Reconstructing Sea Surface Dynamics Using a Linear Koopman Kalman Filte

Said Ouala[1], Ronan Fablet[1], Ananda Pascual[2], Bertrand Chapron[3], Fabrice Collard[4] and Lucile Gaultier[4]

1) IMT-Atlantique, Brest, France
2) IMEDEA, Esporles, Spain
3) Ifremer, Plouzané, France
4) ODL, Locmaria-Plouzané, France

April, 2021

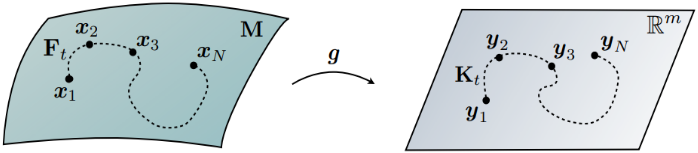
Reduced order models and Koopman formalism

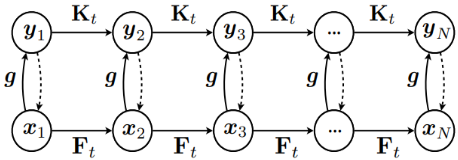
- Given the following state space model

$$\begin{cases} \dot{\mathbf{z}}_t &= f(\mathbf{z}_t) \\ \mathbf{x}_t &= \mathcal{H}(\mathbf{z}_t, \Omega_t, \epsilon_t) \end{cases} \tag{1}$$

- The resolution of the dynamical equation for forecast and data assimilation is challenging
- The main goal of all this work: reduced order models
- Koopman representations can provide a good short term forecast, and are suitable for data assimilation
- Data driven Koopman for forecast and data assimilation !

Koopman formalism





$$\mathbf{F}_t : x_k \mapsto x_{k+1}$$

$$g : x_k \mapsto y_k$$

$$\mathbf{K}_t : y_k \mapsto y_{k+1}$$

Source: Brunton et al. (2016)

Koopman formalism

Vol. 17, 1931 *MATHEMATICS: B. O. KOOPMAN* 315

HAMILTONIAN SYSTEMS AND TRANSFORMATIONS IN HILBERT SPACE

By B. O. Koopman

Department of Mathematics, Columbia University

Communicated March 23, 1931

In recent years the theory of Hilbert space and its linear transformations has come into prominence.[1] It has been recognized to an increasing extent that many of the most important departments of mathematical physics can be subsumed under this theory. In classical physics, for example in those phenomena which are governed by linear conditions—linear differential or integral equations and the like, in those relating to harmonic analysis, and in many phenomena due to the operation of the laws of chance, the essential rôle is played by certain linear transformations in Hilbert space. And the importance of the theory in quantum mechanics is known to all. It is the object of this note to outline certain investigations of our own in which the domain of this theory has been extended in such a way as to include classical Hamiltonian mechanics, or, more generally, systems defining a steady n-dimensional flow of a fluid of positive density.

Consider the dynamical system of n degrees of freedom, the canonical equations of which are formed from the Hamiltonian $H(q, p) = H(q_1, \ldots, q_n, p_1, \ldots, p_n)$, which we will assume to be single-valued, real, and

- Can new artificial intelligence techniques help finding good finite dimensional approximations of this operator ?

Koopman formalism

Modern state of the art

- Let us assume the following state space model

$$\left\{ \begin{array}{lcl} \dot{\mathbf{z}}_t & = & f(\mathbf{z}_t) \\ \mathbf{x}_t & = & \mathcal{H}(\mathbf{z}_t, \Omega_t, \epsilon_t) \end{array} \right. \tag{2}$$

Dynamic Mode Decomposition (DMD) (Schmid (2010)).

$\dot{\mathbf{x}}_t = A\mathbf{x}_t$

Koopman formalism

Modern state of the art

- Let us assume the following state space model

$$\begin{cases} \dot{\mathbf{z}}_t &= f(\mathbf{z}_t) \\ \mathbf{x}_t &= \mathcal{H}(\mathbf{z}_t, \Omega_t, \epsilon_t) \end{cases} \tag{2}$$

Dynamic Mode Decomposition (DMD) (Schmid (2010)).

$\dot{\mathbf{x}}_t = A\mathbf{x}_t$

Extended Dynamic Mode Decomposition (EDMD) (Li et al. (2017)).

$\dot{\mathbf{u}}_t = A\mathbf{u}_t$ with
$\mathbf{u}_t^T = [\mathbf{x}_t^T, f_p(\mathbf{x}_t)^T]$

Koopman formalism

Modern state of the art

- Let us assume the following state space model

$$\begin{cases} \dot{\mathbf{z}}_t & = & f(\mathbf{z}_t) \\ \mathbf{x}_t & = & \mathcal{H}(\mathbf{z}_t, \Omega_t, \epsilon_t) \end{cases} \quad (2)$$

Dynamic Mode Decomposition (DMD) (Schmid (2010)).	Extended Dynamic Mode Decomposition (EDMD) (Li et al. (2017)).	Deep learning based approaches (Lusch et al. (2017))
$\dot{\mathbf{x}}_t = A\mathbf{x}_t$	$\dot{\mathbf{u}}_t = A\mathbf{u}_t$ with $\mathbf{u}_t^T = [\mathbf{x}_t^T, f_p(\mathbf{x}_t)^T]$	$\dot{\mathbf{u}}_t = A\mathbf{u}_t$ with $\mathbf{u}_t^T = f_{NN}(\mathbf{x}_t)^T$

Koopman formalism

Modern state of the art

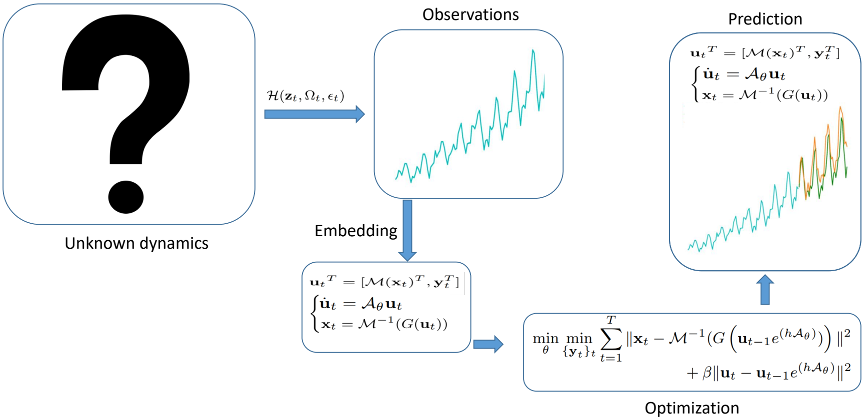
- Let us assume the following state space model

$$\left\{ \begin{array}{lcl} \dot{\mathbf{z}}_t & = & f(\mathbf{z}_t) \\ \mathbf{x}_t & = & \mathcal{H}(\mathbf{z}_t, \Omega_t, \epsilon_t) \end{array} \right. \quad (2)$$

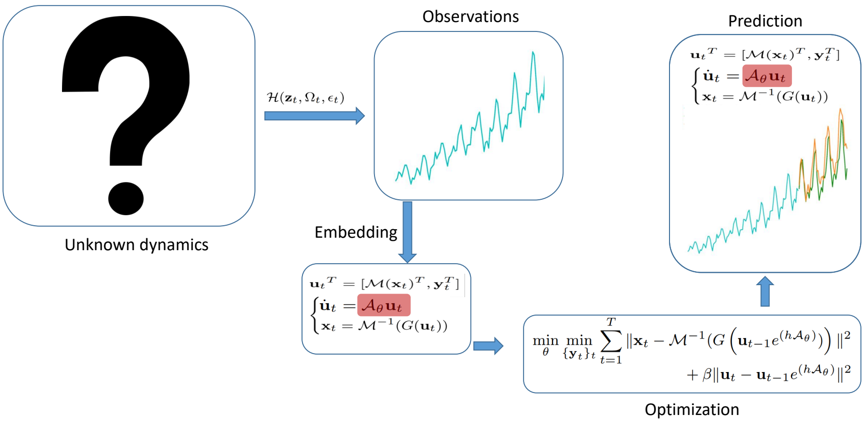
Dynamic Mode Decomposition (DMD) (Schmid (2010)).	Extended Dynamic Mode Decomposition (EDMD) (Li et al. (2017)).	Deep learning based approaches (Lusch et al. (2017))
$\dot{\mathbf{x}}_t = A\mathbf{x}_t$	$\dot{\mathbf{u}}_t = A\mathbf{u}_t$ with $\mathbf{u}_t^T = [\mathbf{x}_t^T, f_p(\mathbf{x}_t)^T]$	$\dot{\mathbf{u}}_t = A\mathbf{u}_t$ with $\mathbf{u}_t^T = f_{NN}(\mathbf{x}_t)^T$

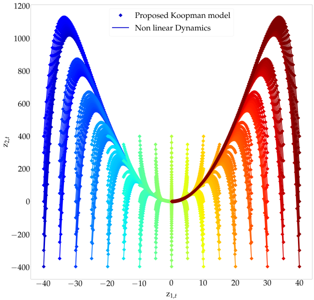
- Does not account for missing processes or use Takens
- The augmented space depends on predefined parametric family.

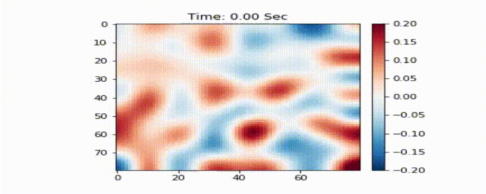
Proposed augmented Koopman framework

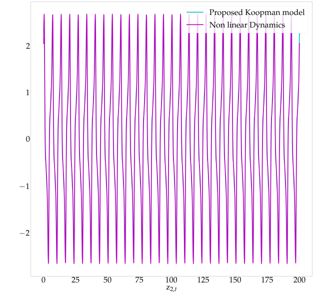


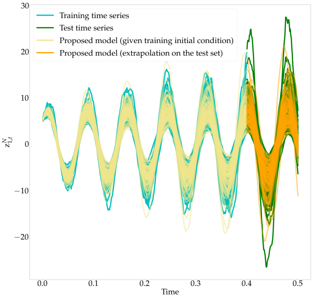
Proposed augmented Koopman framework




Proposed Koopman model
Non linear Dynamics
$z_{2,t}$
$z_{1,t}$


Time: 0.00 Sec


Proposed Koopman model
Non linear Dynamics
$z_{2,t}$

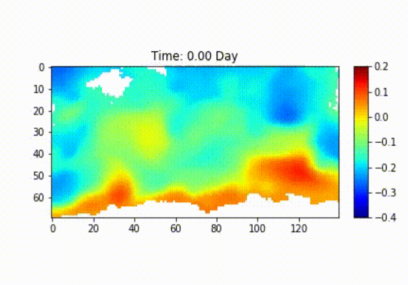

Training time series
Test time series
Proposed model (given training initial condition)
Proposed model (extrapolation on the test set)
$z^N_{1,t}$
Time

Proposed augmented Koopman framework

WMOP case study

- The considered case study area.

WMOP simulation

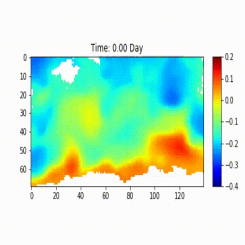


Proposed augmented Koopman framework

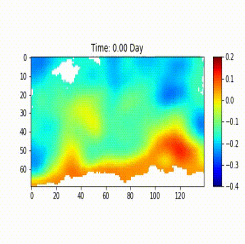
WMOP case study

- Can provide a good short term forecast on chaotic dynamics.

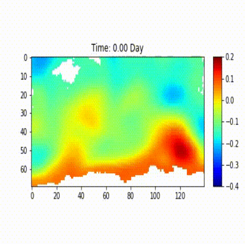
WMOP



Projection



Koopman

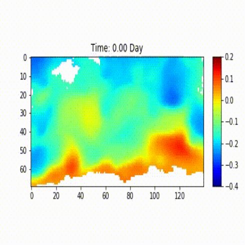


Proposed augmented Koopman framework

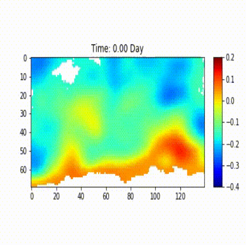
WMOP case study

- Can provide a good short term forecast on chaotic dynamics.

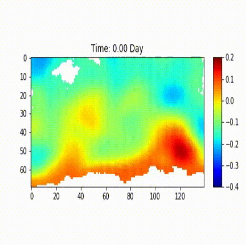
WMOP



Projection

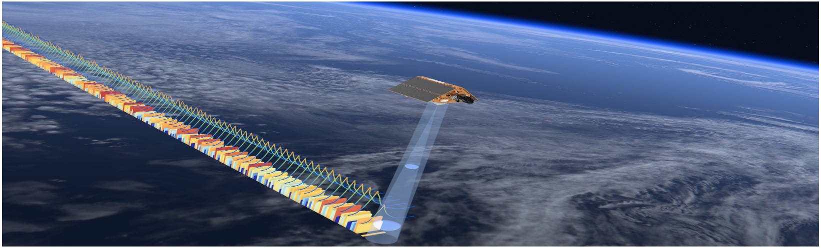


Koopman



Spatio-temporal interpolation of the Seal Level Anomaly

Along track data sampling gap

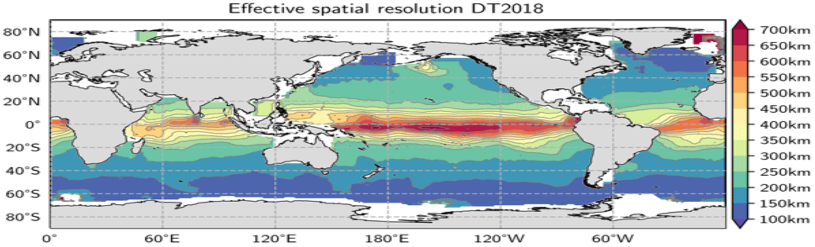


Source: esa.int

- These observations involve very high missing data rates
- The spatio-temporal interpolation of SLA fields from along track data involves several altimeters.

Spatio-temporal interpolation of the Seal Level Anomaly

Along track data sampling gap

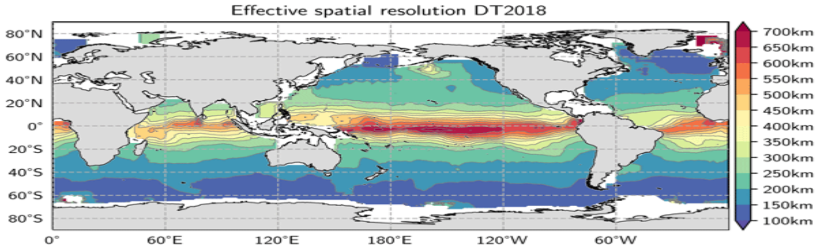


Source: Ballarotta et al. (2019)

- The resolved scales from operational products using state of the art optimal interpolation are large.

Spatio-temporal interpolation of the Seal Level Anomaly

Along track data sampling gap



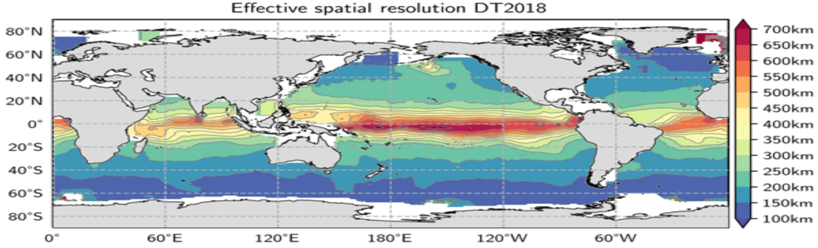
Source: Ballarotta et al. (2019)

- The resolved scales from operational products using state of the art optimal interpolation are large.

Rely on new sensing missions

Spatio-temporal interpolation of the Seal Level Anomaly

Along track data sampling gap



Source: Ballarotta et al. (2019)

- The resolved scales from operational products using state of the art optimal interpolation are large.

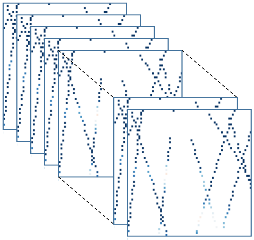
Rely on new sensing missions

Exploit learning based approaches

Spatio-temporal interpolation of the Seal Level Anomaly

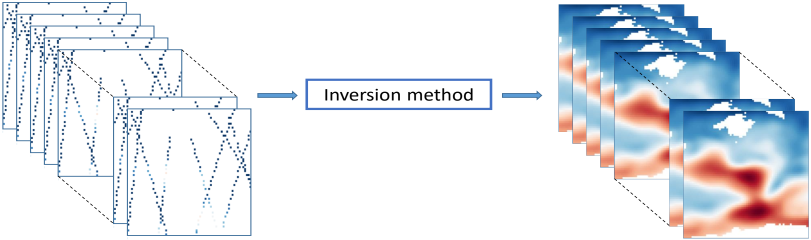
State of the art

The spatio-temporal SLA interpolation problem can be defined as :



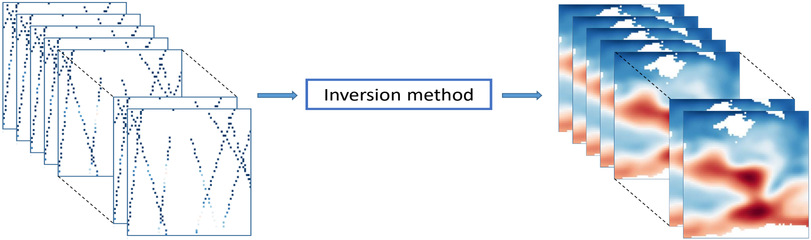
Spatio-temporal interpolation of the Seal Level Anomaly

State of the art



Spatio-temporal interpolation of the Seal Level Anomaly

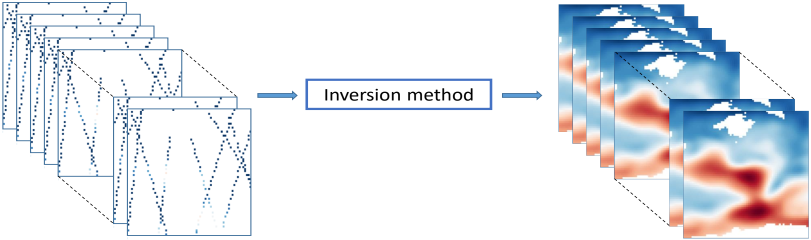
State of the art



Plug-and-play schemes.

Spatio-temporal interpolation of the Seal Level Anomaly

State of the art

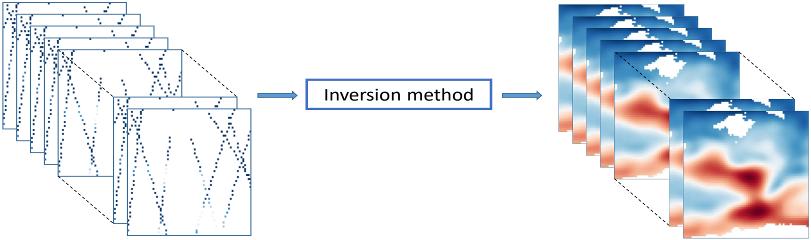


Plug-and-play schemes.

End-to-end learning schemes.

Spatio-temporal interpolation of the Seal Level Anomaly

State of the art



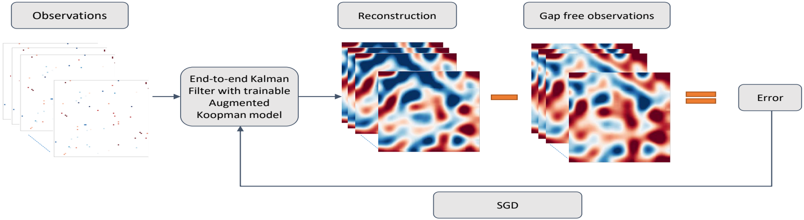
Plug-and-play schemes.

End-to-end learning schemes.

End-to-end sequential filtering.

Ocean surface data assimilation

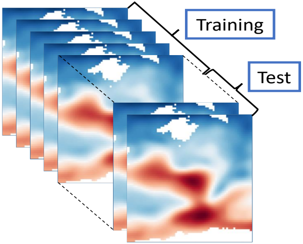
Augmented Koopman Kalman filter



- A linear model that propagates the observations in time
- A Kalman filter that assimilates along-track data to the model

Augmented Koopman Kalman filter

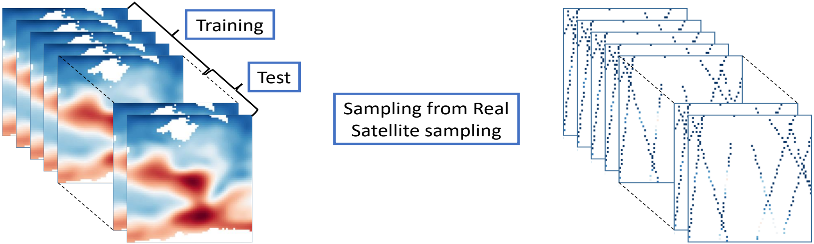
OSSE based case study



- OSSE based on realistic high-resolution ocean simulation data in the Western Mediterranean sea from WMOP configuration (Juza et al. (2016)).
- The data from January 2009 to December 2014 were used as training and we tested our approach on the first 347 days of the year 2015.

Augmented Koopman Kalman filter

OSSE based case study



- $\mathbf{x}_{\Omega_t,t}$ is generated from real satellite tracks from a four-altimeter sampling configuration in 2014.

Augmented Koopman Kalman filter

WMOP case study, results vs OI and AnDA

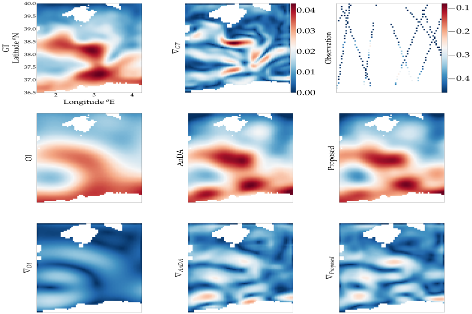
- Quantitative analysis, RMSE of the reconstructed fields and their gradients.

Model	Entire map				Missing data areas			
	RMSE		Correlation		RMSE		Correlation	
	$SLA(m)$	$\nabla SLA(m^{\circ})$	SLA	∇SLA	$SLA(m)$	$\nabla SLA(m/^{\circ})$	SLA	∇SLA
Proposed, E2EKF	**0.021**	**0.0041**	**96.22%**	**77.51%**	**0.022**	**0.0043**	**97.95%**	**79.59%**
LAF-EnKF	0.023	0.0043	95.79%	75.78%	0.025	0.0044	97.51%	77.54%
OI	0.036	0.0062	90.84%	60.01%	0.037	0.0063	94.50%	62.98%

Augmented Koopman Kalman filter

WMOP case study, results vs OI and AnDA

- Qualitative analysis, visual reconstruction example on the February 19, 2015.



Takeaway messages

- We proposed a new algorithm for the identification of linear approximations of high dimensional non linear dynamics.
- The proposed framework is relevant in the forecasting of various dynamical regimes
- The linearity of the model makes it relevant when considering spatio-temporal interpolation applications

Conclusion

Better then non-linear filtering ?

How to estimate model and observation errors ?

Bibliography I

M. Ballarotta, C. Ubelmann, M.-I. Pujol, G. Taburet, F. Fournier, J.-F. Legeais, Y. Faugère, A. Delepoulle, D. Chelton, G. Dibarboure, and N. Picot. On the resolutions of ocean altimetry maps. *Ocean Science*, 15(4):1091–1109, 2019. doi: 10.5194/os-15-1091-2019. URL `https://os.copernicus.org/articles/15/1091/2019/`.

S. L. Brunton, B. W. Brunton, J. L. Proctor, and J. N. Kutz. Koopman invariant subspaces and finite linear representations of nonlinear dynamical systems for control. *PloS one*, 11(2):e0150171, 2016.

Bibliography II

M. Juza, B. Mourre, L. Renault, S. Gómara, K. Sebastián, S. Lora, J. P. Beltran, B. Frontera, B. Garau, C. Troupin, M. Torner, E. Heslop, B. Casas, R. Escudier, G. Vizoso, and J. Tintoré. Socib operational ocean forecasting system and multi-platform validation in the western mediterranean sea. *Journal of Operational Oceanography*, 9(sup1):s155–s166, 2016. doi: 10.1080/1755876X.2015.1117764. URL `https://doi.org/10.1080/1755876X.2015.1117764`.

Q. Li, F. Dietrich, E. M. Bollt, and I. G. Kevrekidis. Extended dynamic mode decomposition with dictionary learning: A data-driven adaptive spectral decomposition of the koopman operator. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 27(10):103111, 2017.

B. Lusch, S. L. Brunton, and J. N. Kutz. Data-driven discovery of koopman eigenfunctions using deep learning. *APS*, pages M1–006, 2017.

Bibliography III

P. J. Schmid. Dynamic mode decomposition of numerical and experimental data. *Journal of fluid mechanics*, 656:5–28, 2010.