

High-orders methods and high-precision arithmetics make direct scattering transform for the Korteweg-De Vries equation robust

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We consider the KdV equation for a real-valued function $u(x, t)$ with x and t playing the role of space and time coordinates:

$$u_t + 6uu_x + u_{xxx} = 0 \quad (1)$$

The model (1) is [integrable](#) using the IST method and the wave field can be fully characterized by its [scattering data](#) (IST spectrum). Historically the IST method was discovered using the KdV equation^{1,2}. The authors of these works revealed the key role of the operator $\mathcal{L}$:

$$\mathcal{L}\phi = \lambda^2\phi, \quad \mathcal{L} = -\partial_{xx} - u(x, t), \quad (2)$$

where $\phi(x, t) = \psi(x, t_0)$ is the wave function and $\lambda = k + i\kappa$ is the time-independent spectral parameter, $u(x) = u(x, t_0)$ playing the role of the potential. This is a well-known problem for [stationary Schrödinger equation](#) in quantum mechanics. For spatially localized potentials $u(x)$ the eigenvalues of the operator $\mathcal{L}$ represent a finite number of discrete points $\{\lambda_n = i\kappa_n\}$, $n = 1, \dots, N$ and the real line $\lambda = k \in \mathbb{R}$.

¹C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, Physical Review Letters, 1967

²P. D. Lax, Communications on pure and applied mathematics, 1968 

We rewrite Eq. (1) in the form of a 2×2 system of the first order linear equations for the vector wave function $\Psi = (\phi, \phi')^T$:

$$\Psi' = A\Psi, \quad A = \begin{pmatrix} 0 & 1 \\ -\lambda^2 - u(x) & 0 \end{pmatrix}. \quad (3)$$

We impose the following boundary conditions for real values of the spectral parameter corresponding to the right scattering problem:

$$\Psi|_{x \rightarrow -\infty} = \begin{pmatrix} e^{-ikx} \\ -ike^{-ikx} \end{pmatrix}, \quad \Psi|_{x \rightarrow \infty} = \begin{pmatrix} a(k)e^{-ikx} + b(k)e^{ikx} \\ -ika(k)e^{-ikx} + ikb(k)e^{ikx} \end{pmatrix}, \quad (4)$$

where $a(k)$ and $b(k)$ are the scattering coefficients.

The scattering coefficients are connected to the wave field scattering data $\{\lambda_n, \rho_n; r\}$ as follows:

$$a(\lambda_n) = 0, \quad \rho_n = (b_n / \partial_\lambda a)|_{\lambda=\lambda_n}; \quad r(k) = b(k)/a(k). \quad (5)$$

Scattering coefficients and S -matrix

For convenience we define the following extended 4×4 linear system for the vector wave function $\Phi = (\Psi, \partial_\lambda \Psi)^T$:

$$\Phi' = \bar{A}\Phi, \quad \bar{A} = \begin{pmatrix} A & 0 \\ \partial_\lambda A & A \end{pmatrix}. \quad (6)$$

We begin our theoretical analysis of the DST moving the asymptotics from infinity to $[-L/2, L/2]$, where the wave field $u(x)$ is well localized. We introduce a 4×4 S -matrix which transfers the wave function Φ :

$$\Phi|_{x \rightarrow \frac{L}{2}} = \underbrace{\begin{pmatrix} \Sigma & 0 \\ \partial_\lambda \Sigma & \Sigma \end{pmatrix}}_S \Phi|_{x \rightarrow -\frac{L}{2}}, \quad (7)$$

where $\Sigma(\lambda)$ is a 2×2 matrix for Ψ , i.e. $\Psi|_{x \rightarrow \frac{L}{2}} = \Sigma \Psi|_{x \rightarrow -\frac{L}{2}}$.

Solving the system (7) we obtain:

$$a = e^{i\lambda L}[-i\lambda S_{12} - S_{21}/i\lambda + S_{11} + S_{22}]/2,$$

$$b = [S_{11} - S_{22} - i\lambda S_{12} + S_{21}/i\lambda]/2,$$

$$\begin{aligned} \partial_\lambda a = e^{i\lambda L}[-iS_{21} + \lambda(-LS_{21} + \lambda^2(LS_{12} - iS_{32}) + iS_{41} + \\ + \lambda(-iS_{12} + iL(S_{11} + S_{22}) + S_{31} + S_{42}))/ (2\lambda^2), \end{aligned} \tag{8}$$

$$\partial_\lambda b = [-S_{21} + \lambda(S_{41} + \lambda(\lambda S_{32} + S_{12} + iS_{31} - iS_{42}))]/(2i\lambda^2).$$

Magnus expansion

We discretize the interval $[-L/2, L/2]$ into M bins and denote the width of m th bin as Δx_m with its center at x_m . To derive a general solution of Eq. (3) within the interval $[x_c, x_p]$, where $x_c = x_m - \Delta x_m/2$ and $x_p = x_m + \Delta x_m/2$, we use the Magnus expansion³:

$$\Psi(x_p) = U(x_m)\Psi(x_c), \quad U(x_m) = e^{\Omega(x_m)}, \quad (9)$$

where Ω represents an infinite series:

$$\Omega(x_m) = \sum_{j=1}^{\infty} \Omega_j(x_m), \quad (10)$$

with the following four terms:

$$\begin{aligned} \Omega_1(x_m) &= \int_{x_c}^{x_p} A(x) dx, \\ \Omega_2(x_m) &= \frac{1}{2} \int_{x_c}^{x_p} dx_1 \int_{x_c}^{x_1} dx_2 [A_1, A_2], \\ \Omega_3(x_m) &= \frac{1}{6} \int_{x_c}^{x_p} dx_1 \int_{x_c}^{x_1} dx_2 \int_{x_c}^{x_2} dx_3 ([A_1, [A_2, A_3]] + [A_3, [A_2, A_1]]), \\ \Omega_4(x_m) &= \frac{1}{12} \int_{x_c}^{x_p} dx_1 \int_{x_c}^{x_1} dx_2 \int_{x_c}^{x_2} dx_3 \int_{x_c}^{x_3} dx_4 \times ([[[A_1, A_2], A_3], A_4] + \\ &\quad + [A_1, [[A_2, A_3], A_4]] + [A_1, [A_2, [A_3, A_4]]] + [A_2, [A_3, [A_4, A_1]]]). \end{aligned} \quad (11)$$

³S. Blanes, F. Casas, J. A. Oteo and J. Ros, Physics Reports, 2009

Magnus expansion

The potential appearing in the matrix A can be represented using the Taylor series within m th bin:

$$u(x) = u(x_m) + u'(x_m)(x - x_m) + u''(x_m)(x - x_m)^2/2 + \dots \quad (12)$$

These expressions lead to Σ multiplying the results within each bin:

$$\Sigma = \prod_{m=1}^M U(x_m) = \prod_{m=1}^M e^{\Omega(x_m)}. \quad (13)$$

The trace-vanishing feature of A leads to the form of matrix U as follows:

$$e^{\Omega(x_m)} = \begin{pmatrix} \cosh \kappa_m + \zeta_m \sinh \kappa_m / \kappa_m & \phi_m \sinh \kappa_m / \kappa_m \\ \beta_m \sinh \kappa_m / \kappa_m & \cosh \kappa_m - \zeta_m \sinh \kappa_m / \kappa_m \end{pmatrix}, \quad (14)$$

where the terms for second-, fourth- and sixth-order schemes are retained:

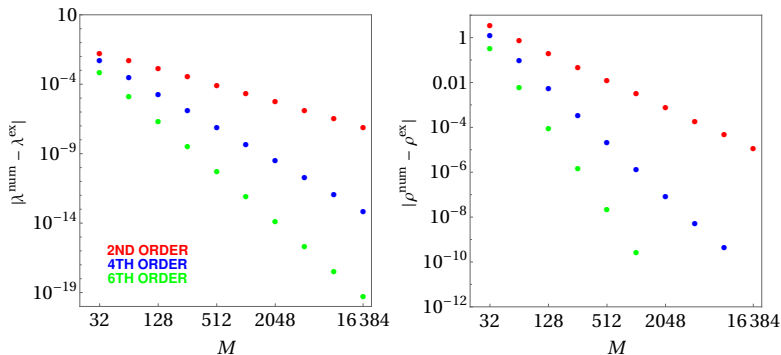
$$\begin{aligned} \phi_m &= \underbrace{\Delta x^2}_{2\text{nd} + 4\text{th}} + \underbrace{u_m''/360 \Delta x^6}_{6\text{th order}}, \\ \zeta_m &= \underbrace{0}_{2\text{nd}} + \underbrace{(u_m')/12 \Delta x^4}_{4\text{th order}} + \underbrace{(8u_m'(\lambda^2 + u_m) + 3u_m^{(3)})/1440 \Delta x^6}_{6\text{th order}}, \\ \kappa_m^2 &= \underbrace{-(\lambda^2 + u_m) \Delta x^2}_{2\text{nd}} + \underbrace{[-u_m''/24] \Delta x^4}_{4\text{th order}} + \underbrace{[(8u_m'^2 + 3u_m^{(4)})/5760] \Delta x^6}_{6\text{th order}}. \end{aligned} \quad (15)$$

One-soliton solution

The solution describing one soliton has the following form⁴:

$$u(x) = \frac{2\kappa^2}{\cosh^2[\kappa(x - x_0)]}, \quad x_0 = \frac{1}{2\kappa} \log\left(\frac{\rho}{2i\kappa}\right), \quad (16)$$

where x_0 represents the spatial position of a soliton. To test the numerical algorithm we set the parameters $\lambda = i\kappa = i$, $\rho = 60i$ and perform the numerical DST on the interval $L = 32$. Figure below shows the influence of the numerical scheme order on the numerical error against the number of discretization:



⁴S. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov, Theory of Solitons: The Inverse Scattering Method, 1984

Dressing method⁵

To construct N -soliton fields for KdV, we use the dressing method that allows us to obtain new solvable Sturm-Liouville equations:

$$\begin{aligned}\Psi[1] &= \left(\frac{d}{dx} - \sigma_1\right)\psi_0, \text{ where } \sigma_1 = \psi_{1x}\psi^{-1}, \\ \Psi[2] &= \left(\frac{d}{dx} - \frac{\Psi_{2,x}[1]}{\Psi_2[1]}\right)\Psi[1], \text{ where } \Psi_2[1] = \left(\frac{d}{dx} - \frac{\psi_{1,x}}{\psi_1}\right)\psi_2, \\ \Psi[N+1] &= \left(\frac{d}{dx} - \frac{\Psi_{N+1,x}[N]}{\Psi_{N+1}[N]}\right)\Psi[N].\end{aligned}\tag{17}$$

The Cram theorem gives the construction for all reflectionless smooth and real potentials through transformation functions ψ_N :

$$\psi_N = \begin{cases} \cosh(\kappa_N(x - x_N)) = C_N^{1/2} e^{(\kappa_N x)} + C_N^{-1/2} e^{(-\kappa_N x)}, & N = 2k + 1, \\ \sinh(\kappa_{N+1}(x - x_{N+1})) = C_N^{1/2} e^{(\kappa_N x)} - C_N^{-1/2} e^{(-\kappa_N x)}, & N = 2k. \end{cases}\tag{18}$$

Assuming large enough L , we obtain the following general expressions:

$$\begin{aligned}\Psi[N]_{|x=-L} &= \prod_{i=1}^N (\kappa + \kappa_i) e^{-\kappa L}, \\ \Psi[N]_{|x=+L} &= \prod_{i=1}^N \frac{(\kappa - \kappa_i)}{(\kappa + \kappa_i)} e^{\kappa L} + a_N \sum_{j=1}^N (-1)^{N+j} \frac{\prod_{k=1}^N (\kappa_j + \kappa_k)}{\prod_{k \neq j}^N (\kappa_j - \kappa_k)} \frac{e^{(\kappa - \kappa_j)L}}{C_j(\kappa - \kappa_j)} e^{-\kappa L}.\end{aligned}\tag{19}$$

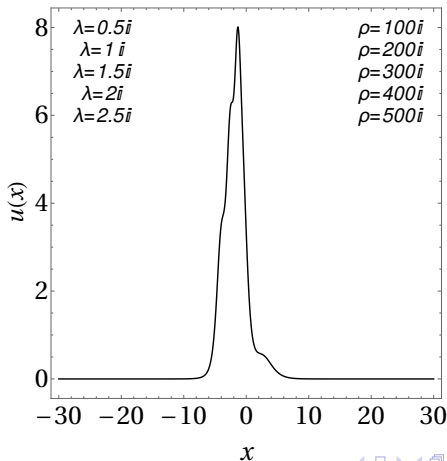
⁵V. B. Matveev, M. A. Salle, Darboux transformations and solitons, 1991. 

Dressing method

The potential of the linear Sturm-Liouville equation corresponding to $\Psi[N + 1]$ is given by

$$u[N + 1] = u[N] + 2 \frac{\partial^2 \ln(\Psi[N])}{\partial x^2} \quad (20)$$

We constructed $N = 5$ -soliton field through the dressing method with the spectral data shown in the picture below:

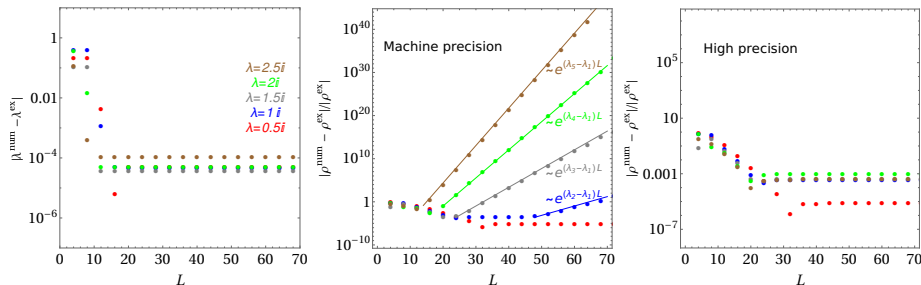


Anomalous numerical errors

The analytic formula of scattering coefficient b_N shows that at large L even a small deviation δ of the corresponding λ_k can lead to large errors for ρ_k :

$$b_N(\varkappa_k + \delta) \approx \rho_k \prod_{i \neq k}^N \frac{(\varkappa_k - \varkappa_i)}{(\varkappa_k + \varkappa_i)} \frac{1}{2\varkappa_k} + \delta \sum_{j \neq k}^N \rho_j \frac{e^{(\varkappa_k - \varkappa_j)L}}{(\varkappa_k - \varkappa_j)2\varkappa_k} \prod_{i \neq k}^N \frac{(\varkappa_k - \varkappa_i)}{(\varkappa_k + \varkappa_i)}. \quad (21)$$

The figure below shows the DST results for our field with added noise as a machine epsilon. As expected, the divergence of norming constants grows gradually with the increase in the size of the computational domain exponentially.



Conclusions and further perspectives

1. The test based on a single-soliton solution verified the declared orders of numerical schemes.
2. We reveal a class of instabilities of the DST leading to anomalous numerical errors of soliton norming constants growing exponentially with L , making solitons in some sense “uncatchable” when standard machine precision arithmetic is used.
3. To mitigate the anomalous errors, we propose to use high precision arithmetic that compensates the behavior of the coefficient b_N .
4. Simultaneous application of the high precision arithmetic and high order algorithms makes the direct scattering transform for the KdV equation robust
5. The developed algorithm can be used for fast and efficient analysis of experimental and numerical data in the presence of noise.
6. One of the key perspectives is to apply the developed algorithm to complex wave fields containing a large number of solitons when the advantages of the high-order approaches are especially pronounced.