kaup and P.J. Hansen, "The forced nonlinear Schrödinger equation", Physica D: Nonlinear Phenomena, 1986.
- ✓ M.N. Bussac, P. Lochak, C. Meunier, A. Heron-Gourdin, "Soliton generation in the forced non-linear Schrödinger equation", Physica D: Nonlinear Phenomena, 1985.

Solitonic content of stochastic wavefields (theory and numerics):

- ✓ V. Konotop and L. Vazquez, "Nonlinear Random Waves", (World Scientific, 1994).
- ✓ S.A. Derevyanko and J.E. Prilepsky, "Soliton generation from randomly modulated return-to-zero pulses", Optics Communications, 2008.
- ✓ S.A. Derevyanko and J.E. Prilepsky, "Random input problem for the nonlinear Schrödinger equation", Physical Review E, 2008.

Zakharov–Shabat problem

The scattering data can be found with the direct scattering transform (DST) based on the Zakharov–Shabat (ZS) equation representing an auxiliary linear system for a vector wave function $\Phi = (\phi_1, \phi_2)^T$

$$\mathcal{L}\Phi - \zeta\Phi = 0, \quad \mathcal{L} = \begin{pmatrix} i\partial_x & -iq(x) \\ -iq^*(x) & -i\partial_x \end{pmatrix}, \quad (4)$$

where $\zeta = \xi + i\eta$ is the time-independent complex spectral parameter with real ξ and η , the superscripts T and the star stand for a transposition and complex conjugation. Eq. (4) is typically solved for a fixed moment of time t_0 with $q(x) = q(t_0, x)$ playing the role of a potential.

In case of potentials with compact support the wave function has the following asymptotics:

$$\Phi|_{x \rightarrow -\infty} = (e^{-i\zeta x}, 0)^T, \quad \Phi|_{x \rightarrow \infty} = (ae^{-i\zeta x}, be^{i\zeta x})^T. \quad (5)$$

The scattering coefficients $a(\zeta)$ and $b(\zeta)$ are connected to the scattering data $\{\zeta_n, \rho_n; r\}$ as follows:

$$a(\zeta_n) = 0, \quad \rho_n = b(\zeta_n)/a'(\zeta_n); \quad r(\xi) = b(\xi)/a(\xi), \quad (6)$$

where $\{\zeta_n, \rho_n\}$ is a countable set of eigenvalues (discrete spectrum) and associated norming constants, while $r(\xi)$ is the reflection coefficient defined on a real axis (continuous spectrum).

Perturbation theory: eigenvalues

With the inner product of two vectors $\langle \Psi, \Phi \rangle = \int_{-\infty}^{\infty} \Psi^* \Phi dx$ we derive an eigensystem adjoint to (4):

$$\mathcal{L}^\dagger \Phi^\dagger - \zeta^* \Phi^\dagger = 0, \quad \mathcal{L}^\dagger = \begin{pmatrix} i\partial_x & iq \\ iq^* & -i\partial_x \end{pmatrix}, \quad (7)$$

where the adjoint operator satisfies the relation $\langle \Phi^\dagger, \mathcal{L}\Phi \rangle = \langle \mathcal{L}^\dagger \Phi^\dagger, \Phi \rangle$. Note that $\Phi^\dagger = (\phi_2^*, \phi_1^*)^T$ satisfies Eq. (7). We are interested in the variation of $\{\zeta_n, \rho_n\}$ and r associated with a small perturbation $\delta q(x)$ of the potential. Let us take the variation of Eq. (4):

$$\delta(\mathcal{L}\Phi - \zeta\Phi) = (\delta\mathcal{L} - \delta\zeta)\Phi + (\mathcal{L} - \zeta)\delta\Phi = 0. \quad (8)$$

To cancel out the second term in the last expression we take the inner product of Eq. (8) with $\Phi^\dagger$ resulting in $\langle \Phi^\dagger, (\delta\mathcal{L} - \delta\zeta)\Phi \rangle = 0$. Extracting $\delta\zeta$, we end up with the following expression:

$$\delta\zeta = \frac{\langle \Phi^\dagger, \delta\mathcal{L}\Phi \rangle}{\langle \Phi^\dagger, \Phi \rangle}, \quad \delta\mathcal{L} = -i \begin{pmatrix} 0 & \delta q \\ \delta q^* & 0 \end{pmatrix}, \quad (9)$$

where $\langle \Phi^\dagger, \delta\mathcal{L}\Phi \rangle = -i \int_{-\infty}^{\infty} (\phi_1^2 \delta q^* + \phi_2^2 \delta q) dx$ and $\langle \Phi^\dagger, \Phi \rangle = \int_{-\infty}^{\infty} 2\phi_1 \phi_2 dx$.

Perturbation theory: norming constants (1)

A deviation in ζ leads to small changes in Φ and as a consequence to non-zero δa and δb as well as their derivatives with respect to ζ . To find the perturbation $\delta\rho$, we take the variation of ρ :

$$\delta\rho = \delta(b/a') = \delta b/a' - b\delta a'/a'^2 = \rho(\delta b/b - \delta a'/a'). \quad (10)$$

The values b and a' at some $\{\zeta_n\}$ are assumed to be already known. According to boundary conditions, see (5), in order to obtain δb and $\delta a'$ we have to explore the variation of the solution Φ and $\Phi' = \partial_\zeta \Phi$ at $x \rightarrow \infty$. However, a multiplier $e^{i\zeta x}$ in boundary conditions does not make it straightforward. We rewrite the ZS system using a new variable:

$$\tilde{\Phi} = e^{i\zeta \Lambda x}, \quad \text{where } \Lambda = \text{diag}(1, -1), \quad (11)$$

leading to the system

$$\partial_x \tilde{\Phi} = \tilde{Q}_- \tilde{\Phi}, \quad \tilde{Q}_\pm = \begin{pmatrix} 0 & qe^+ \\ \pm q^* e^- & 0 \end{pmatrix}, \quad (12)$$

where we use the notation $e^\pm = e^{\pm 2i\zeta x}$ and $\tilde{Q}_\pm$.

We arrive to a modified set of boundary conditions:

$$\tilde{\Phi}|_{x \rightarrow -\infty} = (1, 0)^T, \quad \tilde{\Phi}|_{x \rightarrow \infty} = (a, b)^T. \quad (13)$$

Perturbation theory: norming constants (2)

This important simplification let us express δb and $\delta a'$ using variations $\delta\tilde{\Phi}$ and $\delta\tilde{\Phi}'$ at $x \rightarrow \infty$:

$$\delta\tilde{\Phi}|_{x \rightarrow \infty} = (\delta a, \delta b)^T, \quad \delta\tilde{\Phi}'|_{x \rightarrow \infty} = (\delta a', \delta b')^T. \quad (14)$$

Thus, to obtain $\delta\rho$ we have to compute $\delta\tilde{\Phi}(x)$ and $\delta\tilde{\Phi}'(x)$.

Taking the variation of Eq. (12), we obtain:

$$\partial_x \delta\tilde{\Phi} = \tilde{Q}_- \delta\tilde{\Phi} + \delta\tilde{Q}_- \tilde{\Phi}, \quad (15)$$

arriving to a nonhomogeneous equation for $\delta\tilde{\Phi}$.

To derive the equation for $\delta\tilde{\Phi}'$ we first differentiate Eq. (12) with respect to ζ and then take the variation since these operations do not commute:

$$\partial_x \delta\tilde{\Phi}' = \tilde{Q}_- \delta\tilde{\Phi}' + \delta\tilde{Q}_- \tilde{\Phi}' + \tilde{Q}'_- \delta\tilde{\Phi} + \delta\tilde{Q}'_- \tilde{\Phi}. \quad (16)$$

According to (13), at $x \rightarrow -\infty$ zero boundary conditions have to be imposed for $\delta\tilde{\Phi}$ and $\delta\tilde{\Phi}'$. The full expressions for the matrices are as follows:

$$\begin{aligned} \tilde{Q}'_- &= 2ix\tilde{Q}_+, \quad \delta\tilde{Q}_- = \begin{pmatrix} 0 & \delta q e^+ \\ -\delta q^* e^- & 0 \end{pmatrix} + 2i\delta\zeta x \tilde{Q}_+, \\ \delta\tilde{Q}'_- &= 2ix \begin{pmatrix} 0 & \delta q e^+ \\ \delta q^* e^- & 0 \end{pmatrix} - 4\delta\zeta x^2 \tilde{Q}_-. \end{aligned} \quad (17)$$

Perturbation theory: norming constants (3)

We extend the treatment for Eqs. (15), (16) to find δb and $\delta a'$ appearing in Eq. (10). Using both independent solutions of the ZS system, i.e. $\Phi = (\phi_1, \phi_2)^T$ and

$$\Psi = (\psi_1, \psi_2)^T = (-\phi_2^*, \phi_1^*)^T|_{\zeta=\zeta^*}, \quad (18)$$

we represent the solution of Eq. (15) as

$$\delta\tilde{\Phi} = f_1(x)\tilde{\Phi} + f_2(x)\tilde{\Psi}, \quad (19)$$

where variables with tilde are obtained in agreement with Eq. (11) and $f = (f_1, f_2)$ is to be determined.

Substituting $\delta\tilde{\Phi}$ (19) to Eq. (15) and using Eq. (12) and the notation $\mathcal{W}$, we obtain:

$$f_1'\tilde{\Phi} + f_2'\tilde{\Psi} = \mathcal{W}f' = \delta\tilde{Q}_-\tilde{\Phi}, \quad \mathcal{W} = (\tilde{\Phi}^T, \tilde{\Psi}^T). \quad (20)$$

The solution of Eq. (20) is as follows $f(x) = \int_{-\infty}^x \mathcal{W}^{-1}(y)\delta\tilde{Q}_-(y)\tilde{\Phi}(y)dy$, where the integration constant is zero due to zero boundary conditions of $\delta\tilde{\Phi}$ at $x \rightarrow -\infty$. Now we can recover the solution for $\delta\tilde{\Phi}$, see (19).

A similar scheme can be applied to Eq. (16) with the form $\delta\Phi' = g_1\Phi + g_2\Psi$ where for $g = (g_1, g_2)$ we can obtain $g(x) = \int_{-\infty}^x \mathcal{W}^{-1}(\delta\tilde{Q}_-\tilde{\Phi}' + \tilde{Q}'_-\tilde{\delta\Phi} + \delta\tilde{Q}'_-\tilde{\Phi})dy$.

Perturbation theory: reflection coefficient

The perturbation of the reflection coefficient is expressed as follows:

$$\delta r = \delta(b/a) = \delta b/a - b\delta a/a^2 = r(\delta b/b - \delta a/a). \quad (21)$$

Note that δr is defined on the real axis ξ , thus, $\delta\zeta = 0$ in Eqs. (15), (16), see also expressions (17).

A box potential (1)

As the basic unperturbed potential we consider a box function $q_{\square}(x) = A$ for $|x| < L/2$ where A is a real-valued constant, while $q_{\square} = 0$ otherwise. The scattering coefficients are as follows [Manakov, 1973]:

$$a(\zeta) = e^{iL\zeta} (\cos(\chi L) - i\zeta \sin(\chi L)/\chi), \quad (22)$$

$$b(\zeta) = -A \sin(\chi L)/\chi, \quad \chi = \sqrt{A^2 + \zeta^2}. \quad (23)$$

The wave function in the region $|x| < L/2$ represents:

$$\Phi = e^{\frac{i\zeta L}{2}} \frac{\nu}{1 - \nu^2} \begin{pmatrix} \nu^{-1} e^{-i\mu} - \nu e^{i\mu} \\ e^{-i\mu} - e^{i\mu} \end{pmatrix}, \quad (24)$$

$$\nu = i(\zeta - \chi)/A, \quad \mu = \chi L/2 + \chi x, \quad (25)$$

while for $|x| > L/2$ it corresponds to the asymptotics (5).

A box potential (2)

We recall that each eigenvalue $\zeta_n = \xi_n + i\eta_n$ corresponds to a soliton. The condition $a(\zeta_n) = 0$ with $\{\eta_n\} > 0$ for $n = 1, \dots, N$ guaranties the decay of the wave function according to asymptotics (5) leading to physically meaningful soliton eigenvalues $\{\zeta_n\}$:

$$\Phi|_{x \rightarrow -\infty} = (e^{-i\zeta x}, 0)^T, \quad \Phi|_{x \rightarrow \infty} = (ae^{-i\zeta x}, be^{i\zeta x})^T. \quad (26)$$

At the same time, the condition $a(\zeta_n) = 0$ can also be satisfied for $\{\eta_n\} < 0$ with $n = -1, -2, \dots$ with the exponentially growing wave function (26). We refer to these $\{\zeta_n\}$ distinguished by negative indexes n as nonphysical zeros of $a(\zeta)$ or virtual soliton eigenvalues, the number of which can be infinite.

The condition $a(\zeta_n) = 0$ for both cases $\{\eta_n\} > 0$ and $\{\eta_n\} < 0$ leads to the transcendental equation:

$$\tan(\chi_n L) = -i\chi_n/\zeta_n, \quad \chi_n = \sqrt{A^2 + \zeta_n^2}. \quad (27)$$

Using (22)–(27) we express norming constants (6) as:

$$\rho_n = -i\chi_n^2 e^{-i\zeta_n L} / [A(1 - i\zeta_n L)]. \quad (28)$$

Note that the number of solitons in the box is limited by $N = \text{Integer}[1/2 + AL/\pi]$ and all the eigenvalues are aligned on the imaginary axis, i.e. $\{\zeta_n\} = i\{\eta_n\}$ for $n = 1, \dots, N$ and the solitons have zero velocities.

A box potential (3)

First we study the evolution of the roots of $e^{-iL\zeta}a(\zeta)$ according to Eq. (22) for different values of A and $L = 10$. Fig. 1 illustrates a situation when a simple box-like perturbation moves a nonphysical zero to the region $\eta > 0$ leading to the emergence of a soliton from its virtual counterpart.

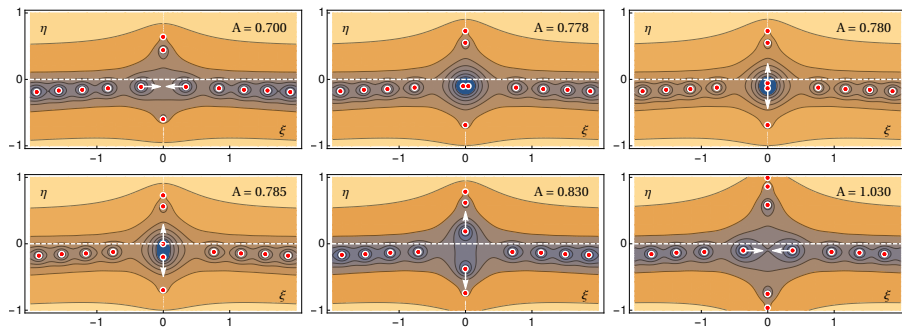


Figure 1: Contours of $|e^{-iL\zeta}a(\zeta)|$ for different values of A and $L = 10$. Red-in-white points denote the roots of this expression – physical $\{\eta_n\} > 0$ and virtual $\{\eta_n\} < 0$ soliton eigenvalues. White dashed line shows the real axis, while white arrows show the direction of a pair of roots in focus with the increase of A .

Harmonic perturbations

As a perturbation we consider real- and imaginary-valued Fourier modes:

$$\delta q^{\text{re}} = \varepsilon \cos(kx + \varphi), \quad \delta q^{\text{im}} = i\delta q^{\text{re}}, \quad (29)$$

where ε is a small parameter while k and φ are the wavelength and phase, respectively. The following exact expressions are obtained:

$$\delta \zeta_n^{\text{re}} = i\varepsilon h^{\text{re}}(k, \zeta_n) \cos \varphi, \quad \delta \zeta_n^{\text{im}} = \varepsilon h^{\text{im}}(k, \zeta_n) \sin \varphi, \quad (30)$$

$$\delta \rho_n^{\text{re}} = i\varepsilon [s_1^{\text{re}}(k, \zeta_n) \cos \varphi + s_2^{\text{re}}(k, \zeta_n) \sin \varphi], \quad (31)$$

$$\delta \rho_n^{\text{im}} = \varepsilon [s_1^{\text{im}}(k, \zeta_n) \cos \varphi + s_2^{\text{im}}(k, \zeta_n) \sin \varphi], \quad (32)$$

where

$$h^{\text{re}} = \frac{2\chi_n^2(\{k^2 - 4A^2\} \sin \varkappa - 2k\eta_n \cos \varkappa)}{Ak^3(1 + \beta_n L)(1 - \gamma_n^2)}, \quad h^{\text{im}} = -\frac{2\chi_n^2(2\eta_n \sin \varkappa + k \cos \varkappa)}{Ak^2(1 + \beta_n L)(1 - \gamma_n^2)},$$

$$s_1^{\text{re}} = \frac{2(p_{1,n} \sin \varkappa + p_{2,n} \cos \varkappa)}{Ak^3(1 + \eta_n L)^2(1 - \gamma_n^2)^2}, \quad s_1^{\text{im}} = \frac{2i(k\eta_n \sin \varkappa + 2\chi_n^2 \cos \varkappa)}{Ak^2(1 - \gamma_n^2)},$$

$$s_2^{\text{re}} = \frac{2\eta_n(2\eta_n \sin \varkappa + k \cos \varkappa)}{Ak^2(1 - \gamma_n^2)}, \quad s_2^{\text{im}} = -\frac{2i\eta_n(p_{3,n} \sin \varkappa + p_{4,n} \cos \varkappa)}{Ak^2(1 + \eta_n A)^2(1 - \gamma_n^2)^2},$$

with $\gamma_n^2 = 4\chi_n^2/k^2$, $\varkappa = kL/2$ and

$$p_{1,n} = \eta_n[4A^2(3 + 2\eta_n L - 2\chi_n^2 L^2 - \gamma_n^2\{1 + \eta_n L - \chi_n^2 L^2\}) + k^2(-2 + A^2 L^2 - \eta_n L) + 4\eta_n^3 L],$$

$$p_{2,n} = -2k[\chi_n^2(1 - \gamma_n^2) + \eta_n(1 + \eta_n L)\{A^2 L(1 - \gamma_n^2) - 2\eta_n\}]$$

$$p_{3,n} = 2\{3\eta_n(1 + L\eta_n) + L\chi_j^2 - (LA^2 + \eta_n)\gamma_j^2\},$$

$$p_{4,n} = k\{2 - L^2 A^2 + L\eta_n + L(LA^2 + \eta_n)\gamma_j^2\}.$$

Numerical verification of the perturbation theory

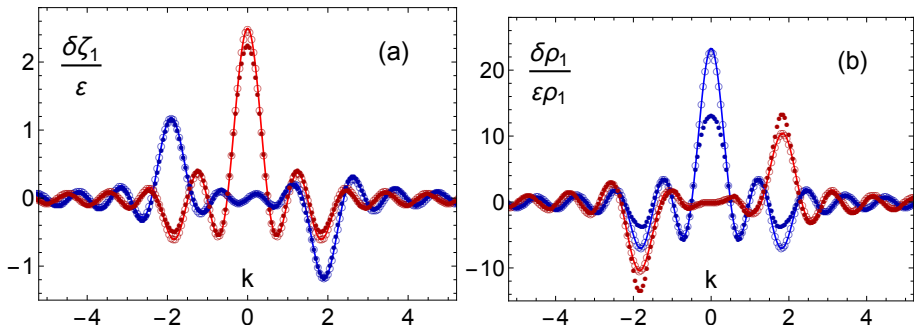


Figure 2: Numerical verification of the perturbation theory using the first of 4 solitons contained the box with $A = 1$ and $L = 12$ perturbed by a complex harmonics εe^{ikx} with different ε and k . (a): real (blue) and imaginary (red) parts of the soliton eigenvalue deviation $\delta\zeta_1$ normalized by ε . (b): real (blue) and imaginary (red) part of the soliton norming constant deviation $\delta\rho_1$ normalized by $\varepsilon\rho_1$. On the both pictures the solid lines show theoretical predictions, while circles and dots demonstrate precise numerical solution for $\varepsilon = 0.001$ and $\varepsilon = 0.05$ correspondingly.

See for the numerical approaches our recent papers: [R. Mullyadzhanov and A. Gelash, Optics Letters (2019)] and [A. Gelash and R. Mullyadzhanov, Physical Review E, 2020].

Numerical verification of the perturbation theory at large L

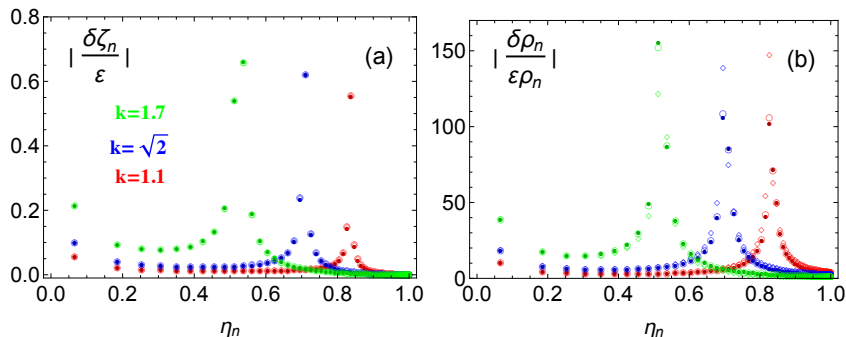


Figure 3: Numerical verification of the perturbation theory using 64 solitons contained in the box having $A = 1$ and $L = 200$, perturbed by a complex harmonics εe^{ikx} with different ε and $k = 1.1, \sqrt{2}, 1.7$. (a): absolute value of the soliton eigenvalue deviation $\delta\zeta_n$ depending on η_n and normalized by ε . (b): absolute value of the soliton norming constant deviation $\delta\rho_n$ depending on η_n and normalized by $\varepsilon\rho_n$. On the both pictures dots show theoretical predictions, while circles and diamond demonstrate precise numerical solution for $\varepsilon = 0.5 \cdot 10^{-4}$ and $\varepsilon = 0.5 \cdot 10^{-3}$ correspondingly.

We consider a sum of modes (29) with random phases and distributed as $\mathcal{F}(k)$ with respect to k . Integrating Eqs. (30)-(32) over φ , we obtain the following expressions for standard deviations:

$$(\sigma_{\zeta,n}^{\text{re/im}})^2 = \frac{\varepsilon^2}{2} \int_{-\infty}^{\infty} \mathcal{F}(k) |h^{\text{re/im}}|^2 dk, \quad (33)$$

$$(\sigma_{\rho,n}^{\text{re/im}})^2 = \frac{\varepsilon^2}{2} \int_{-\infty}^{\infty} \mathcal{F}(k) (|s_1^{\text{re/im}}|^2 + |s_2^{\text{re/im}}|^2) dk. \quad (34)$$

These expressions describe the effect of noise on the discrete spectrum for the box potential. The convergence of (33), (34) is guaranteed by an algebraic decay of $\delta\zeta_n$ and $\delta\rho_n$ for large k .

These integrals were evaluated analytically for the white noise model, i.e. $\mathcal{F}(k) = 1$:

$$(\sigma_{\zeta,n}^{\text{re}})^2 = \frac{\pi\varepsilon^2\chi_n^2(\eta_n + 2LA^2)}{2A^2(1 + L\eta_n)^2} + \frac{3\pi\varepsilon^2\eta_n}{2(1 + L\eta_n)}, \quad (35)$$

$$(\sigma_{\zeta,n}^{\text{im}})^2 = \pi\varepsilon^2\chi_n^2(\eta_n + LA^2)/[2A(1 + L\eta_n)^2], \quad (36)$$

while $\sigma_{\rho,n}$ is rather cumbersome and omitted here.

Statistical description: numerical verification

We simulate a white noise as the normalized collection of M modes with random phases $\varphi_m^{\text{ns},j}$:

$$\delta q_{\text{ns},j}(x) = \varepsilon \sqrt{\Delta k} \sum_{m=1}^M \cos(x \Delta k m + \varphi_m^{\text{ns},j}), \quad (37)$$

where the subscript 'ns, j ' denotes a particular j th set of random phases.

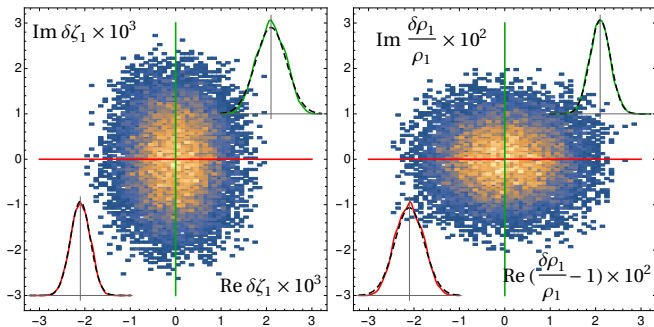


Figure 4: Soliton scattering data deviations induced by 10^4 realizations of noise $\delta q(x) = \delta q_{\text{ns},1} + i\delta q_{\text{ns},2}$ superimposed on the box with $A = 1$ and $L = 12$ computed numerically. The insets show numerical (green solid) and theoretical (grey dashed) PDFs for soliton parameters. The unperturbed values of ζ_1 and ρ_1 are computed using (27) and (28).

Noise-induced soliton generation

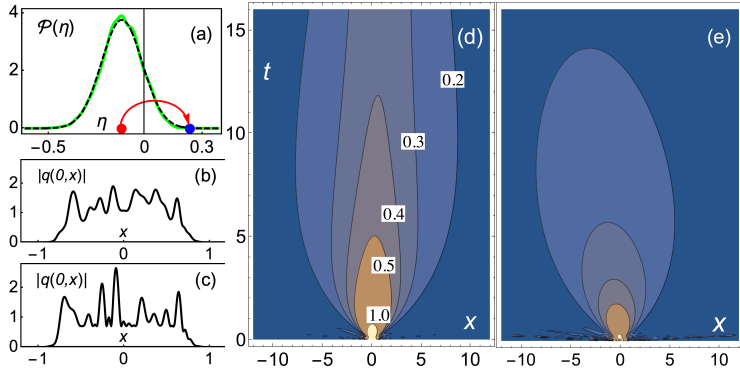


Figure 5: (a) The shift of the nonphysical root to the upper ζ -plane for a particular realization of a real-valued noise within a numerical (green solid) and theoretical (grey dashed) amplitude PDF. Contour plots of $|q(t, x)|$ from numerical simulations of NLSE. Evolution of box with $A = 1$, $L = 1.46$ perturbed by real (b, d) and imaginary (c, e) noise when a soliton is induced and not, respectively. The theoretically predicted and numerically obtained parameters of the induced soliton are $\zeta_{\text{pr}} = 0.223i$, $\zeta_{\text{num}} = 0.232i$ with $\rho_{\text{pr}} = -1.275i$, $\zeta_{\text{act}} = -1.311i$.

Conclusions

- ✓ We presented a complete theoretical framework to evaluate first-order corrections of the full set of scattering data within the NLSE model and applied it to a classic box potential, which can be generalised to other integrable systems and wavefields.
- ✓ In addition to the classical result for eigenvalues as in (9), we derived general expressions, see Eqs. (15), (16), leading to the knowledge of soliton phase and position sensitivity.
- ✓ Starting from a single Fourier mode we obtained statistical integrals (33), (34) allowing to determine the impact of a random-phase noise on soliton parameters which is important in the studies of the spontaneous modulation instability and in a number of applications.
- ✓ The introduction of a concept of a virtual soliton with nonphysical zeros of $a(\zeta)$ allowed us to accurately predict the noise-induced emergence of a soliton.
- ✓ A similar concept to describe soliton emergence can be further developed for the NLSE model with external pumping.

We thank Dr D. Agafontsev for fruitful discussions on virtual soliton eigenvalues.

Mullyadzhyanov R., Gelash A. Solitons in a box-shaped wavefield with noise: perturbation theory and statistics. arXiv:2008.08874, August 2020.