

A priori and a posteriori analysis in Large eddy simulation of the two-dimensional decaying turbulence using Explicit filtering approach

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Goals

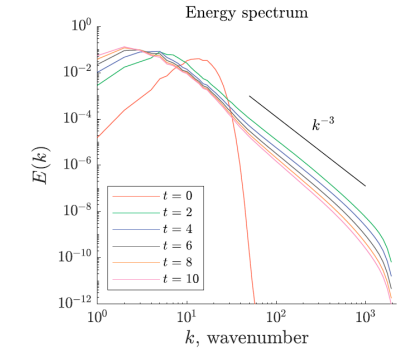
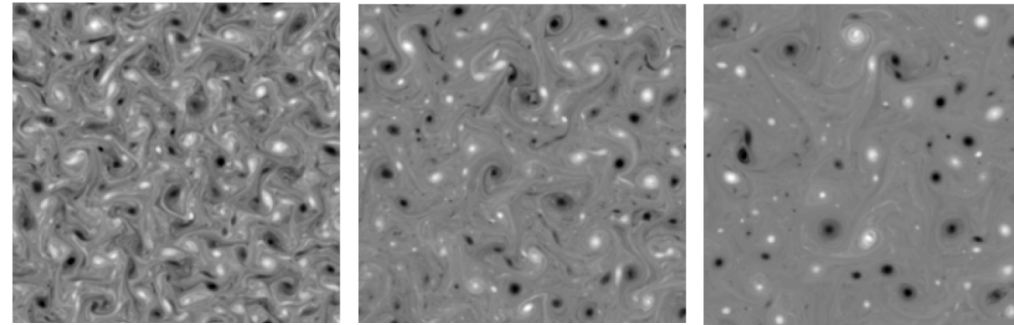
- Construct a LES model describing subgrid **energy** and **enstrophy transfer spectra**.

Methods

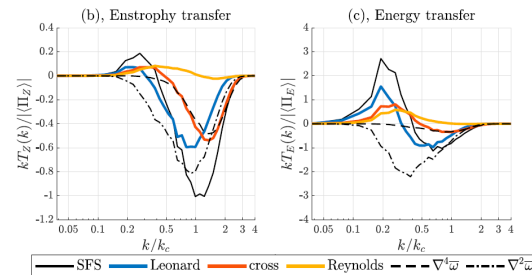
- Explicit filtering, i.e. $\bar{\Delta} = \sqrt{6}\Delta_g \rightarrow$ exclude discretization errors
- Germano 1986 decomposition
- Partial reconstruction of SGS stress
- Dynamic procedure

Motivation

LES models in 2D turbulence **are rare**. It is especially difficult to describe both the **dissipation of enstrophy** and **backscatter of energy (KEB)**.



A priori analysis

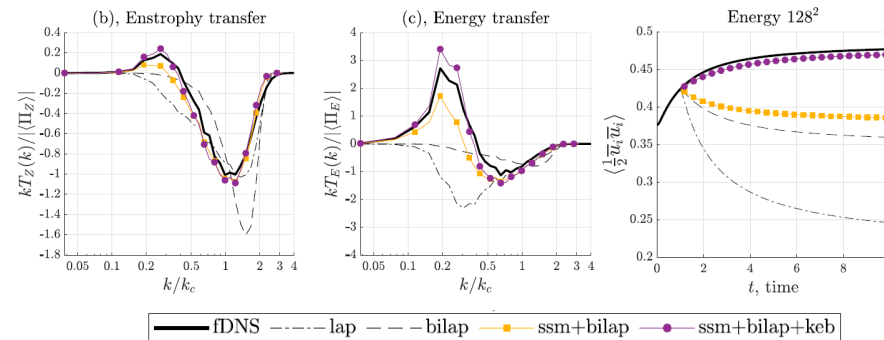


- Leonard stress is directly computed
- Cross stress is approximated by **bilaplacian viscosity**
- An approximation to Reynolds stress is a **backscatter**
- An estimate of **upscale energy flux**:

$$\langle \Pi_E \rangle = -\frac{\bar{\Delta}^2}{12} \langle \Pi_Z \rangle$$

Results

Free of parameters LES model describing **transfer spectra** and **energy growth** in **a posteriori** experiment.



Conclusion

Proposed LES model describes **energy** and **enstrophy exchange** with subfilter scales.
For a discussion, see a full version.

Acknowledgments

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References

- Decaying turbulence problem**: Maulik, R., & San, O. (2017). Int. J. of Comp. Fluid Dyn., 31(2), 69-92.
- Explicit filtering**: Winckelmans, G. S. et al. (2001). Phys. of Fluids, 13(5), 1385-1403.
- Our research**: The same title, submitted to JFM.

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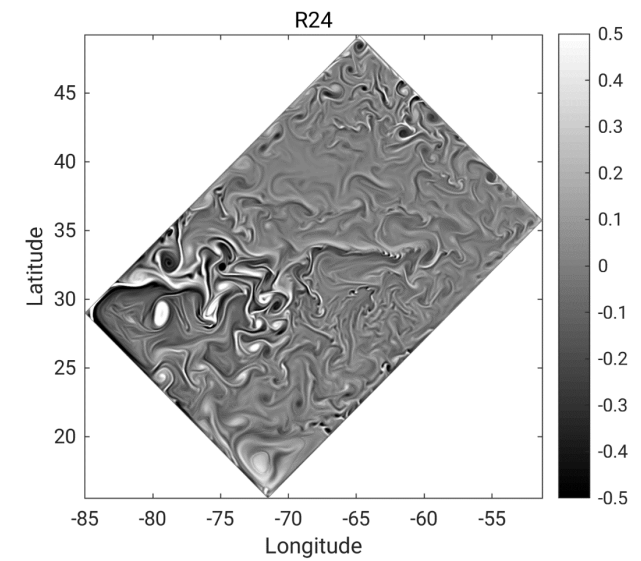
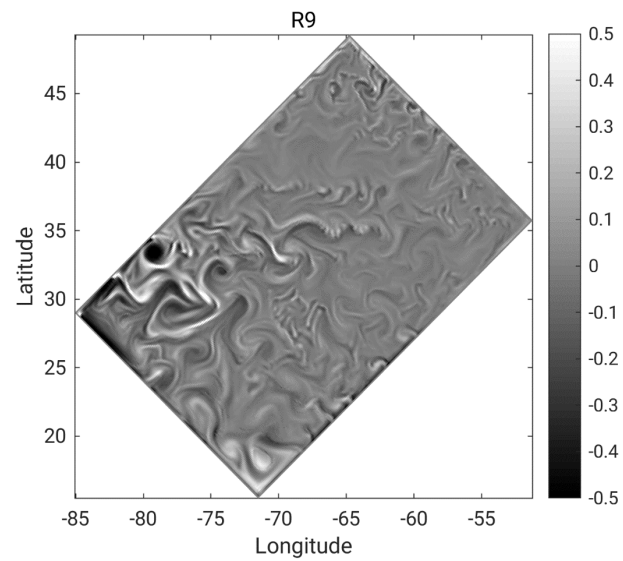
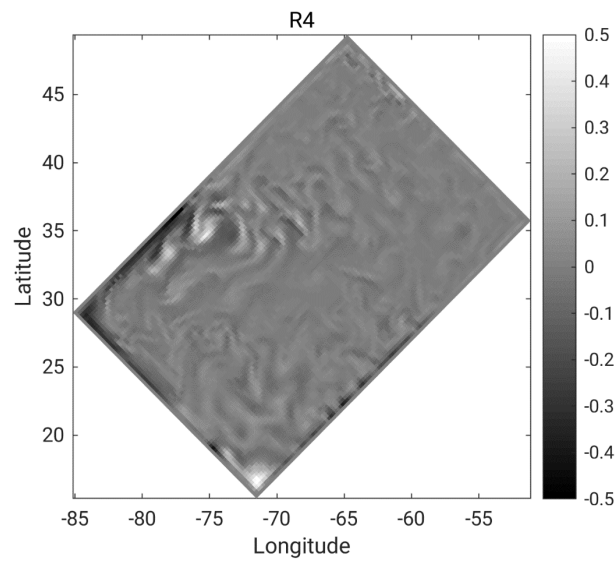
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Motivation

- Resolved quasi-2d turbulence strongly depends on the resolution.
- Idea: **subgrid parameterizations** can improve the simulation of resolved turbulence, but in some characteristics they fail. One of possible ways to understand their deficiency is to compare them directly with **subgrid forces**, that is nonlinear interaction with **subgrid scales**



Types of quasi-2d turbulence parameterizations in ocean models

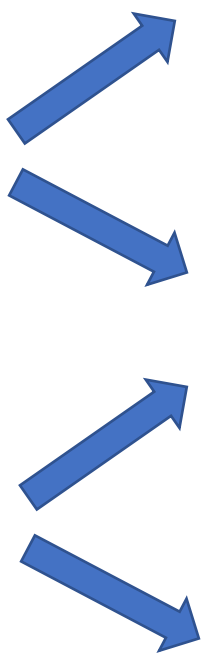
- Historically, parameterizations were aimed to reproduce mean fields, and they can be assumed to be of **RANS** (Reynolds Averaged Navier-Stokes) type. That is, they parameterize the whole eddying field, which is unresolved.

- $\frac{\partial \mathbf{U}_h}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U}_h + \mathbf{cor}_h = -\frac{1}{\rho_0} \nabla_h p + F_U \longrightarrow \text{momentum mixing}$

- $\frac{\partial T}{\partial t} + \mathbf{U} \cdot \nabla T = F_T, \frac{\partial S}{\partial t} + \mathbf{U} \cdot \nabla S = F_S \longrightarrow \text{scalars mixing}$

Types of quasi-2d turbulence parameterizations in ocean models

- For now there are generally four types of them.

$$\begin{aligned} & \bullet \frac{\partial \mathbf{U}_h}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U}_h + \mathbf{cor}_h = -\frac{1}{\rho_0} \nabla_h p + F_U \\ & \bullet \frac{\partial T}{\partial t} + \mathbf{U} \cdot \nabla T = F_T, \frac{\partial S}{\partial t} + \mathbf{U} \cdot \nabla S = F_S \end{aligned}$$


KE dissipation

KE backscatter

Isopycnal diffusion

Gent-McWilliams

How parameterizations for momentum and scalars should be coupled?

- There are two viewpoints:
 - Scalars and velocity are strongly coupled through **thermal wind relation**. And there should be a unique closure for them.
 - Isopycnal diffusion and especially GM are of RANS type and they should be replaced with something else in eddy-resolving simulations.
- In my opinion:
 - Each parameterization describes a unique physical process that is partially unresolved on the grid
 - All of them can be used in eddy-resolving simulation, but with the restriction: each parameterization should be of the same – **subgrid type**, that is its strength should decrease when refining resolution at proper rate ($\nu_e \sim \Delta^n$, Δ – grid step)

KE dissipation and backscatter in physical space

- KE dissipation and backscatter are different sides of energy exchange with subgrid scales:

$$\tau_{i,j} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j \quad - \text{SGS stress tensor}$$

$$\Pi_E = -\tau_{ij} S_{ij} \quad - \text{local energy exchange}$$

- According to Leonard 1997 expansion for 2D fluid:

$$\tau_{ij} \approx \tau_{ij}^{NGM} = \frac{\bar{\Delta}^2}{12} \frac{\partial \bar{u}_i}{\partial x_k} \frac{\partial \bar{u}_j}{\partial x_k}$$

$$\Pi_E \approx -\tau_{ij}^{NGM} S_{ij} = 0$$

- KE dissipation and backscatter approximately in local equilibrium in physical space

KE dissipation and backscatter in Fourier space

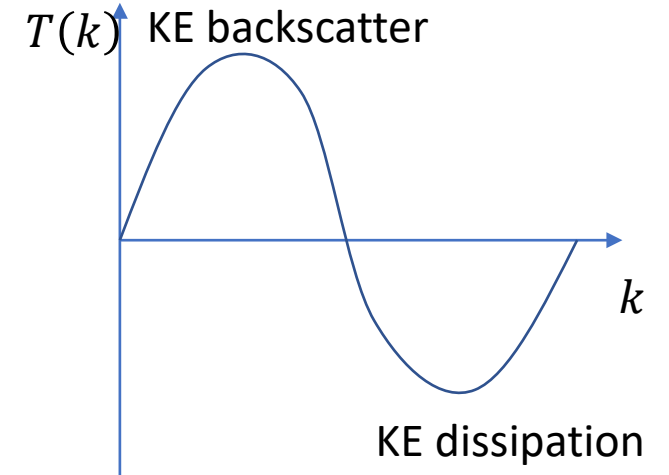
- Let $T(k)$ be an energy transfer spectrum of SGS stress:

$$T(k) = \sum_{|k| \in [k, k+\Delta k)} \Re(\mathcal{F}(\tau_{ij})^* \mathcal{F}(S_{ij}))$$

- Then energy conservation and enstrophy dissipation:

$$\int T(k) dk = 0$$
$$\int T(k) k^2 dk < 0$$

- This gives two wavenumber intervals: **longwave energy backscatter** and **shortwave energy dissipation**



These two properties give the simplest backscatter parameterization for quasi-2D fluid

$$\frac{\partial u_i}{\partial t} = \dots + f_i$$

- f_i – is a backscatter parameterization such that
 - It locally equilibrates dissipation
 - It is larger than eddy viscosity in spatial scale
- For a general ocean circulation model many parameters remains to be specified by hand:
 - Operator returning energy and its spatial scale
 - Backscatter rate
 - Vertical structure
 - Behavior near boundary

Historical overview

- Longwave kinetic energy backscatter is known since at least

Kraichnan 1976 “Eddy viscosity in two and three dimensions”

- However, the notion of “stochastic backscatter” was used for the parameterization which excites shortwave band of wavenumbers

Leslie & Quarini 1979 “The application of turbulence theory to the formulation of subgrid modelling procedures”

Frederiksen & Davies 1997 “Eddy Viscosity and Stochastic Backscatter Parameterizations on the Sphere for Atmospheric Circulation Models”

- Possibly since the following works

Thuburn et.al. 2014 “Cascades, backscatter and conservation in numerical models of two-dimensional turbulence”

Jansen Held 2014 “Parameterizing subgrid-scale eddy effects using energetically consistent backscatter”

longwave energy generation came to be called backscatter.

Parameterizations of dissipation

- In 3D turbulence parameterizations based on dynamic procedure and scale-similarity assumption are widespread

$$\begin{aligned}\tau_{ij} &= \overline{u_i u_j} - \bar{u}_i \bar{u}_j \\ T_{ij} &= \widehat{\overline{u_i u_j}} - \widehat{\bar{u}_i \bar{u}_j} \\ T_{ij} - \widehat{\tau_{ij}} &= \widehat{\overline{u_i u_j}} - \widehat{\bar{u}_i \bar{u}_j} = l_{ij}\end{aligned}$$

- In ocean models a typical parameterization is biharmonic Smagorinsky:

$$\frac{\partial u_i}{\partial t} = \dots - \nabla \left(\sqrt{\nu} \nabla \left(\nabla (\sqrt{\nu} \nabla u_i) \right) \right), \nu = 0.06 \Delta^4 |S|$$

- Dynamic models for quasi-2D fluids were investigated in studies

Bachman et.al. 2017 “A scale-aware subgrid model for quasi-geostrophic turbulence”

Maulik & San 2017 “A stable and scale-aware dynamic modeling framework for subgrid-scale parameterizations of two-dimensional turbulence”

- Remaining questions are:
 - Can dynamic model improve the simulation?
 - Why numerical noise often arises?

LES models of dissipation and backscatter in 2D decaying turbulence

- Goals:

- Simulate both dissipation and backscatter with transfer spectra similar to a priori data
- Distinguish between numerical errors and inaccuracy of LES model



- Methods:

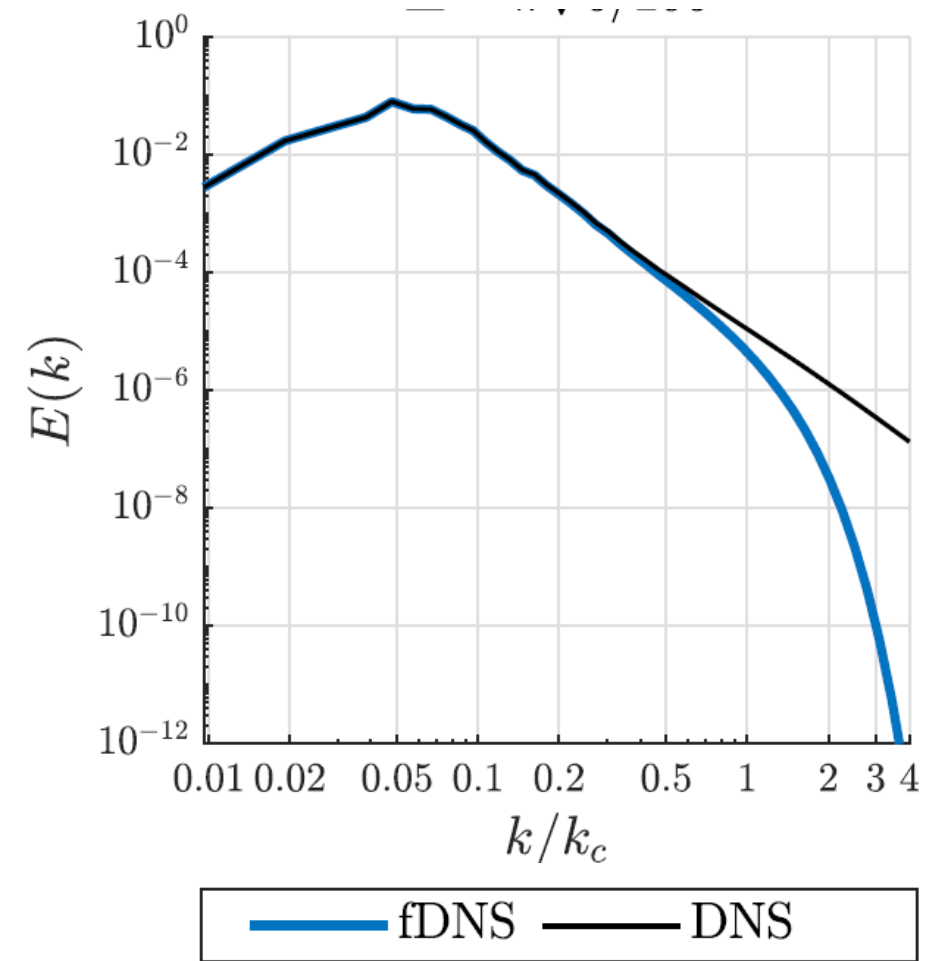
- Partial reconstruction of SGS stress
- Explicit filtering approach

- Materials:

Perezhogin & Glazunov “A priori and a posteriori analysis in large eddy simulation of the two-dimensional decaying turbulence using an explicit filtering approach”, submitted to JFM

Explicit filtering

- LES model should reproduce filtered DNS
- LES models do not contain mesh step (Δ_g)
- Instead, filter width ($\bar{\Delta}$) is defined
- Filter width is typically $\bar{\Delta} > \Delta_g$
- For a fixed $\bar{\Delta}$, grid convergence can be studied ($\Delta_g \rightarrow 0$)



Governing equations, LES closure, Germano decomposition

$$\frac{\partial \omega}{\partial t} + \frac{\partial(u_j \omega)}{\partial x_j} = \frac{1}{Re} \Delta \omega$$

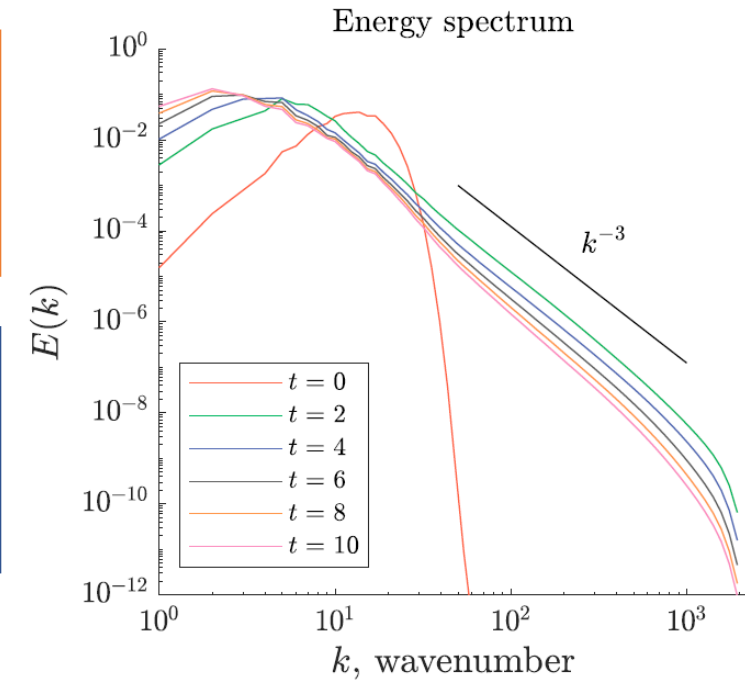
$$\bar{\omega}(\mathbf{x}) = \int G(\mathbf{x}, \mathbf{y}) \omega(\mathbf{y}) d\mathbf{y}$$

$$\frac{\partial \bar{\omega}}{\partial t} + \frac{\partial(\bar{u}_i \bar{\omega})}{\partial x_j} = \frac{1}{Re} \Delta \bar{\omega} - \frac{\partial \sigma_j}{\partial x_j}$$

G – Gaussian filter

$$\sigma_j = \overline{u_i \omega} - \bar{u}_i \bar{\omega}$$

$$\sigma_j = F(\bar{\omega}, \bar{u}_i)$$



Germano 1986 decomposition:

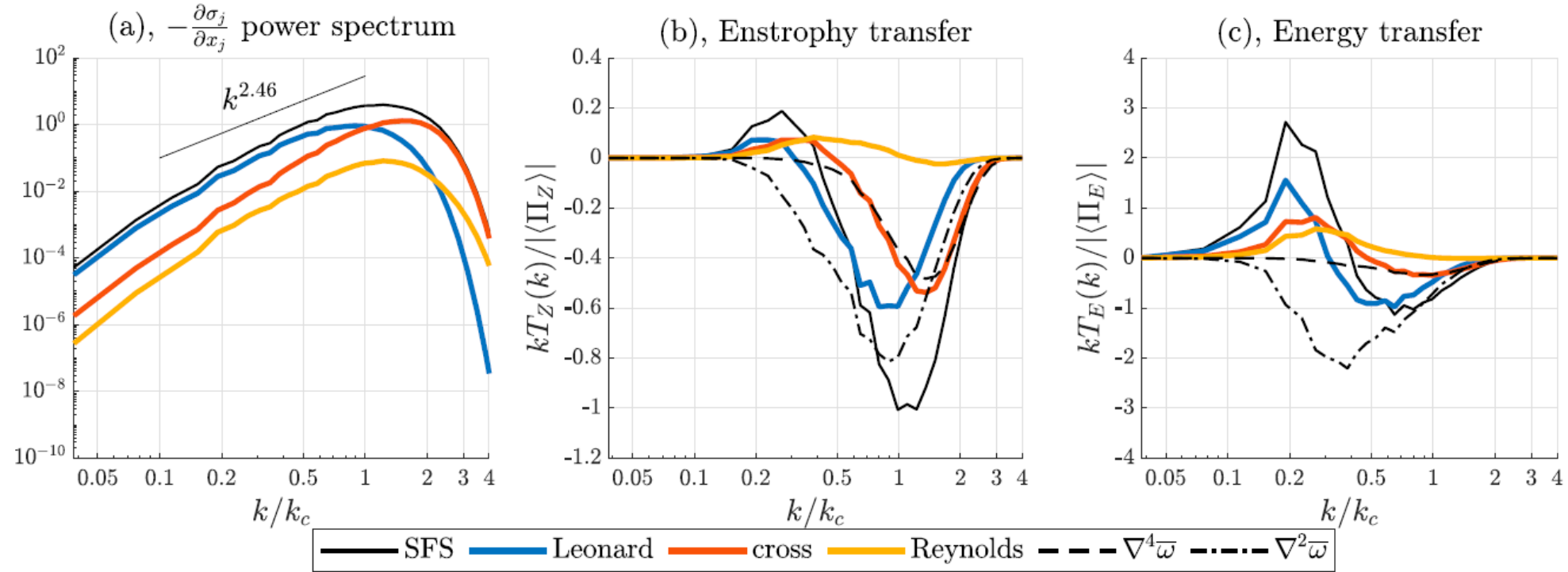
$$\sigma = \underbrace{\overline{\bar{u}_j \bar{\omega}} - \bar{\bar{u}}_j \bar{\bar{\omega}}}_{\text{Leonard stress}} + \underbrace{\overline{\bar{u}_j \omega'} + \overline{u'_j \bar{\omega}} - \bar{\bar{u}}_j \bar{\omega}' - \bar{u}'_j \bar{\bar{\omega}}}_{\text{cross stress}} + \underbrace{\overline{u'_j \omega'} - \bar{u}'_j \bar{\omega}'}_{\text{Reynolds stress}}$$

Leonard
stress

cross
stress

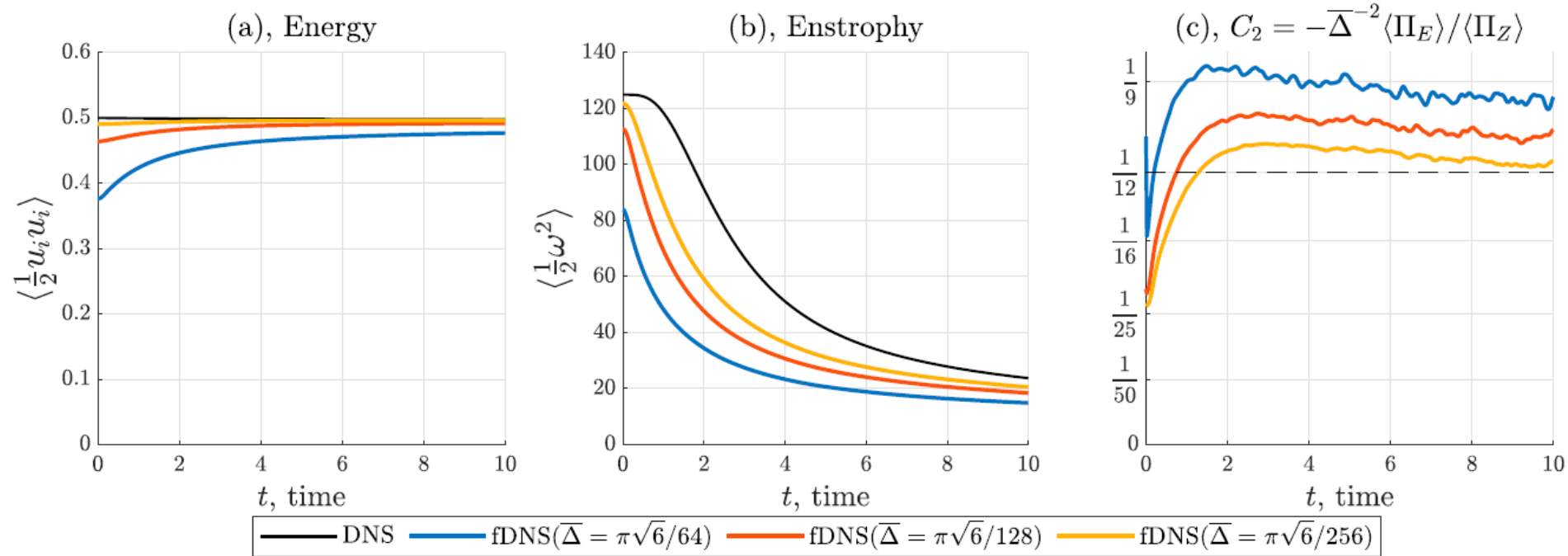
Reynolds
stress

Spectral analysis in Germano decomposition



- Enstrophy dissipation:
 - Represented by Leonard and cross stresses
 - Leonard stress is computed directly, cross stress is approximated by bilaplacian
- Energy backscatter:
 - Represented by all three stresses
 - Reynolds stress injects energy, but does not influence on small-scale enstrophy transfer

Estimation of upscale energy flux (at high Re)



- Resolved enstrophy decays
- Resolved energy grows
- Energy and enstrophy exchanges with subfilter scales are directed oppositely:

$$\langle \Pi_E \rangle = -\frac{\bar{\Delta}^2}{12} \langle \Pi_Z \rangle, \bar{\Delta} - \text{filter width}$$

- The formula is based on the estimation of subfilter energy:

$$e = \frac{1}{2} \text{tr}(\tau_{ij}) \approx \frac{1}{2} \text{tr}(\tau_{ij}^{NGM}) = \frac{\bar{\Delta}^2}{48} (|\bar{S}|^2 + \bar{\omega}^2)$$

ssm+bilap+keb model

$$\bullet \sigma_j^{mod} = \underbrace{\overline{\overline{u_j \omega}} - \overline{\overline{u_j}} \overline{\overline{\omega}}}_{\text{SSM}} + \underbrace{(C_S \overline{\Delta})^4 |\overline{S}| \frac{\partial \nabla^2 \overline{\omega}}{\partial x_j}}_{\text{bilap. Smag.}} + \underbrace{C_{back} (\overline{u_j'' \omega''} - \overline{\overline{u_j''}} \overline{\overline{\omega''}})}_{\text{backscatter (KEB)}}$$

- KEB model is an approximation to Reynolds stress with:

$$\begin{aligned} \omega'' &= \overline{\omega} - \overline{\overline{\omega}} \\ u_j'' &= \overline{u_j} - \overline{\overline{u_j}} \end{aligned}$$

- Smagorinsky constant C_S is estimated using dynamic procedure:

$$C_S^4 = \frac{\langle (l_j - h_j) m_j \rangle}{\langle m_j m_j \rangle}$$

$$l_j = \widehat{\overline{u_j \omega}} - \widehat{\overline{u_j}} \widehat{\overline{\omega}}, m_j - \text{viscous part}, h_j - \text{ssm part}$$

- “Backscatter amplitude” C_{back} is determined from:

$$\langle \Pi_E \rangle = -\frac{\overline{\Delta}^2}{12} \langle \Pi_Z \rangle$$

lap:

$$\sigma_j^{mod} = -(C_S \overline{\Delta})^2 |\overline{S}| \frac{\partial \overline{\omega}}{\partial x_j}$$

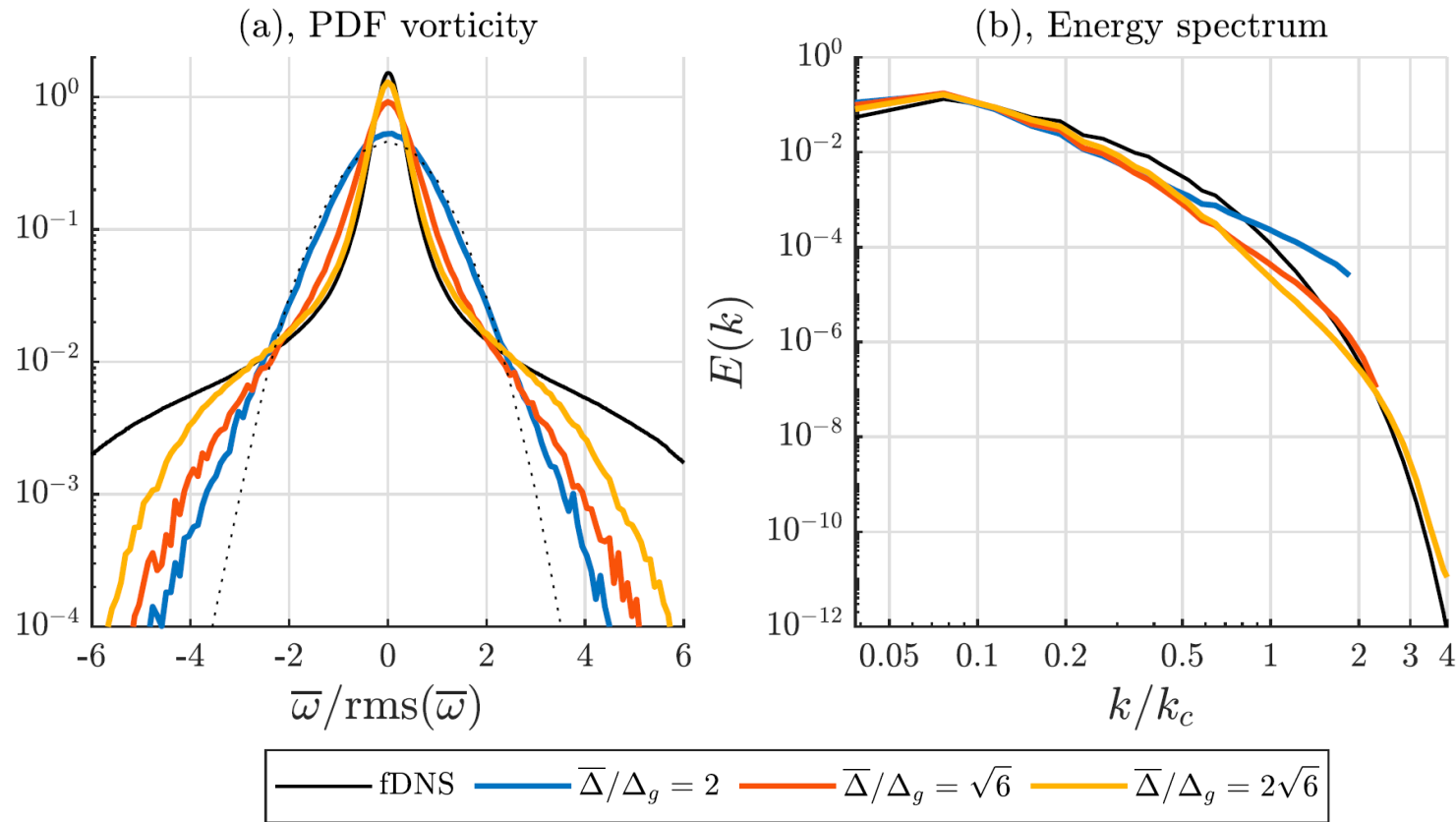
bilap:

$$\sigma_j^{mod} = (C_S \overline{\Delta})^4 |\overline{S}| \frac{\partial \nabla^2 \overline{\omega}}{\partial x_j}$$

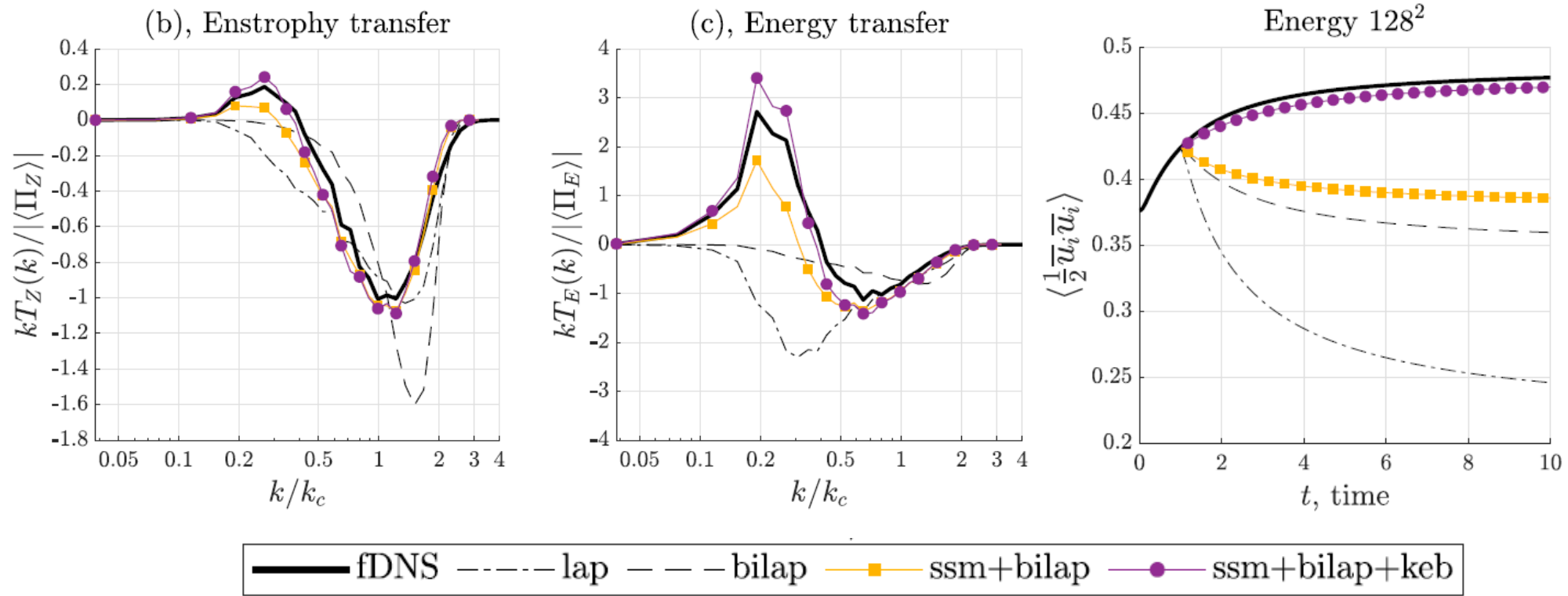
ssm+bilap:

$$\sigma_j^{mod} = \overline{\overline{u_j \omega}} - \overline{\overline{u_j}} \overline{\overline{\omega}} + (C_S \overline{\Delta})^4 |\overline{S}| \frac{\partial \nabla^2 \overline{\omega}}{\partial x_j}$$

Grid convergence for fixed filter width ($\bar{\Delta}$). ssm+bilap+keb model.

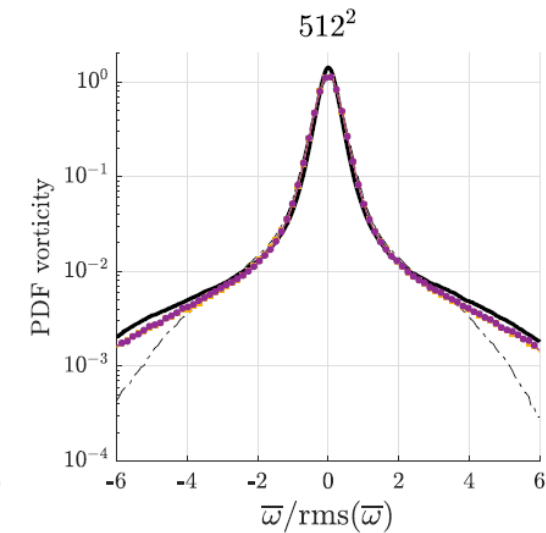
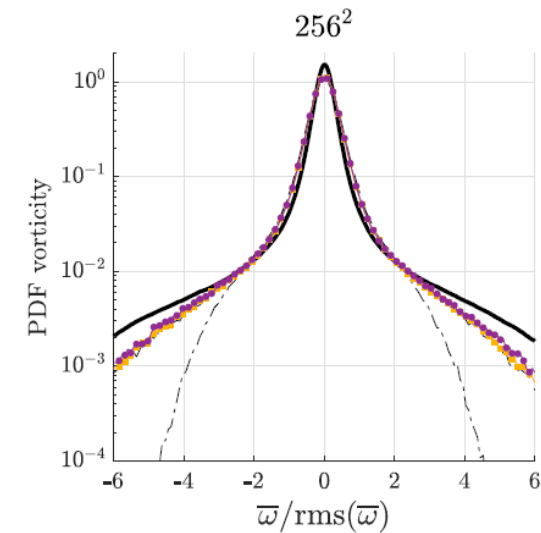
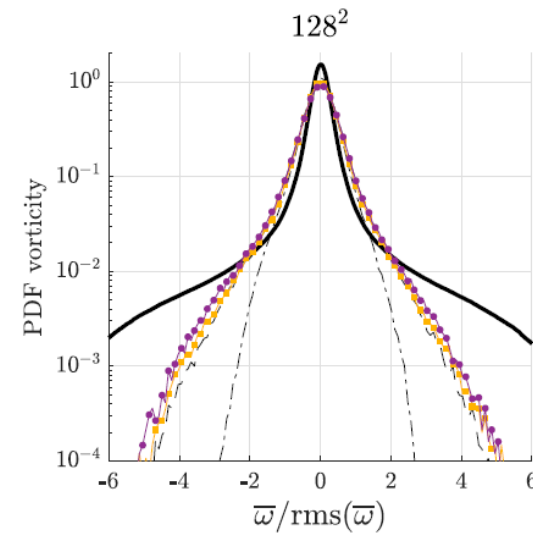
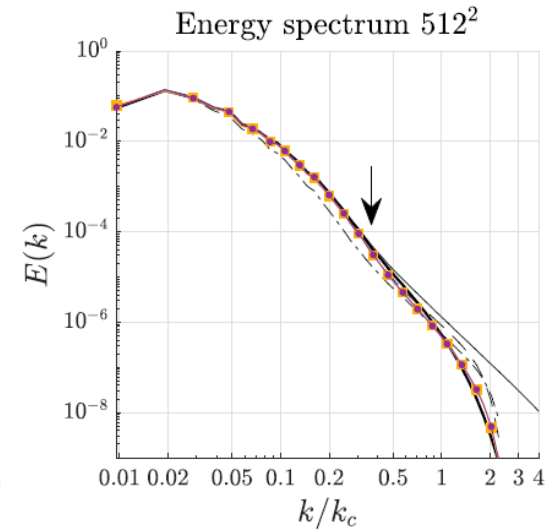
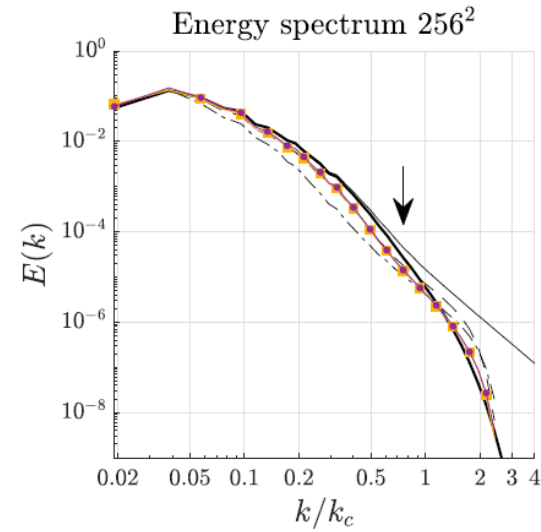
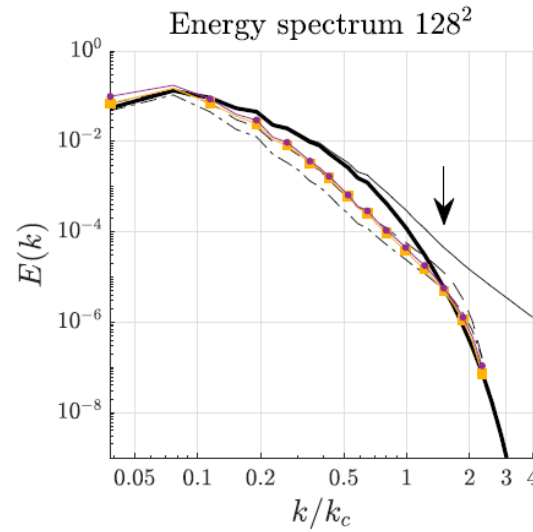
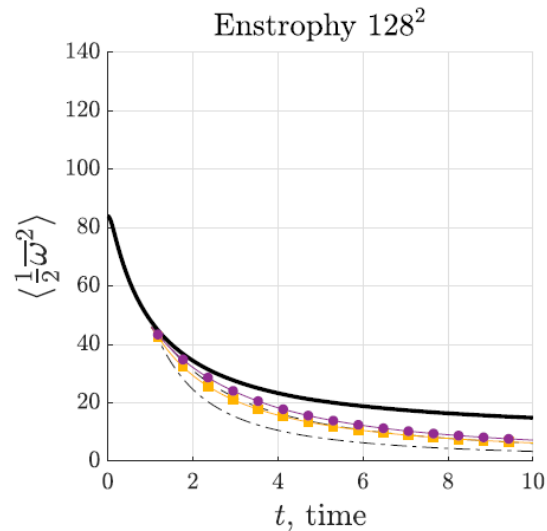


- At $\frac{\bar{\Delta}}{\Delta_g} = 2$ numerical noise arises
- The minimum required filter to grid width ratio is $\frac{\bar{\Delta}}{\Delta_g} = \sqrt{6}$



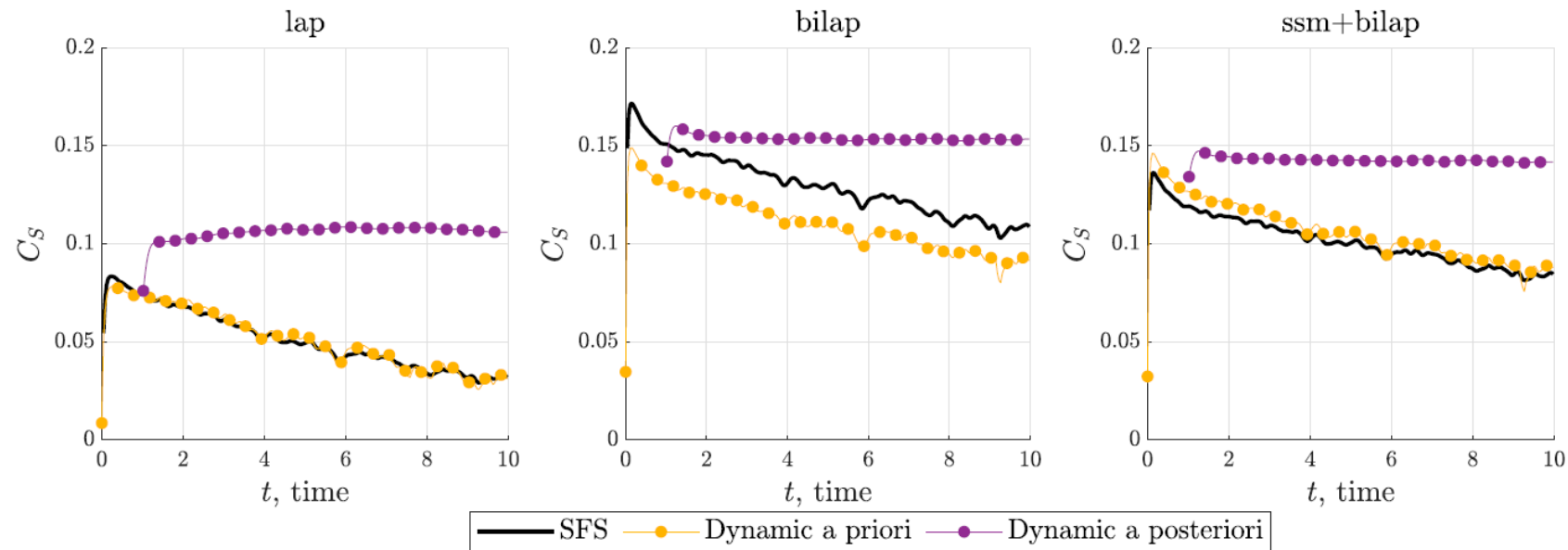
- Eddy viscosity models (lap and bilap) cannot simulate enstrophy and energy transfer spectra
- ssm+bilap reproduces enstrophy transfer
- ssm+bilap+keb introduces additional backscatter
- ssm+bilap+keb reproduces growth of resolved energy

- As $\bar{\Delta} \rightarrow 0$, all models converge to filtered DNS
- Slowest convergence for Laplacian model
- When coherent vortices size is close to filter width, all models underestimate energy in middle scales and dissipate enstrophy too fast

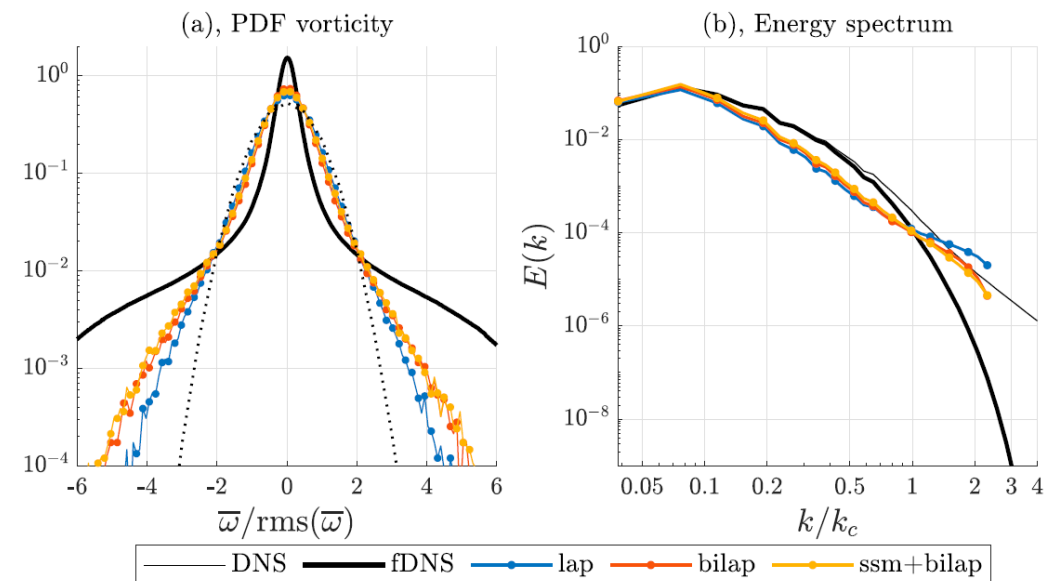


— fDNS - - - lap - - - bilap - - - ssm+bilap - - - ssm+bilap+keb

- Smagorinsky constant should decrease as coherent vortices develop (black line)
- Dynamic model in a priori analysis predicts this decrease (yellow)
- But in a posteriori experiments C_S is independent of time (violet)



- Experiments with C_S given by a priori data lead to noisy solutions



Conclusions

- General:
 - It is insufficient to reproduce energy and enstrophy transfer spectra to obtain correct representation of coherent structures
- About dissipation:
 - In a general case, dissipation is likely to be overestimated compared to a priori data
 - Dynamic model and scale-similarity model cannot improve dissipation compared with biharmonic Smagorinsky
- About backscatter:
 - Partial reconstruction of SGS stress can help to construct backscatter parameterization
 - It cannot improve dynamics of coherent structures
- About dynamic models:
 - They are accurate, but only in a priori analysis
 - They need base filter to be sufficiently large compared to grid step ($\bar{\Delta}/\Delta_g = \sqrt{6}$)
 - They predict the same Smagorinsky constant in any flow regime. Similar results can be achieved with the fixed constant
- About subgrid models:
 - It seems that cross stress is responsible for reproducing of coherent structures, but keeping in mind its very small spatial scale a model for cross stress will suffer from numerical instability