Poroelastoplastic modeling of borehole shear bands on high order curvilinear meshes using CUDA technology

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Plan

1. **Problem statement. Math models.**
2. Numerical solution using spectral element method.
3. Parallel implementation on GPU.
4. Examples.

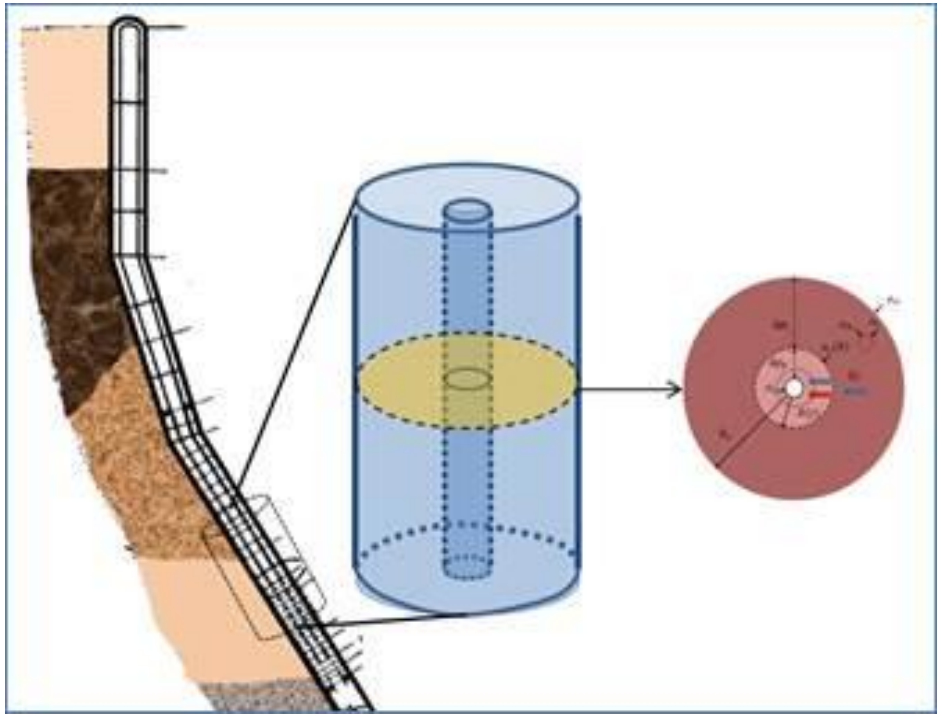
Wellbore stability problem

One of the key problems of geomechanics is the determination of technological parameters, for which the wellbore will maintain its stability.

Different rock’s properties (modulus of elasticity, Poisson's ratio, density, friction and dilatancy angles, strength and yield strengths for tension and compression, adhesion, porosity, permeability, compressibility, etc) should be taken into account.

In addition, the rock is prestressed, which is determined by the components of the generally anisotropic nonuniform stress tensor.

When drilling, in general, a bit and mud generates a pressure on the rock, thereby deforming it and **redistributing** the stresses (superposition of generally finite strains), causing the reaction of the rock on the applied impact. This may lead to the formation and development of localized zones of plastic shear bands.



Math model

Nonstationary poroelastoplasticity, two-way coupling
(O. Coussy's "Poromechanics", 2004; S. Pride, 2005):

$$\begin{cases} \nabla \cdot \sigma = \rho(\ddot{u} + d_u \dot{u}) + \rho_f(\ddot{w} + d_{uw}\dot{w}) \\ -\nabla p_f = \rho_f(\ddot{u} + d_{uw}\dot{u}) + \dfrac{\alpha\rho_f}{\phi}(\ddot{w} + d_w\dot{w}) + F_{fr} \end{cases}$$

$F_{fr} = \dfrac{\eta}{k}\dot{w}$ − interphase friction force (for low frequency case)

$$w = \phi(U - u)$$

$$\rho = \rho_s(1 - \phi) + \rho_f\phi$$

u – displacement vector of solid skeleton,
U – displacement vector of saturating fluid,
ϕ - porosity,
ρ_s, ρ_f - densities of porous solid skeleton and saturating fluid,
η - viscosity of saturating fluid,
α - tortuosity of pore channels,

$k(\phi) = k_0\left(\dfrac{\phi}{\phi_0}\right)^3\left(\dfrac{1-\phi_0}{1-\phi}\right)^2$ - dependency of permeability on porosity (Kozeny-Carman equation),

d_i – damping parameters.

Material model

Constitutive relations:

$$\sigma = \lambda(\varepsilon_e : I)I + 2\mu\varepsilon_e - bp_f I$$

$$p_f = M(\zeta_e - b(\varepsilon_e : I))$$

$b=1-K_d/K_s$ – Biot-Willis coefficient

$K_d(\phi) = K_s \dfrac{1-\phi}{1+c\phi}$ - drained bulk modulus of solid skeleton

$\mu(\phi) = \mu_s \dfrac{1-\phi}{1+3c\phi/2}$ - drained shear modulus of solid skeleton

c – consolidation parameter

$\lambda = K_d - 2\mu/3$ – Lame parameter

K_s – bulk modulus of mineral grains,

μ_s – shear modulus of mineral grains,

K_f – bulk modulus of fluid,

$M(\phi) = \dfrac{1}{\dfrac{\phi}{K_f} + \dfrac{1-\phi}{K_s} - \dfrac{K_d}{K_s^2}}$ - Biot modulus,

ζ_e - change in fluid volume content per unit volume of mixture (change of fluid content)

Dynamic porosity models:

Matrix porosity depends on volumetric strains and initial porosity:

$$\phi = \phi_i(1+\varepsilon_v)$$

$$\varepsilon_v = \frac{\Delta V}{V} = (1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3) - 1 =$$
$$= \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \{\varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_2 + \varepsilon_1\varepsilon_2\varepsilon_3\}$$

For small strains: $\varepsilon_v = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$

$$\phi = \phi_0 e^{\frac{cK_d}{K_s}\left(\frac{p_f}{K_s} + \varepsilon_e : I\right)}$$ (S. Pride, 2005)

$$\phi = \frac{\phi_0 - \nabla \cdot w}{1 + \dfrac{p_f}{K_f}}$$ (Coussy“Poromechanics”,2004)

Poroelastoplastic model

Additive decomposition in case of small strains:

$$\varepsilon_e = \varepsilon - \varepsilon_p = \frac{1}{2}(\nabla u + u\nabla) - \varepsilon_p$$

$$\zeta_e = \zeta - \zeta_p = -\nabla \cdot w - \zeta_p$$

Plastic flow rule:

$$d\varepsilon_p = d\lambda \frac{\partial Q(\sigma, p_f)}{\partial \sigma}; \; d\zeta_p = d\lambda \frac{\partial Q(\sigma, p_f)}{\partial p_f};$$

Non-associative Drucker-Prager model:

$$F(\sigma, p_f) = \sqrt{\frac{1}{2}(\sigma - \frac{\sigma : I}{3} I) : (\sigma - \frac{\sigma : I}{3} I)} - B(\sigma : I + 3p_f) - A$$ - plastic criterion

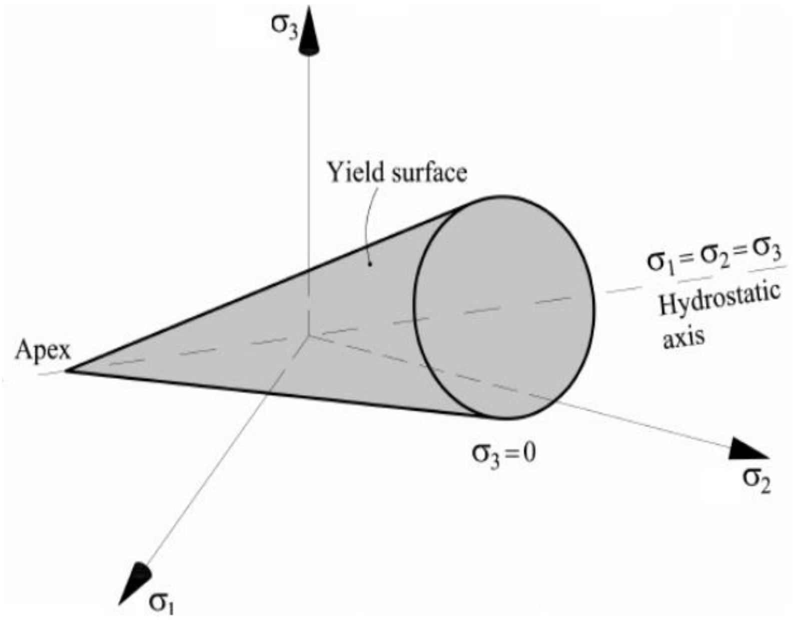
$$Q(\sigma, p_f) = \sqrt{\frac{1}{2}(\sigma - \frac{\sigma : I}{3} I) : (\sigma - \frac{\sigma : I}{3} I)} - C(\sigma : I + 3p_f)$$ - plastic potential

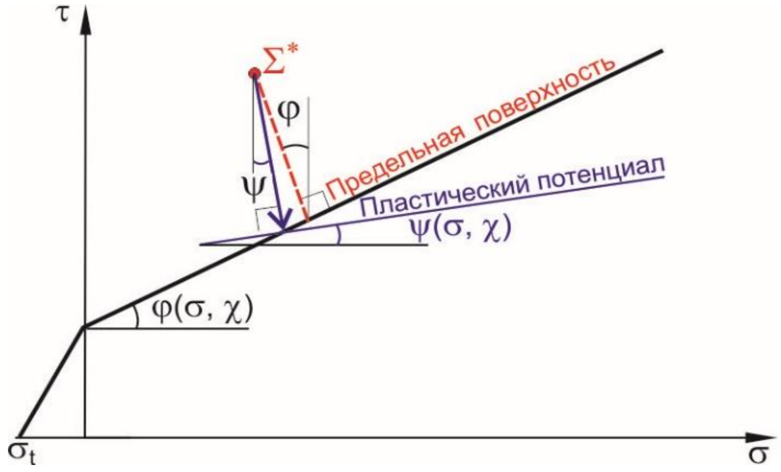
$$A = \frac{6c\cos(\varphi)}{\sqrt{3}(3 - \sin(\varphi))}; \; B = \frac{-2\sin(\varphi)}{\sqrt{3}(3 - \sin(\varphi))}; \; C = \frac{-2\sin(\psi)}{\sqrt{3}(3 - \sin(\psi))}$$

φ – effective internal friction angle,

ψ – effective dilatation angle,

c – cohesion.





Weak formulation

$$\forall N \in \Omega$$

$$\iiint_V (\rho \ddot{u} + \rho_f \ddot{w}) N dV = \iiint_V \nabla \cdot \sigma N dV$$

$$\Leftrightarrow$$

$$\iiint_V (\rho_f \ddot{u} + \frac{\alpha \rho_f}{\phi} \ddot{w} + F_{fr}) N dV = -\iiint_V \nabla p_f N dV$$

$$\Leftrightarrow$$

$$\forall N \in \Omega$$

$$\iiint_V (\rho \ddot{u} + \rho_f \ddot{w}) N dV = -\iiint_V \sigma \nabla N dV + \iint_{S_I} N \vec{\sigma}_n dS + external$$

$$\iiint_V (\rho_f \ddot{u} + \frac{\alpha \rho_f}{\phi} \ddot{w} + F_{fr}) N dV = \iiint_V p_f \nabla N dV + \iint_{S_I} N P_f \vec{n} dS + external$$

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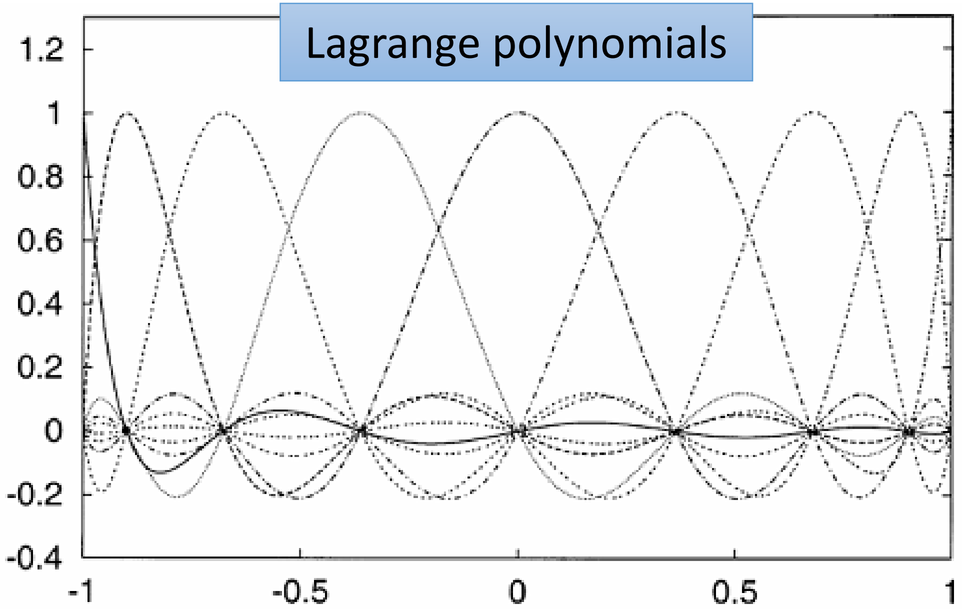
Spectral element method (SEM)

Basis functions — Lagrange polynomials (reconstructed based on roots of Legendre polynomials), providing a spectral convergence in space



$$l_\alpha^{n_l}(\xi) = \frac{(\xi - \xi_0)}{(\xi_\alpha - \xi_0)} \cdots \frac{(\xi - \xi_{\alpha-1})}{(\xi_\alpha - \xi_{\alpha-1})} \frac{(\xi - \xi_{\alpha+1})}{(\xi_\alpha - \xi_{\alpha+1})} \cdots \frac{(\xi - \xi_{n_l})}{(\xi_\alpha - \xi_{n_l})}$$

$$\xi_\alpha \ - \ GLL - nodes$$

$$l_\alpha^{n_l}(\xi_\beta) = \delta_{\alpha\beta}$$

Orthogonal basis using GLL-quadrature

$$\int_{-1}^{1} l_m(\xi) l_n(\xi) d\xi = \frac{2}{2n+1} \delta_{mn}$$

$$f(\mathbf{x}(\xi,\eta,\zeta)) \approx \sum_{\alpha,\beta,\gamma=0}^{n_l} f^{\alpha\beta\gamma} l_\alpha(\xi) l_\beta(\eta) l_\gamma(\zeta)$$

Function decomposition using the basis functions

$$\nabla f(\mathbf{x}(\xi,\eta,\zeta)) \approx \sum_{i=1}^{3} \mathbf{e}_i \sum_{\alpha,\beta,\gamma=0}^{n_l} f^{\alpha\beta\gamma} [l_\alpha{}'(\xi) l_\beta(\eta) l_\gamma(\varsigma) \partial_i \xi + l_\alpha(\xi) l_\beta{}'(\eta) l_\gamma(\varsigma) \partial_i \eta + l_\alpha(\xi) l_\beta(\eta) l_\gamma{}'(\varsigma) \partial_i \varsigma]$$

$$\int_{\Omega_e} f(\mathbf{x}) d^3\mathbf{x} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(\mathbf{x}(\xi,\eta,\varsigma)) J_e(\xi,\eta,\varsigma) d\xi d\eta d\varsigma \approx \sum_{\alpha,\beta,\gamma=0}^{n_l} \omega_\alpha \omega_\beta \omega_\gamma f^{\alpha\beta\gamma} J_e^{\alpha\beta\gamma}$$

GLL cubature rule

Diagonal mass matrix

Weak formulation for the inertia term:

$$\int_{\Omega_e} \rho(\mathbf{x})\upsilon(\mathbf{x})\ddot{u}(\mathbf{x})d^3\mathbf{x} = \int_{\Lambda} \rho(\mathbf{x}(\boldsymbol{\xi}))\upsilon(\mathbf{x}(\boldsymbol{\xi}))\ddot{u}(\mathbf{x}(\boldsymbol{\xi}))J_e(\boldsymbol{\xi})d^3\boldsymbol{\xi} \approx$$

$$\approx \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \rho(\mathbf{x}(\xi,\eta,\zeta))\left[\sum_{i,j,k=0}^{N} \upsilon_{ijk} l_i(\xi) l_j(\eta) l_k(\varsigma)\right]\left[\sum_{l,m,n=0}^{N} \ddot{u}_{lmn} l_l(\xi) l_m(\eta) l_n(\varsigma)\right] J_e(\xi,\eta,\varsigma)\, d\xi\, d\eta\, d\varsigma \approx$$

$$\approx \sum_{r,s,t=0}^{N} \{\rho_{rst}\omega_r\omega_s\omega_t \left[\sum_{i,j,k=0}^{N} \upsilon_{ijk} l_i(\xi_r) l_j(\eta_s) l_k(\varsigma_t)\right]\left[\sum_{l,m,n=0}^{N} \ddot{u}_{lmn} l_l(\xi_r) l_m(\eta_s) l_n(\varsigma_t)\right] J_{rst}\} =$$

$$= \sum_{r,s,t=0}^{N} \{\rho_{rst}\omega_{rst} \left[\sum_{i,j,k=0}^{N} \delta_{ir}\delta_{js}\delta_{kt}\right]\left[\sum_{l,m,n=0}^{N} \ddot{u}_{lmn}\delta_{lr}\delta_{ms}\delta_{nt}\right] J_{rst}\} =$$

$$= \sum_{l,m,n=0}^{N} \ddot{u}_{lmn} \sum_{i,j,k=0}^{N} \sum_{r,s,t=0}^{N} \rho_{rst}\omega_{rst} J_{rst}\delta_{ir}\delta_{js}\delta_{kt}\delta_{lr}\delta_{ms}\delta_{nt} =$$

$$= \sum_{l,m,n=0}^{N} \ddot{u}_{lmn} \sum_{i,j,k=0}^{N} \rho_{ijk}\omega_{ijk} J_{ijk}\delta_{(lmn)(ijk)} = \sum_{\alpha=1}^{(N+1)^3} \ddot{u}_{\alpha} \sum_{\beta=1}^{(N+1)^3} \rho_{\beta}\omega_{\beta} J_{\beta}\delta_{\alpha\beta} = M^e_{\alpha\beta}\ddot{u}_{\alpha}$$

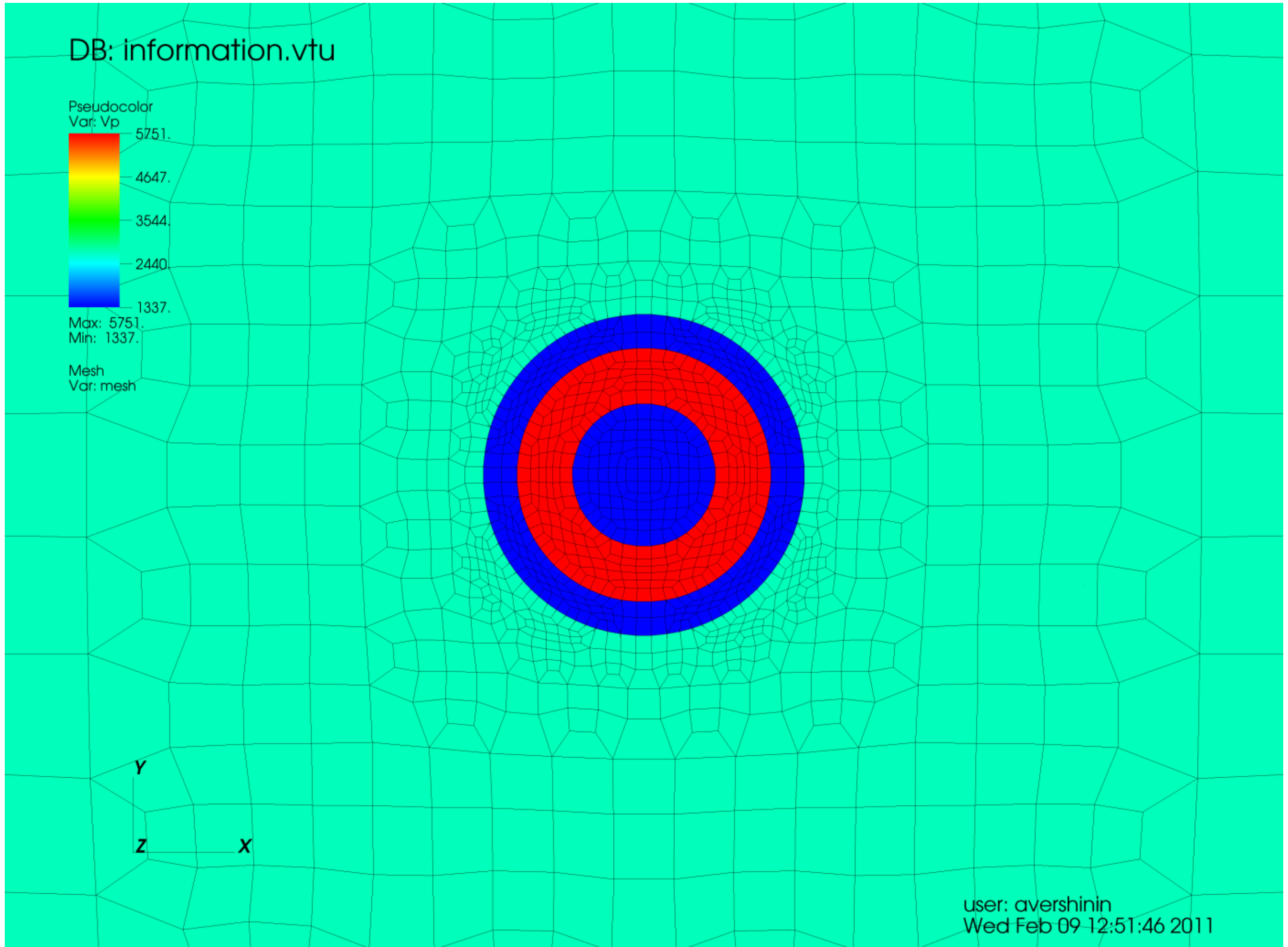
$$M^e_{\alpha\beta} = \rho_{\beta}\omega_{\beta} J_{\beta}\delta_{\alpha\beta} = \rho_{ijk}\omega_{ijk} J_{ijk}\delta_{ijk}$$

Discretization requirements

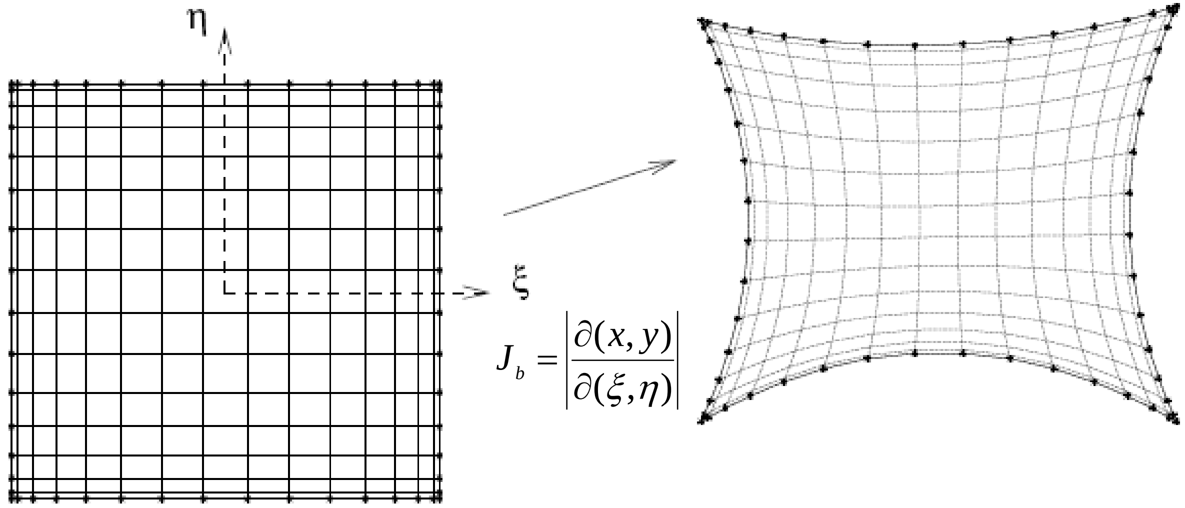
Maximization of the shortest edge in the unstructured SEM mesh used for the discretization of generally curvilinear boundary of a stress concentrator

CFL condition:

$$\Delta t \le C \min_{\Omega_e \in \Omega} \frac{h^e}{V^e_{\max}}$$



Isoparametric SEM

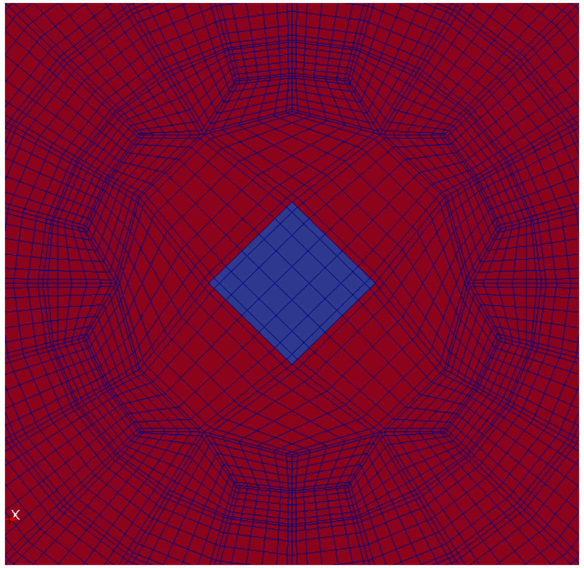


$$J_b = \left| \frac{\partial(x,y)}{\partial(\xi,\eta)} \right|$$

$$\mathbf{x}(\boldsymbol{\xi}) = \sum_{a=1}^{n_a} l_a(\boldsymbol{\xi})\mathbf{x}_a$$

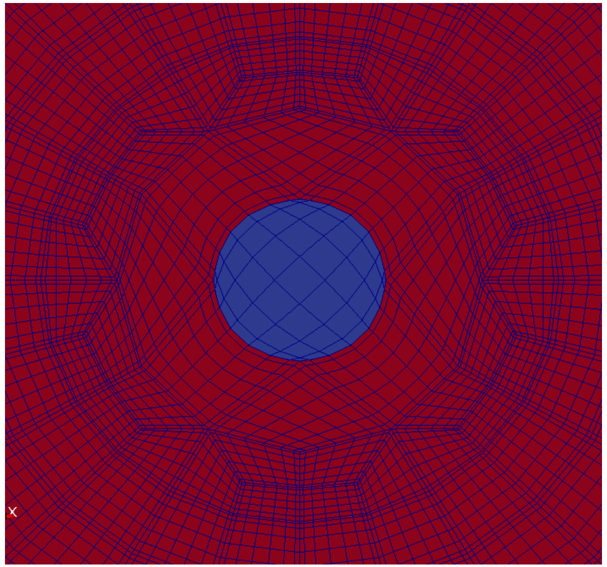
$$\frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}} = \sum_{a=1}^{n_a} \frac{\partial l}{\partial \boldsymbol{\xi}} \mathbf{x}_a$$

Mapping of a reference square element onto curvilinear high order SEM-element using SEM basis functions– isoparametric approximation and discretization.



Linear (on the left) and curvilinear isoparametric (on the right) spectral elements

Spectral convergence of SEM (superconvergence, Bernardi, Maday 1992)
$\|u - u_h\| \leq Ch^N e^{-N}$ in H^1-norm



Time integration scheme

$$\begin{cases} M_u \dfrac{\partial^2 u}{\partial t^2} + M_{uw} \dfrac{\partial^2 w}{\partial t^2} + C_u \dfrac{\partial u}{\partial t} + C_{uw} \dfrac{\partial w}{\partial t} + K_u(u,w) - F_u = 0 \\ M_{uw} \dfrac{\partial^2 u}{\partial t^2} + M_w \dfrac{\partial^2 w}{\partial t^2} + C_{uw} \dfrac{\partial u}{\partial t} + C_w \dfrac{\partial w}{\partial t} + K_w(u,w) - F_w = 0 \end{cases}$$

$$M_u = \int_\Omega N^T \rho\, N d\Omega;\, M_{uw} = \int_\Omega N^T \rho_f\, N d\Omega;\, M_w = \int_\Omega N^T \frac{\alpha \rho_f}{\phi} N d\Omega$$

$$C_u = \int_\Omega N^T \rho d_u\, N d\Omega;\, C_{uw} = \int_\Omega N^T \rho_f d_{uw}\, N d\Omega;\, C_w = \int_\Omega N^T \frac{\alpha \rho_f}{\phi} d_w\, N d\Omega$$

$$K_u(\mathrm{u},\mathrm{w}) = \int_\Omega \sigma(\nabla u, \nabla w) \cdot \nabla N\, \mathrm{d}\Omega;\; K_w(\mathrm{u},\mathrm{w}) = -\int_\Omega p_f(\nabla u, \nabla w) \nabla N\, \mathrm{d}\Omega + \int_\Omega N^T \frac{\eta}{k} N d\Omega \frac{\Delta w}{\Delta T};$$

$$F_u = \oint_\Gamma N^T \sigma_n d\Gamma;\, F_w = \oint_\Gamma N^T P_f \vec{n} d\Gamma;$$

ΔT − physical time step; Δt − pseudotransient time step for the dynamic relaxation method

Solution (displacement vectors u, w) at (n+1)-th time step is sought in the following way using explicit Newmark scheme of 2nd order in time:

$$u(t_{n+1}) = u_{n+1} = u_n + v_{un}\Delta t + a_{un}\frac{\Delta t^2}{2}$$

$$w(t_{n+1}) = w_{n+1} = w_n + v_{wn}\Delta t + a_{wn}\frac{\Delta t^2}{2}$$

$$v_{un+1} = v_{un} + (1-\beta)a_{un}\Delta t + \beta a_{un+1}\Delta t$$

$$v_{wn+1} = v_{wn} + (1-\beta)a_{wn}\Delta t + \beta a_{wn+1}\Delta t$$

The scheme is conditionally (according to CFL condition) stable under $\beta > 0.5$

Integration of plastic flow equations

1) Compute predicted stresses $\sigma_{n+1} = \sigma(\nabla u_{n+1}, \nabla w_{n+1})$ и $p_{fn+1} = p(\nabla u_{n+1}, \nabla w_{n+1})$
2) Check plastic criterion $F_{n+1}=F(\sigma_{n+1}, p_{fn+1}) > 0$
3) If false, go to the next time step.
4) If true, return stresses and fluid pressure back to the yield surface:

$$\Delta\lambda = \frac{F_{n+1}}{\mu + 9BC(K + M(1-b)^2)}$$

$$p_{fn+1} = p_{fn+1} + 3CM\,\Delta\lambda\,(1-b)$$

$$P_{n+1} = \sigma_{n+1}:I + 9C\,\Delta\lambda(K - bM(1-b))$$

$$S = A + B(P_{n+1} + 3p_{fn+1})$$

$$\sigma ii_{n+1} = \frac{(\sigma ii_{n+1} + 3C\,\Delta\lambda(K - bM(1-b)))S + \mu\,\Delta\lambda\frac{P_{n+1}}{3}}{S + \mu\,\Delta\lambda}$$

$$\sigma ij_{n+1} = \frac{\sigma ij_{n+1} S}{S + \mu\,\Delta\lambda} \text{ for } i \neq j$$

And update plastic strains according to the plastic flow rule:

$$\varepsilon_p ii_{n+1} = \varepsilon_p ii_{n+1} + \Delta\lambda\,\left(\frac{\sigma ii_{n+1} - \frac{P_{n+1}}{3}}{2S} - C\right)$$

$$\varepsilon_p ij_{n+1} = \varepsilon_p ij_{n+1} + \Delta\lambda\,\frac{\sigma ij_{n+1}}{2S} \text{ for } i \neq j$$

$$\zeta_{p_{n+1}} = \zeta_{p_{n+1}} - 3C\Delta\lambda$$

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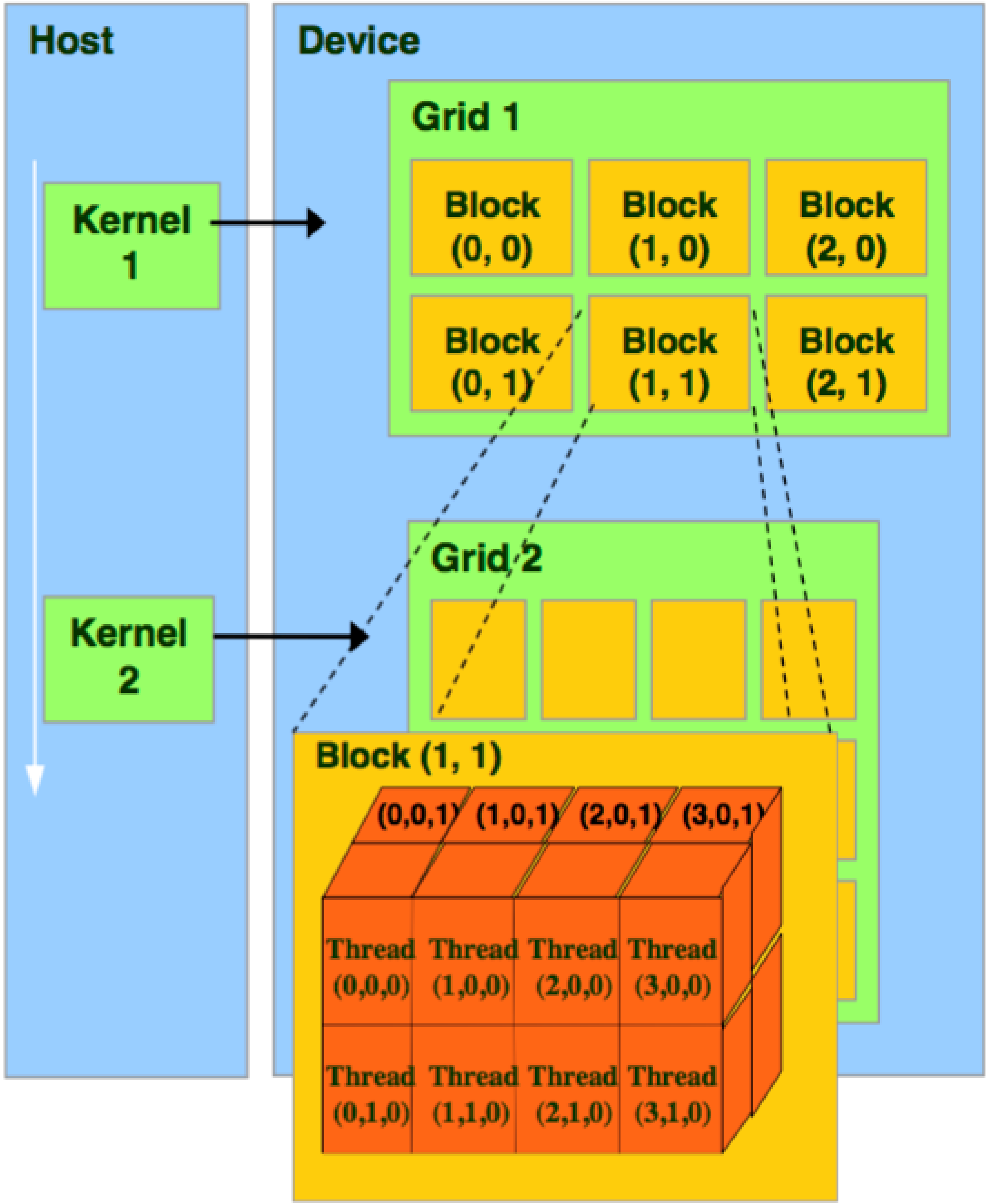
Massively-parallel implementation on GPU

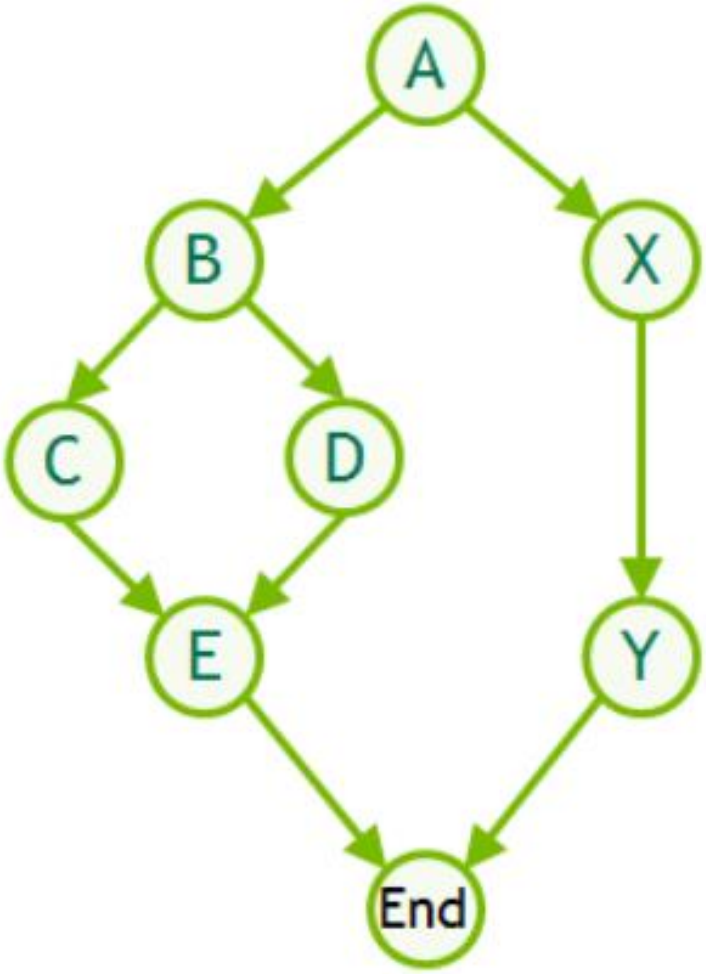
- A set of CUDA kernels: SEM assembly stage, time integration (Newmark scheme), boundary conditions
- SEM-mesh is essentially mapped onto CUDA Grid: each spectral element is processed by a separate CUDA Block, and correspondingly local nodes of SEM element are processed by CUDA Threads inside a block => this allows efficient usage of CUDA Shared memory for SEM element data caching while computing space partial derivatives
- CUDA Block size is automatically defined by SEM approximation order, a number of CUDA Blocks is equal to the number of SEM elements
- Use of atomic operations at the SEM assembly stage (assembly of global force vector and global mass matrix) keeps global GPU memory read-write operations secure



Use of graphs in CUDA 10

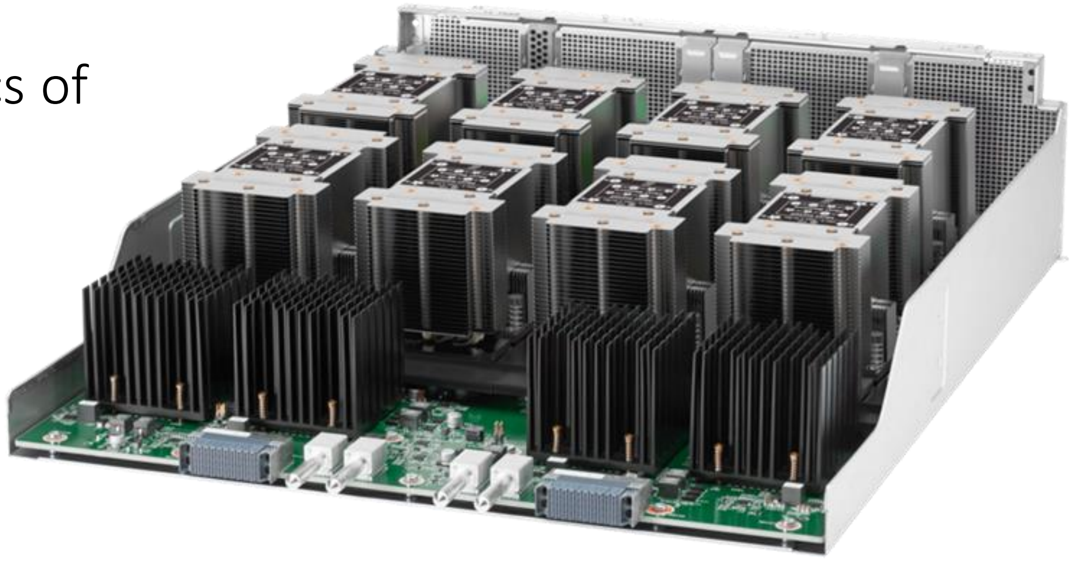
Application of dynamic relaxation method requires multiple (several thousands pseudotransient iterations) kernel calls using constant input data (since the static solution is sought).

In order to optimize kernel calls' overhead CUDA graphs are efficiently used –which allow asynchronous kernel calls and overlap them with global GPU memory operations (memcpy/memset).



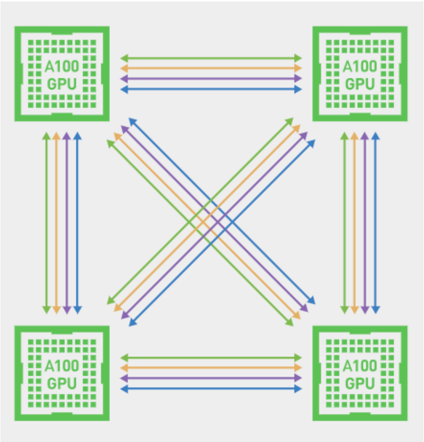
Hybrid HPC platform

(located in the laboratory of computation hydrogeomechanics of the chair of computational mechanics of Lomonosov Moscow State University)



HPE Apollo 6500 Gen10:

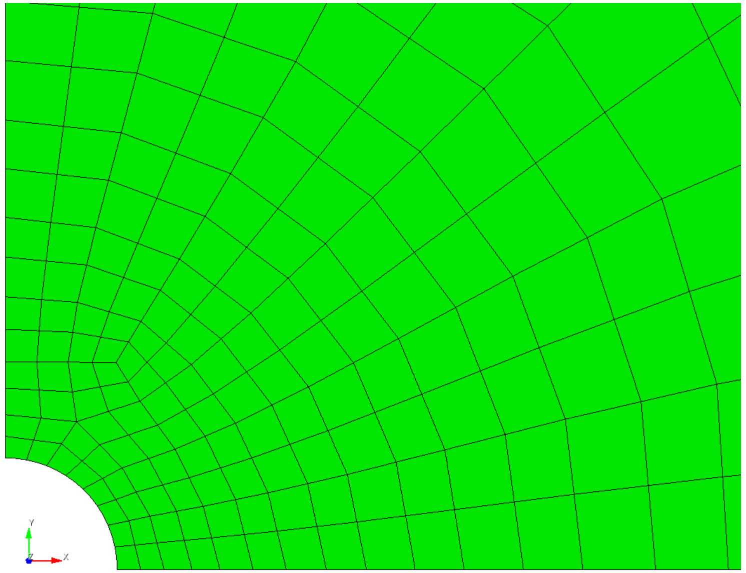
- 8xTesla V100 NVLink 2.0
- InterGPU memory bandwindth using nvLink - 300 Gb/sec
- NVIDIA Grid for remote (RDP) CUDA-computing and 3D rendering



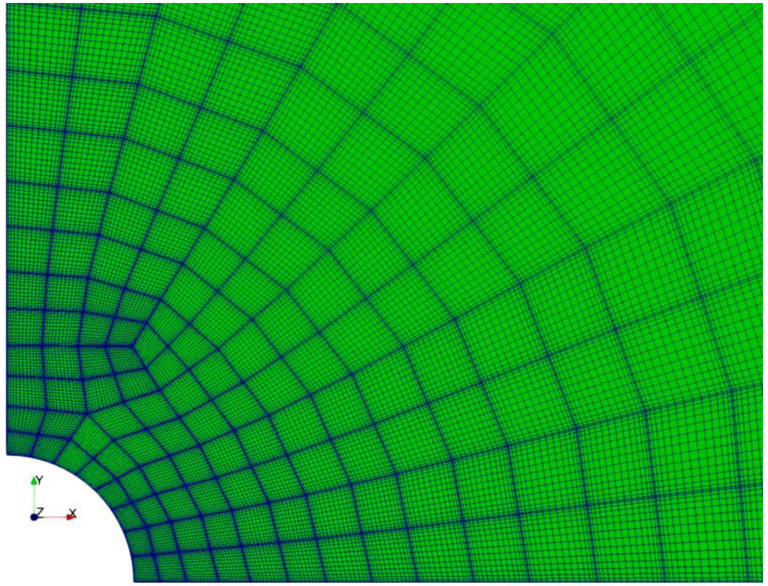
NVIDIA Tesla V100:

- FP64 (double precision) - 7,8 Teraflops
- GPU stack memory HBM2 - **32** Gb
- 80 multiprocessors, **2560** cores FP64
- Global GPU memory bandwidth- 900 Gb/sec
- Support for CUDA 10 and CUDA 11

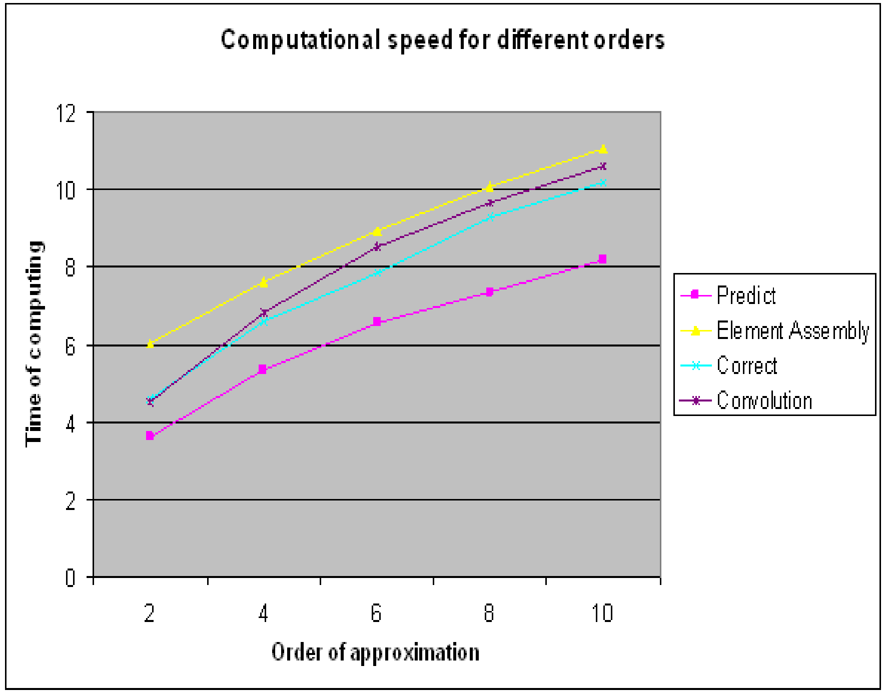
Performance analysis



Unstructured mesh consisting of 208 curvilinear quadrangular elements



SEM mesh of 15th order consisting of 208x16x16=53248 nodes



Numerical error reduces exponentially with respect to SEM order (spectral convergence), while computing time grows linearly due to massively parallel architecture of GPU and memory caching inside a block (spectral element).

10000 pseudotransient time iterations takes about 2 sec.

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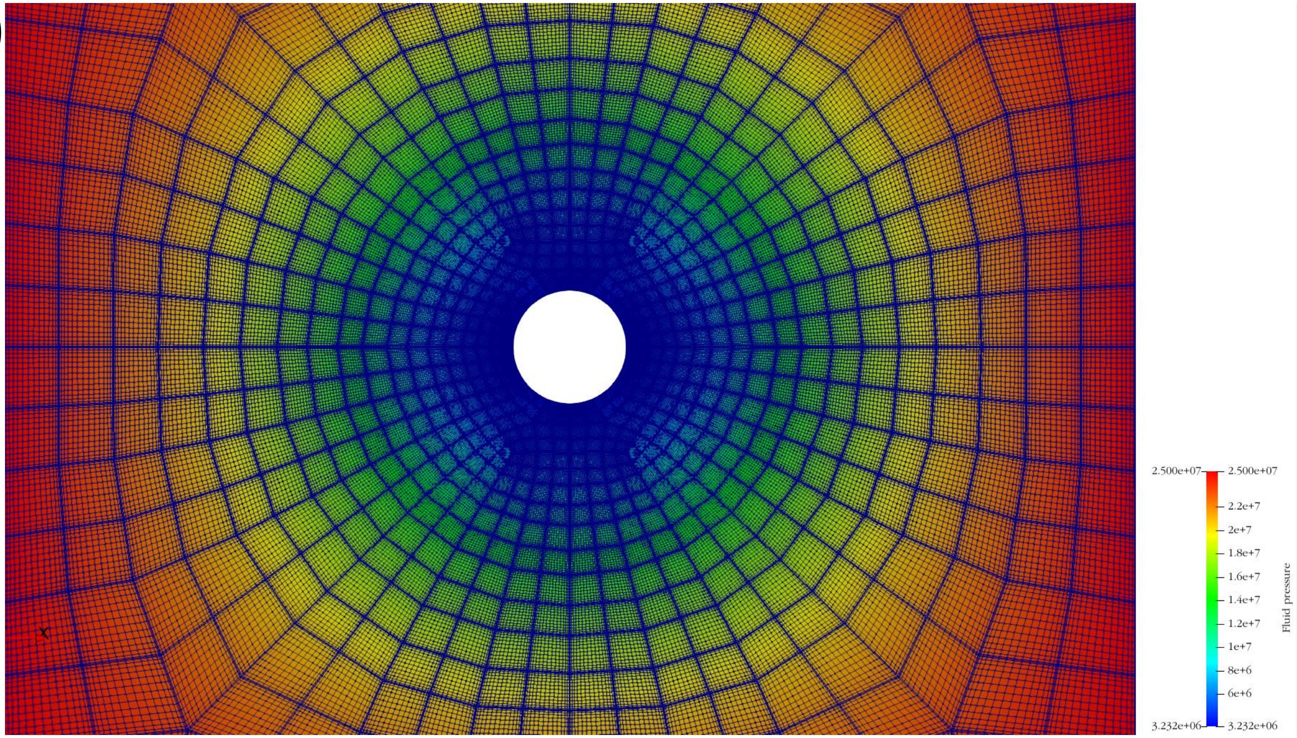
Model problem of a drawdown in the borehole

(H. Wang and M. Sharma, 2016, SPE-181566-MS)

- Model size: 8x8 m
- Borehole radius: 20 sm
- Material parameters:
  - Bulk modulus of solid grains K_s - 2.04 GPa
  - Shear modulus of solid grains μ_s - 1.59 GPa
  - Bulk modulus of fluid K_f - 1.0 GPa
  - Initial porosity ϕ - 0.25
  - Fluid viscosity η - 0.005 Pa*sec
  - Initial permeability k - 1.0e-12 m^2
  - Cohesion c – 2 MPa
  - Internal friction angle φ – 20 degrees
  - Dilatancy angle ψ – 10 degrees
- Boundary condition: pressure in the borehole reduces at the rate 0.6 MPa/hour

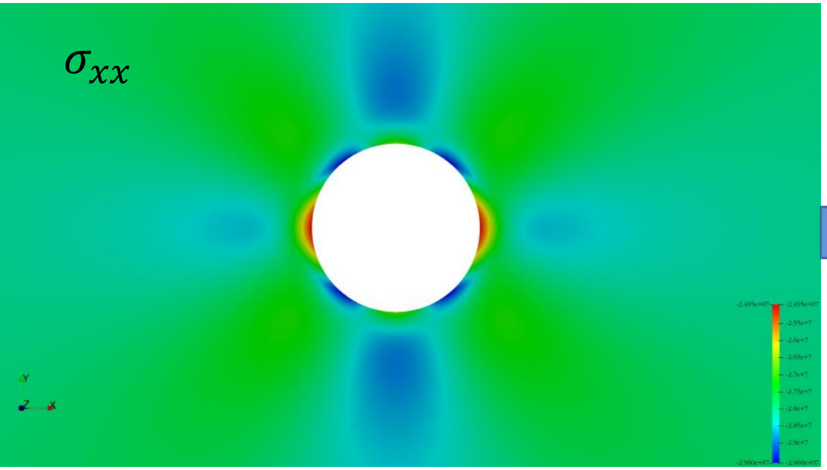


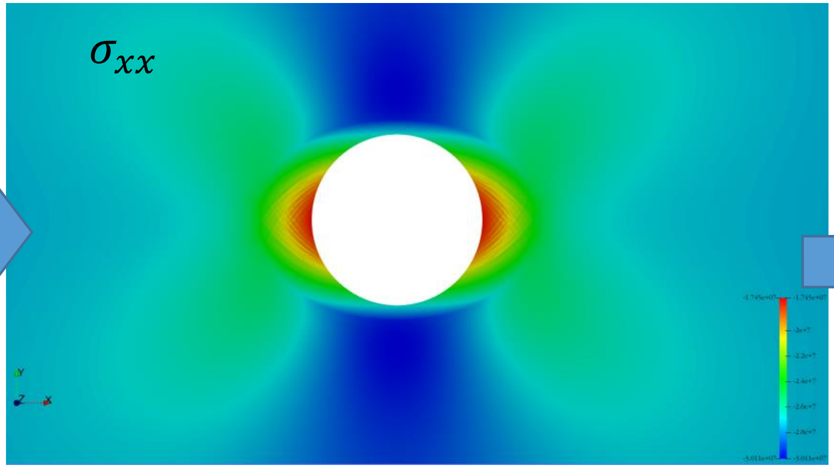
Initial and boundary stresses:

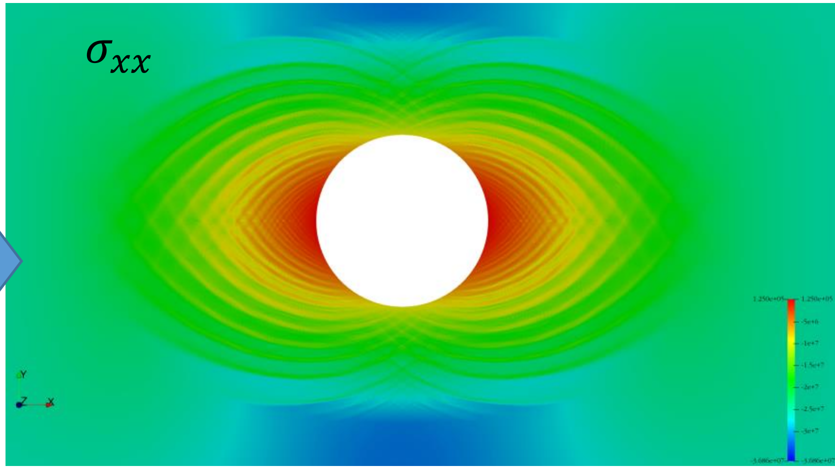
- σ_{xx} = -28 MPa
- σ_{yy} = -32 MPa
- σ_{zz} = -35 MPa
- σ_{xy} = 0

Initial and external pore pressure and initial Pressure in the borehole P_f = 25 MPa

Stresses in different time moments



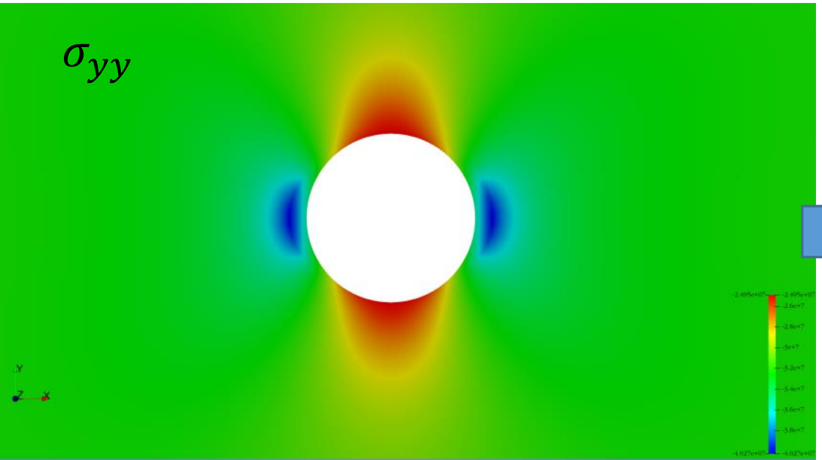


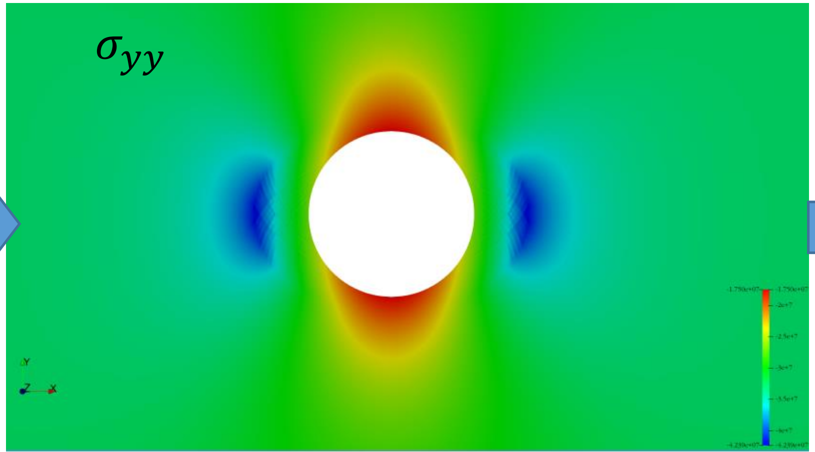


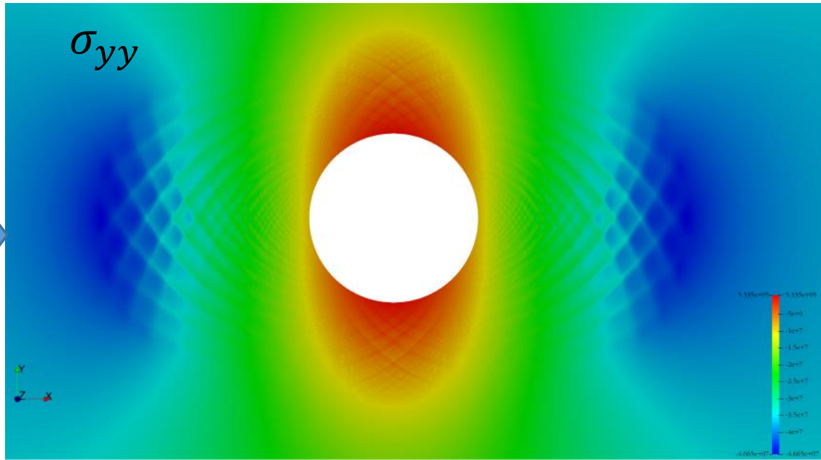
Initial state

Load 30%

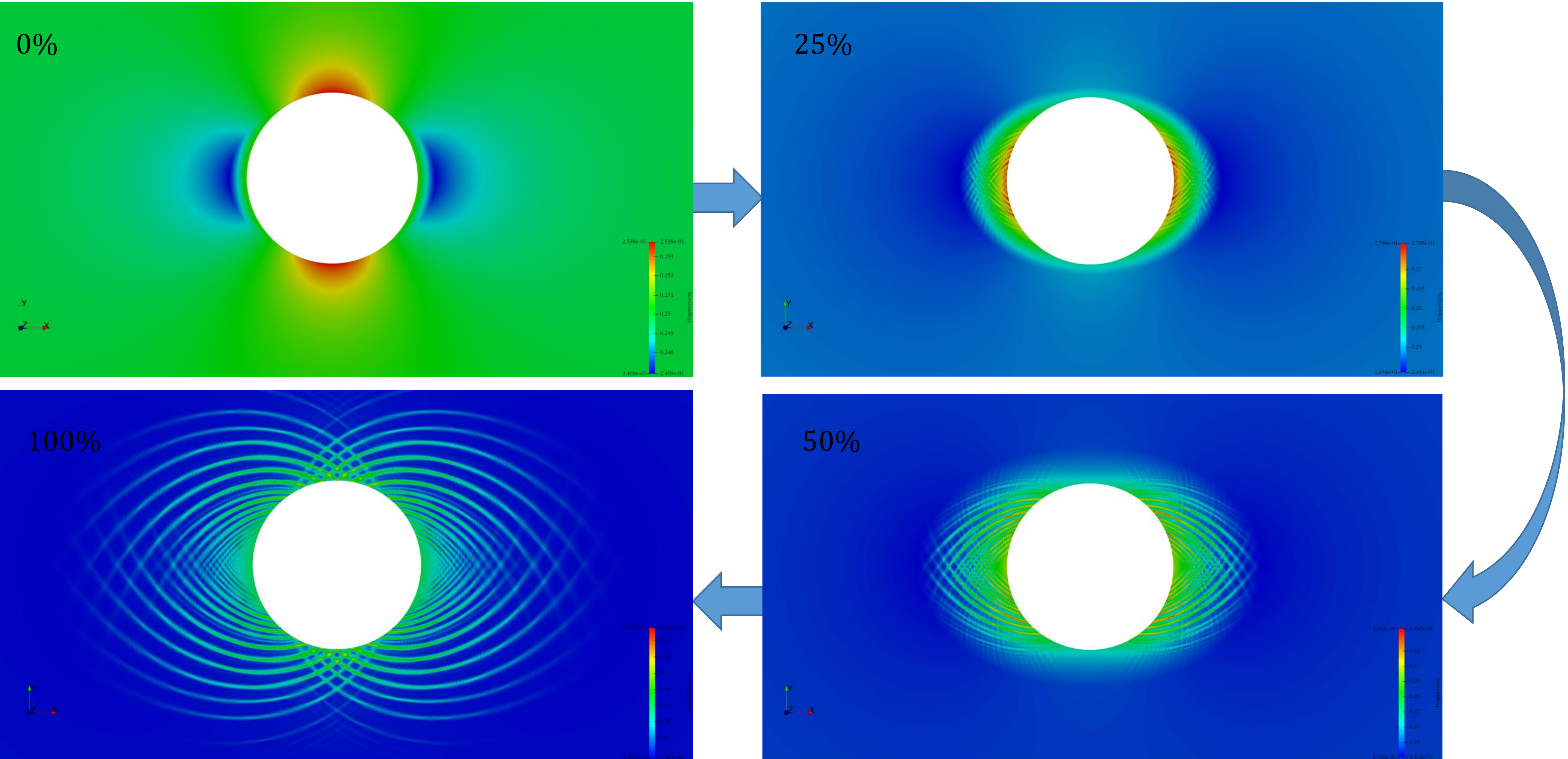
Load 100%



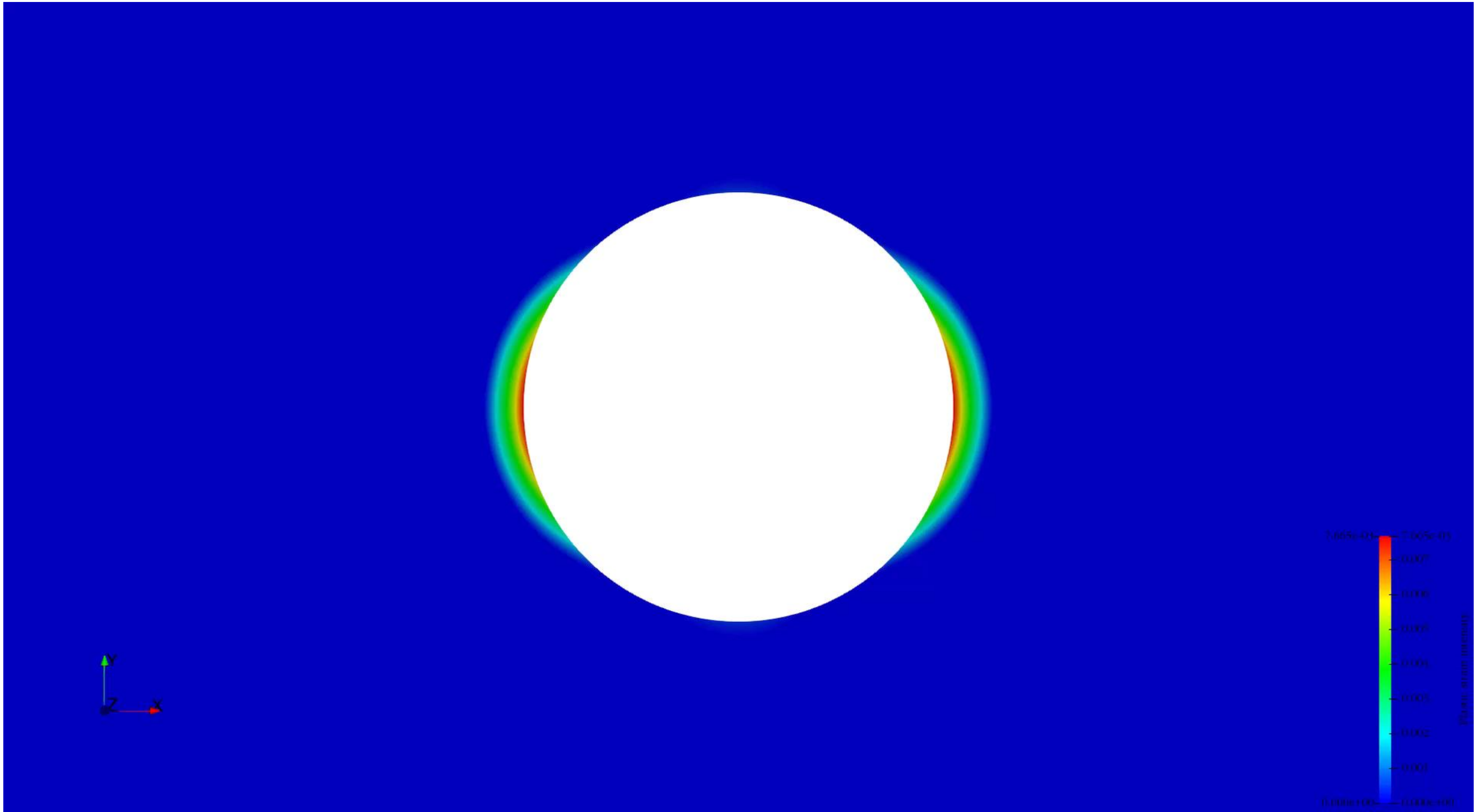




Porosity change during loading process



Development of plastic shear bands



Acknowledgements

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Conclusion

- An algorithm for numerical modeling of poroelastoplastic coupled processes in a fluid-saturated porous medium based on the isoparametric spectral element method is developed
- The algorithm is implemented on a massively parallel GPU architecture using CUDA technology
- For the model problem of an artificial drawdown in a borehole, the process of formation and development of plastic shear bands localization is analyzed.


Thank you
for the
attention!