

Efficient Fourier-collocation method for full scattering data of Zakharov-Shabat periodic problem

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Introduction: integrability of the focusing 1D-NLSE

We consider the focusing 1D NLSE for complex wavefield $u(t, x)$ in its non-dimensional form, where t and x are time and coordinate variables respectively:

$$iu_t + \frac{1}{2}u_{xx} + |u|^2u = 0, \quad (1)$$

Equation (1) can be fully integrated using the inverse scattering transform (IST) [Zakharov, Shabat (1972)], so that the wavefield can be fully characterized by its [scattering data](#).

The first step of the IST for the NLSE (1) is formulation of the auxiliary linear Zakharov – Shabat (Z.-S.) problem for the wavefunction $\vec{\Psi}(x, \lambda) = (\Psi_1, \Psi_2)^T$:

$$\vec{\Psi}_x = \begin{pmatrix} -i\lambda & u \\ -u^* & i\lambda \end{pmatrix} \vec{\Psi}; \quad \vec{\Psi}_t = \begin{pmatrix} -i\lambda^2 + \frac{i}{2}|u|^2 & \lambda u + \frac{i}{2}u_x \\ -\lambda u^* + \frac{i}{2}u_x^* & i\lambda^2 - \frac{i}{2}|u|^2 \end{pmatrix} \vec{\Psi}, \quad (2)$$

$\lambda = \xi + i\eta$ - complex spectral parameter, $u(x) = u(x, t_0)$. From (2) the NLSE can be obtained as a compatibility condition:

$$\vec{\Psi}_{xt} = \vec{\Psi}_{tx} \quad (3)$$

The first equation of Z. – S. system (2) can be reformulated in the form of spectral problem:

$$\widehat{\mathcal{Q}}\Phi = \lambda\Phi, \quad \widehat{\mathcal{Q}} = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial x} - i \begin{pmatrix} 0 & u \\ u^* & 0 \end{pmatrix}, \quad (4)$$

where $u(x)$ plays the role of a potential for “imaginary” scattering problem.

Introduction: applications of the IST for analysis and synthesis of nonlinear wave fields

Analysis. Using **direct scattering transform**, one can decompose a wavefield into a basis of nonlinear non-interacting harmonics.

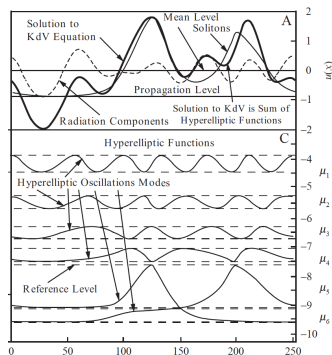


Figure 1: Illustration from the monograph [A. Osborne, Nonlinear ocean waves, (2010)].

Synthesis. Using **inverse scattering transform**, one can construct a wave field with desirable scattering data, which provides an opportunity to control the wavefield physical properties.

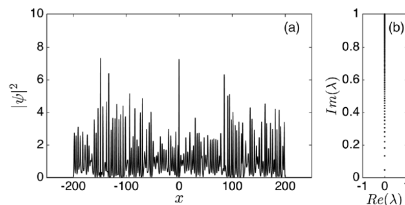


Figure 2: Illustration from the paper [A. Gelash et al, Bound state soliton gas dynamics underlying the spontaneous modulational instability, PRL, (2019)].

Theory: scattering problem for periodic potential

We consider “imaginary” scattering problem on the potential $u(x)$ at certain moment of time.

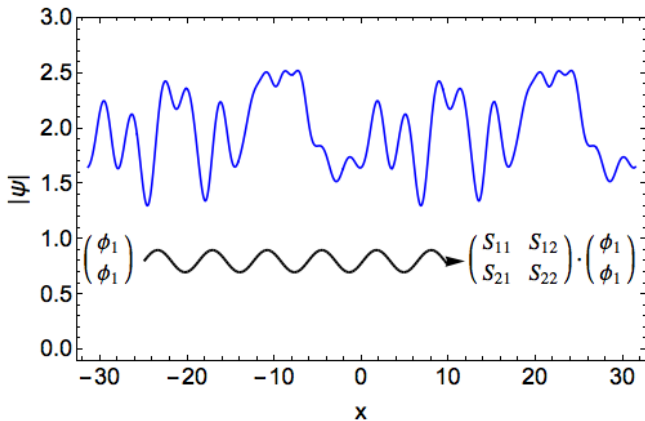


Figure 3: The information about scattering data is contained in the transition matrix

$S(\lambda, x_0) = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$, which transfers solution of the Z. – S. problem from point x_0 to point $x_0 + L$.

Theory: scattering data for periodic potentials, $u(x, t) = u(x + L, t)$

Here we outline the key facts of the periodic IST theory for the 1D-NLSE; see [Ma, Ablowitz, The periodic cubic Schrödinger equation, 1981].

Periodic Z. – S. eigenfunctions represent Bloch functions:

$$\begin{aligned}\widehat{Q}\vec{\Psi} &= \lambda\vec{\Psi}, \\ \vec{\Psi}(x + L, t) &= \kappa(\lambda)\vec{\Psi}(x, t), \quad \kappa(\lambda) = e^{iLp(\lambda)},\end{aligned}\tag{5}$$

where $p(\lambda)$ is quasimomentum. Eigenvalue spectrum of the Z.S. system is determined by the condition:

$$\text{Im}[p(\lambda)] = 0.\tag{6}$$

There are *main* and *auxiliary* spectrum.

Main spectrum represents complex points λ_j , where $\kappa(\lambda_j) = \pm 1$, in which Bloch functions are periodic or antiperiodic:

$$\vec{\Psi}(x + L, t) = \pm \vec{\Psi}(x, t).\tag{7}$$

Alternatively, the main spectrum can be obtained from the following condition:

$$\frac{1}{2} \{ S_{11}(\lambda_j, x_0) + S_{22}(\lambda_j, x_0) \} = \pm 1, \quad \forall x_0.\tag{8}$$

Theory: scattering data for periodic potentials, $u(x, t) = u(x + L, t)$

Auxiliary spectrum $\gamma(x_0)$ represents the points λ , satisfying the condition:

$$S_{11} - S_{22} + iS_{21} + iS_{12} = 0. \quad (9)$$

Auxiliary spectrum $\eta(x_0)$ represents the points λ , satisfying the condition:

$$S_{11} - S_{22} - S_{21} - S_{12} = 0. \quad (10)$$

Auxiliary spectrum depends on x_0 and, in principal, can be located in the whole λ -plane. One can reconstruct the wavefield from its scattering data as follows:

$$u(x) = \frac{1}{2} \sum_i^{2N} \lambda_j - \sum_i^N \gamma_j(x) + i \left\{ \frac{1}{2} \sum_i^{2N} \lambda_j - \sum_i^N \eta_j(x) \right\} \quad (11)$$

Conventional Fourier Collocation method for solving the Z. – S. is well known, see for example the monograph [J. Jang, Nonlinear Waves in Integrable and Nonintegrable Systems, 2010].

The wavefunction $\vec{\Psi} = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$ and the wavefield $u(x)$ are represented as a sum of $2N + 1$ Fourier modes in the interval $[-L/2, L/2]$:

$$\Psi_1 = \sum_{n=-N}^N a_{1,n} e^{iknx}, \Psi_2 = \sum_{n=-N}^N a_{2,n} e^{iknx}, u(x) = \sum_{n=-N}^N c_n e^{iknx}, \quad (12)$$

$$k = \frac{2\pi}{L}.$$

Standard Fourier Collocation method

As a result the initial Z. – S. problem transforms to the following linear matrix problem for eigenvalues and eigenvectors:

$$\begin{pmatrix} -B_1 & B_2 \\ B_2^\dagger & B_1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = i\lambda \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \quad (13)$$

$$B_1 = ik\text{diag}(-N, -N+1, \dots, N), \quad (14)$$

$$B_2 = \begin{pmatrix} c_0 & c_{-1} & \cdots & c_{-N} & 0 & \cdots & 0 \\ c_1 & c_0 & c_{-1} & \ddots & \ddots & & \\ \vdots & c_1 & c_0 & \ddots & \ddots & & \\ c_N & \ddots & \ddots & \ddots & \ddots & \ddots & c_{-N} \\ 0 & c_N & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & c_{-1} \\ 0 & \cdots & \cdots & c_N & \cdots & c_1 & c_0 \end{pmatrix} \quad (15)$$

$$A_1 = (a_{1,-N}, a_{1,-N+1}, \dots, a_{1,N})^T, A_2 = (a_{2,-N}, a_{2,-N+1}, \dots, a_{2,N})^T \quad (16)$$

Now we consider the general case when quasimomentum is non zero, i.e. Ψ is non-periodic. We introduce the new periodic function $\vec{\tilde{\Psi}}$ in the following way:

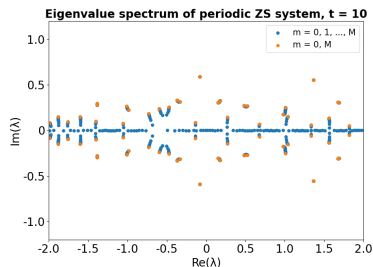
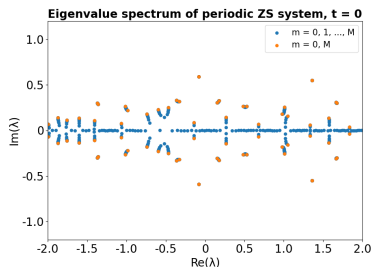
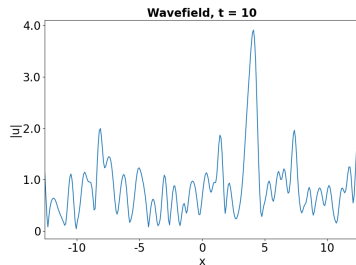
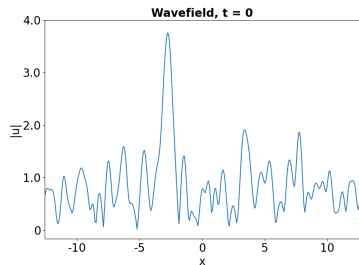
$$\vec{\Psi} = \vec{\tilde{\Psi}} e^{ipx}. \quad (17)$$

Substitution of (17) into the Z.-S. system results in:

$$i\hat{Q}\vec{\tilde{\Psi}}(x, \lambda) + ip \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \vec{\tilde{\Psi}}(x, \lambda) = i\lambda \vec{\tilde{\Psi}}(x, \lambda), \quad (18)$$

The latter means that replacing $B_1 \rightarrow \tilde{B}_1$ in (13), where $\tilde{B}_1 = B_1 + ip_m E$, we obtain Fourier collocation method for certain value of discretized quasi-momentum: $p_m = \frac{\pi m}{LM}$, $m = 0, 1, \dots, M$. There are M eigenproblems to solve. $e^{ipx} = \pm 1$, i.e. $m = 0, M$ corresponds to the main spectrum.

Illustration: evolution of nonlinear wave field and eigenvalue spectrum



Eigenvalue spectrum of periodically perturbed condensate. Motivation

[V. E. Zakharov and A. A. Gelash, Phys. Rev. Lett., Nonlinear Stage of Modulation Instability, 2013.]

[J. M. Soto-Crespo, N. Devine, and N. Akhmediev, Integrable Turbulence and Rogue Waves: Breathers or Solitons? Phys. Rev. Lett., 2016.]

[P. G. Grinevich and P. M. Santini, The finite gap method and the analytic description of the exact rogue wave recurrence in the periodic NLS Cauchy problem. 1, Nonlinearity, 2018.]

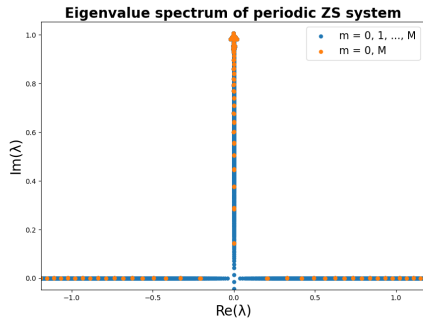
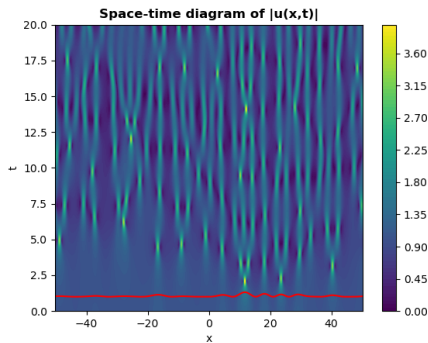
[M. Conforti, S. Li, G. Biondini, and S. Trillo, Automodulation versus breathers in the nonlinear stage of modulational instability, Opt. Lett. 43, 5291 (2018).]

We consider the following initial condition:

$$u(x) = 1 + i\epsilon \sum_{n=1}^{n=N} \sin(k_n x + \theta_n), \quad (19)$$

where k_n correspond to long wave perturbations and $\theta_n \in [0, \pi)$ are random.

Weak condensate perturbation: evolution of wavefield VS periodic eigenvalue spectrum



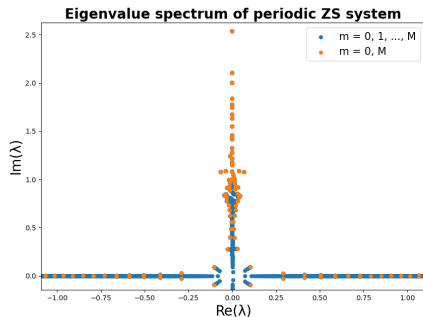
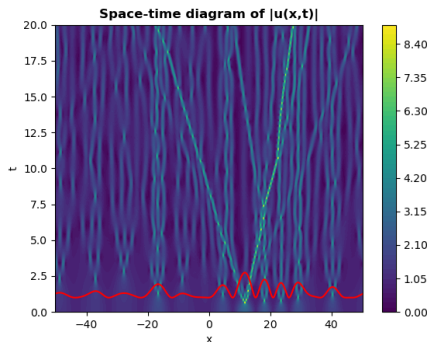
Strong condensate perturbation: evolution of wavefield VS periodic eigenvalue spectrum

NLSE soliton group velocity:

$$V_{Gr}^s = -2\text{Re}[\lambda_j] \quad (20)$$

NLSE breather group velocity:

$$V_{Gr}^{br} = -2\text{Im}[\lambda_j \sqrt{1 + \lambda_j^2}] / \text{Im}[\sqrt{1 + \lambda_j^2}]. \quad (21)$$



Strong condensate perturbation: evolution of wavefield VS periodic eigenvalue spectrum

