ITS3.1/NP0.1/CL4.23 Tipping Points in the Earth System (TiPES)
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Tipping Points Due to Parameter Drift in a Simple Model of the Wind-driven Ocean Circulation

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Pls. see further relevant material at
http://dist.altervista.org/PIERINI/ and
https://dept.atmos.ucla.edu/tcd/people/michael-ghil & https://www.researchgate.net/profile/Michael_Ghil

Motivation

- The self-consistent description, simulation and prediction of climate variability subject to climate change requires an understanding of nonautonomous dynamical systems (NDSs) and their pullback attractors (PBAs).

- A key feature of nonlinearity in classical, autonomous dynamical systems, is the presence of bifurcations that give rise to more complex behavior: multiple equilibria, limit cycles (LCs) and strange attractors.

- The generalization of bifurcations to NDSs is given by tipping points (TPs) that likewise lead to more complex behavior, including the coexistence of multiple PBAs with distinct stability properties.

- A particularly interesting case is that of excitable systems, in which a fixed point or small-amplitude, smooth solution coexists with a large-amplitude, more irregular one, like a relaxation oscillation (RO).

- The paradigmatic example of such a system is the Van der Pol oscillator.

- Interesting climatic examples are given by paleoclimatic oscillators on various time scales (glaciation cycles, Heinrich and Dansgaard-Oeschger events), as well as interannual variability of the wind-driven circulation.

- It is the latter case that is studied here in some detail.

The Simple Wind-driven Ocean Circulation Model – I

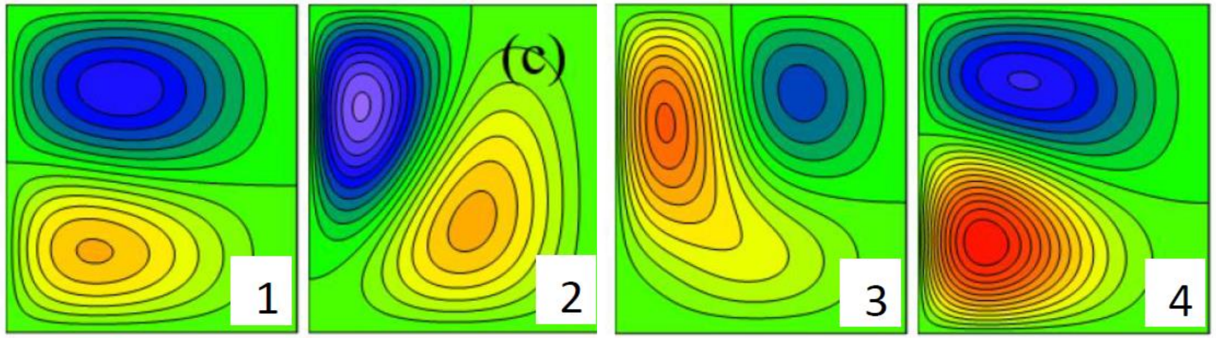
- The quasi-geostrophic (QG) model (Pierini, 2011) is projected onto 4 basis functions that include westward intensification.
- This low-order, spectrally truncated QG model is quadratically **nonlinear and excitable:** for time-independent wind stress a bifurcation leads from a small limit cycle to a large amplitude self-sustained RO.
- We will study its TPs under the action of a smooth drift in the external forcing + periodic perturbation.

$$\frac{\mathrm{d}\boldsymbol{\Psi}}{\mathrm{d}t} + \boldsymbol{\Psi}\mathbf{J}\boldsymbol{\Psi} + \mathbf{L}\boldsymbol{\Psi} = G\left(t\right)\mathbf{w}$$

$$\boldsymbol{\Psi}\left(t\right) = \left(\Psi_1, \Psi_2, \Psi_3, \Psi_4\right)$$

$$\psi(\mathbf{x}, t) = \sum_{i=1}^{4} \Psi_i(t) E_i(x, y)$$

Four successive snapshots of the streamfuntion showing a typical relaxation oscillation:

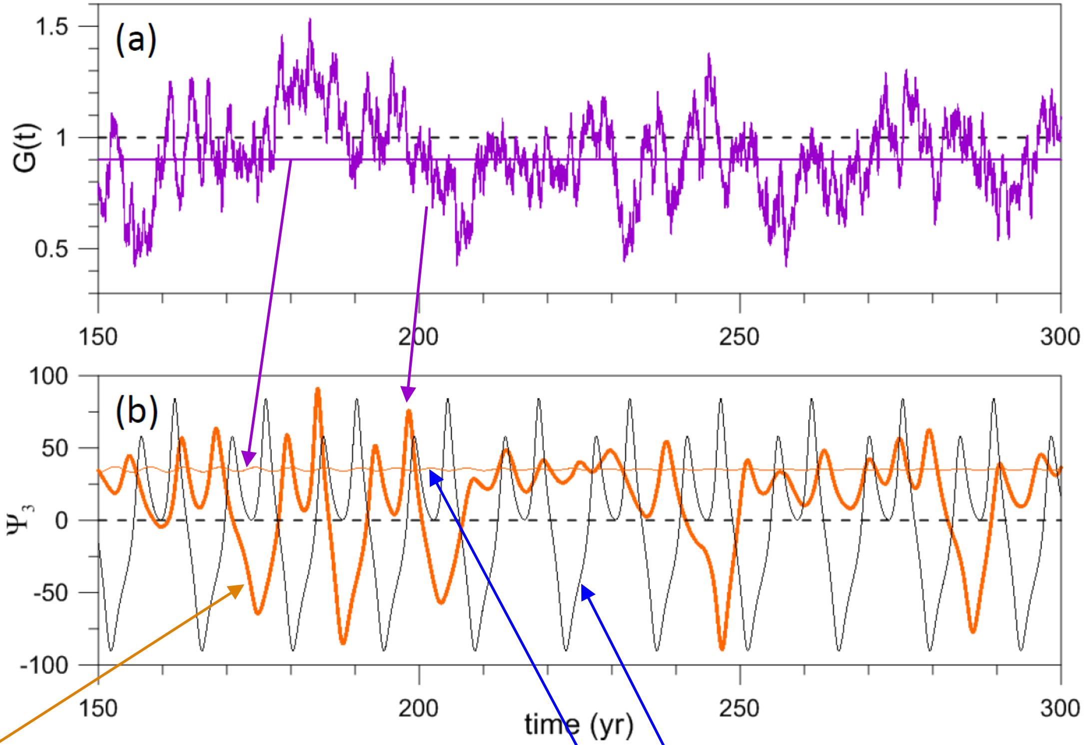


The Simple Wind-driven Ocean Circulation Model – II

- Two examples of **autonomous** behavior are identified by the thin, vertical blue arrows.
- The thick orange line shows that our model possesses the fundamental property of **excitability** when subjected to noisy forcing. This property is common to excitable systems relevant to the climate sciences, ranging from paleoclimatic to multidecadal and down to interannual time scales.

Time dependence of the wind forcing

Model response in terms of Ψ_3

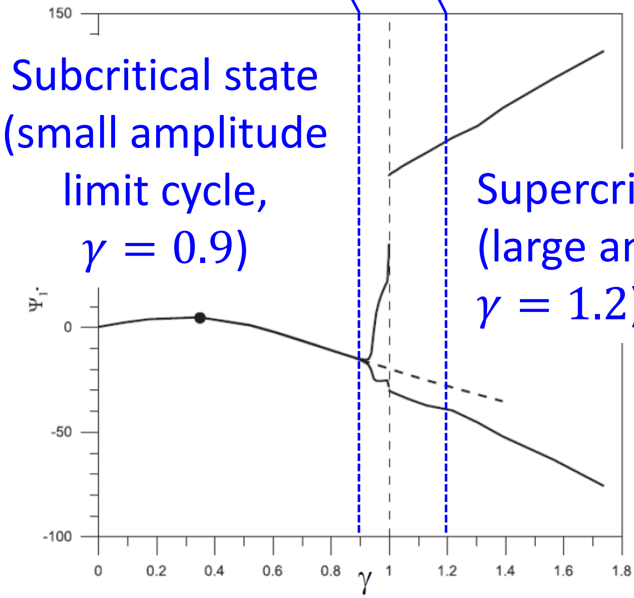


Excited response to a red noise forcing

$G(t) = \gamma + \varepsilon\zeta(t)$

Subcritical state (small amplitude limit cycle, $\gamma = 0.9$)

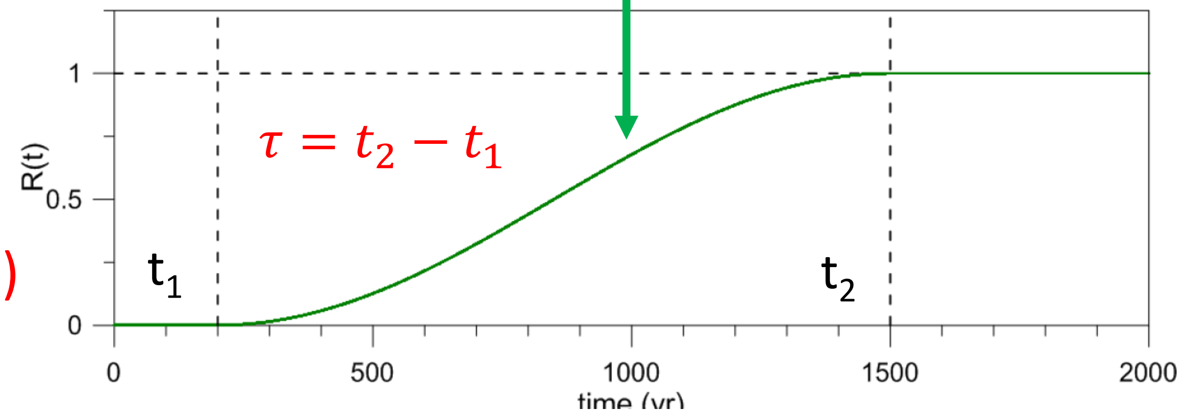
Supercritical state (large amplitude RO, $\gamma = 1.2$)



Bifurcation diagram of the autonomous system showing the range of variability of Ψ_1 vs the normalized forcing amplitude $G = \gamma = const.$ The dashed vertical line shows the sharp transition from small-amplitude oscillations to relaxation oscillations that occurs at $\gamma = 1$.

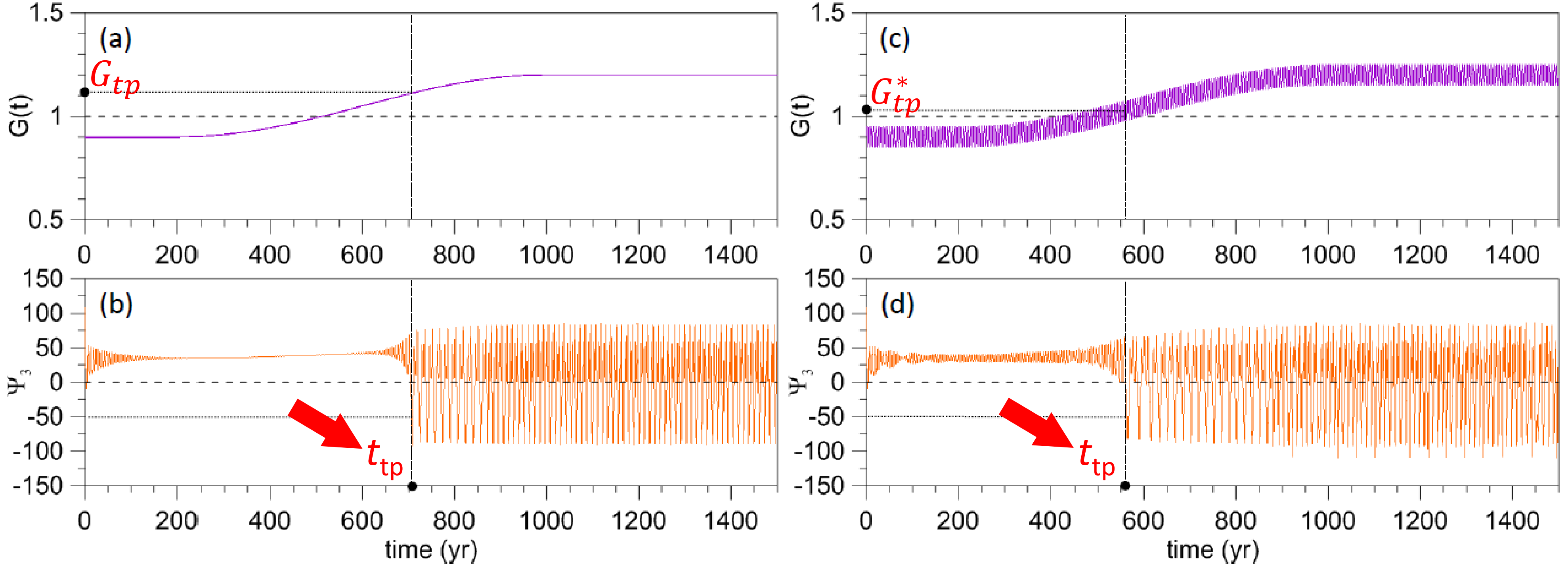
Effects of a Ramp and of a Periodic Perturbation – I

$$G(t) = \gamma + \alpha R_\tau(t) + \beta \sin(\omega t)$$



Definition of the time-dependent forcing

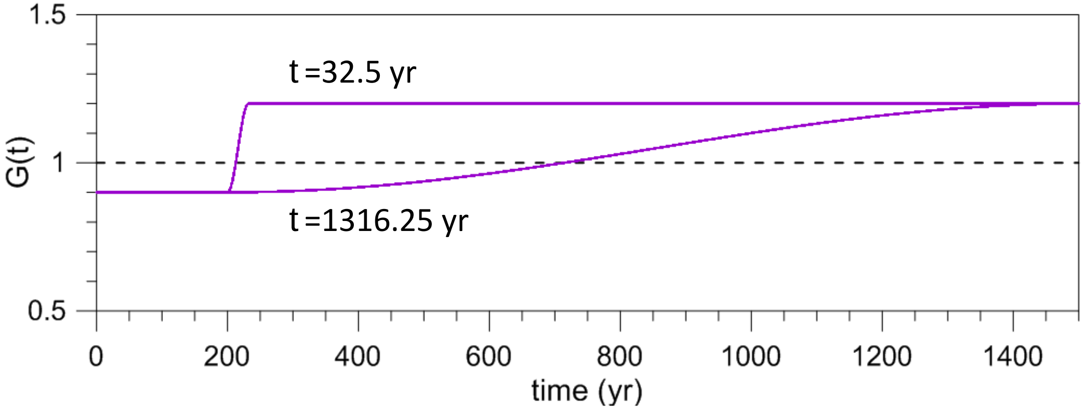
- Ramp $R(t)$ given by half-a-cosine function
- Tipping point time t_{tp}
- Corresponding forcing amplitude G_{tp} $(\beta = 0)$ or G^*_{tp} $(\beta \neq 0)$



Effects of a Ramp and of a Periodic Perturbation – II

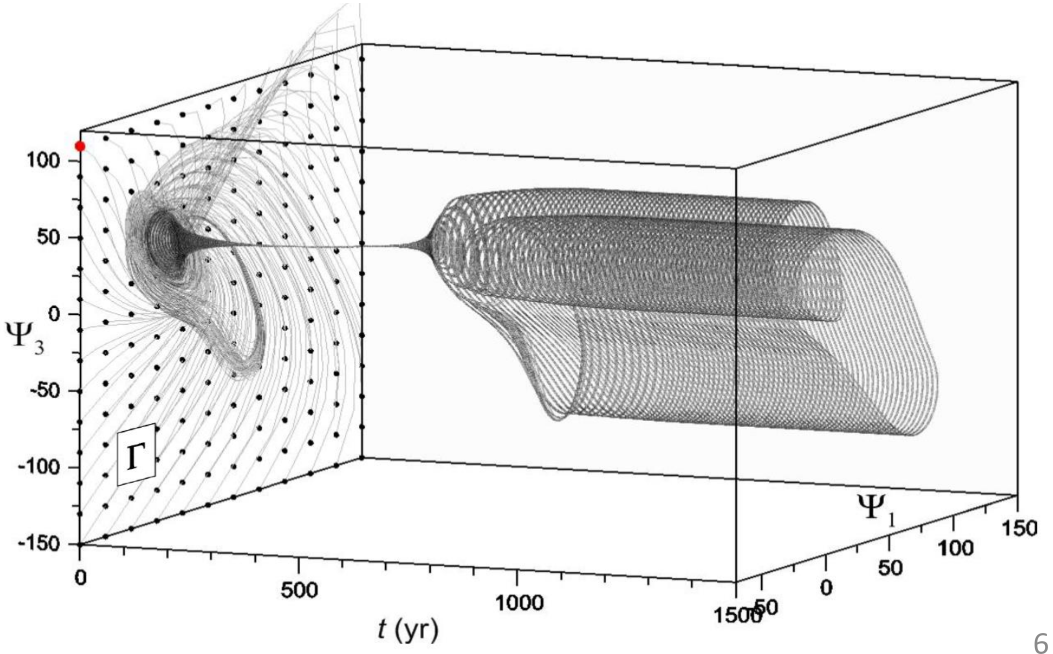
Numerical experiments

In Exp1 – Exp6, τ takes on 80 different values that range from 32.5 yr to 1316.25 yr.



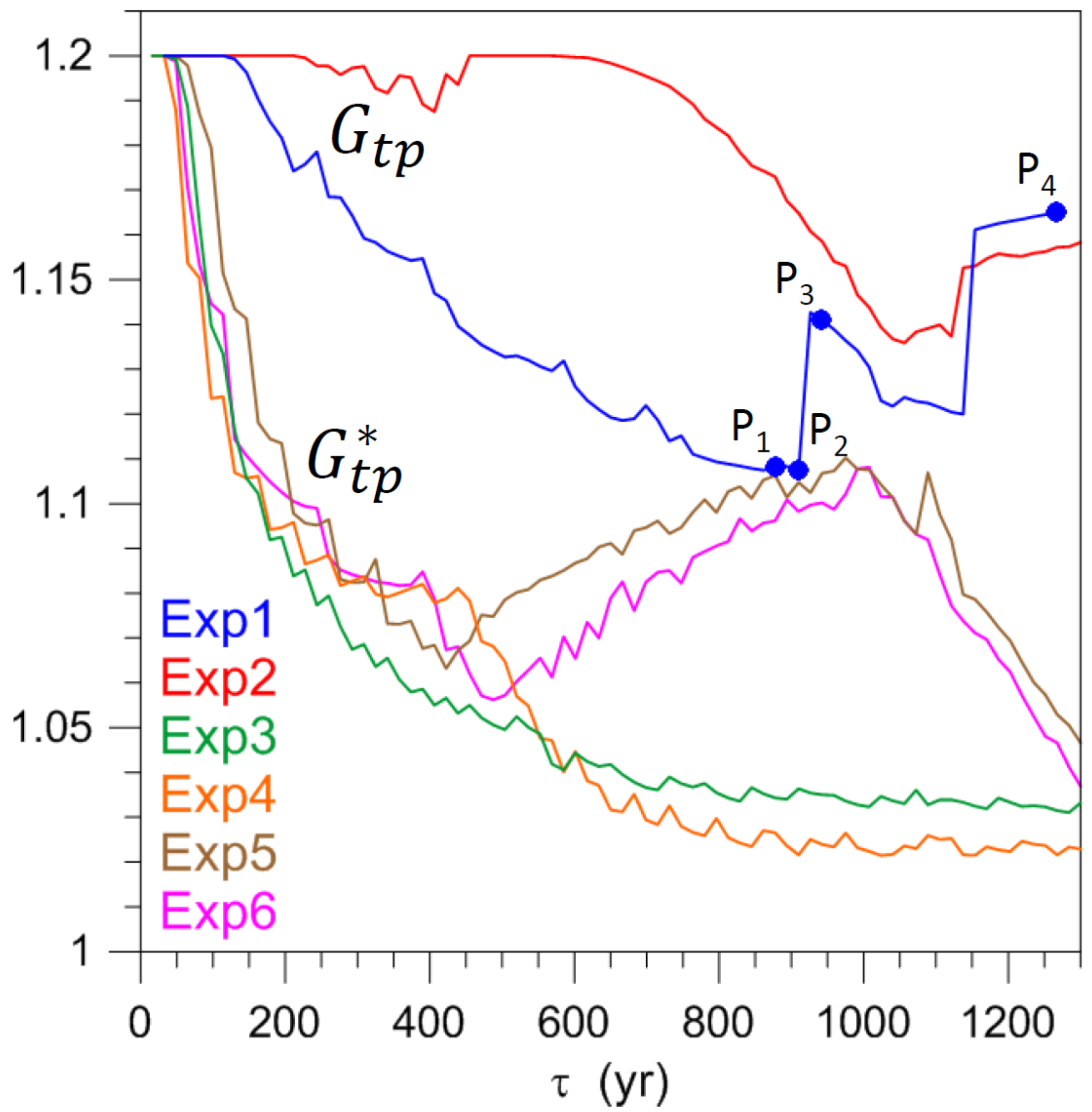
Numerical experiment	γ	α	τ (yr)	β	T (yr)
Exp1	0.9	0.3	32.5 - 1316.25	0	–
Exp2	0.8	0.4	"	"	–
Exp3	0.9	0.3	"	0.025	5
Exp4	"	"	"	0.050	"
Exp5	0.8	0.4	"	0.025	"
Exp6	"	"	"	0.050	"
Exp7	0.9	0.3	800	0.050	1 – 100
Exp8	"	"	"	0.100	"

For each value of τ, an **ensemble simulation (ES)** of $N = 12 \times 14 = 168$ members differing only by their initial points (dots in the Γ-plane on the right) is carried out in order to simulate the irreducible uncertainty associated with the system's internal variability

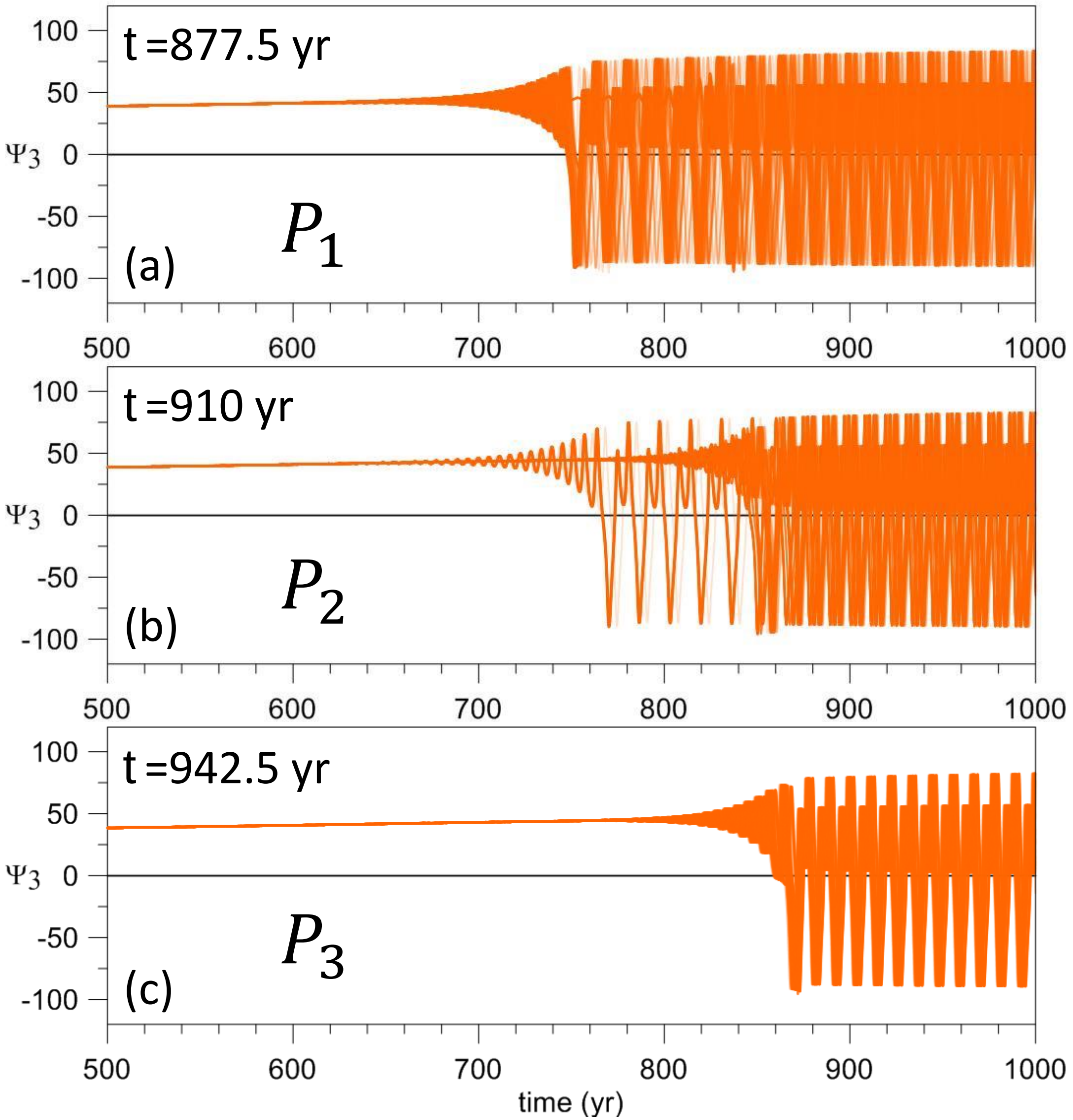


This allows us to estimate the model's **pullback attractor (PBA)**

Effects of a Ramp and of a Periodic Perturbation – III



Value G_{tp} or G^*_{tp} of the forcing at the **TP**, as a function of **ramp length**.



Sample solutions at the transition points $P_1 – P_3$ in G_{tp} at the left.

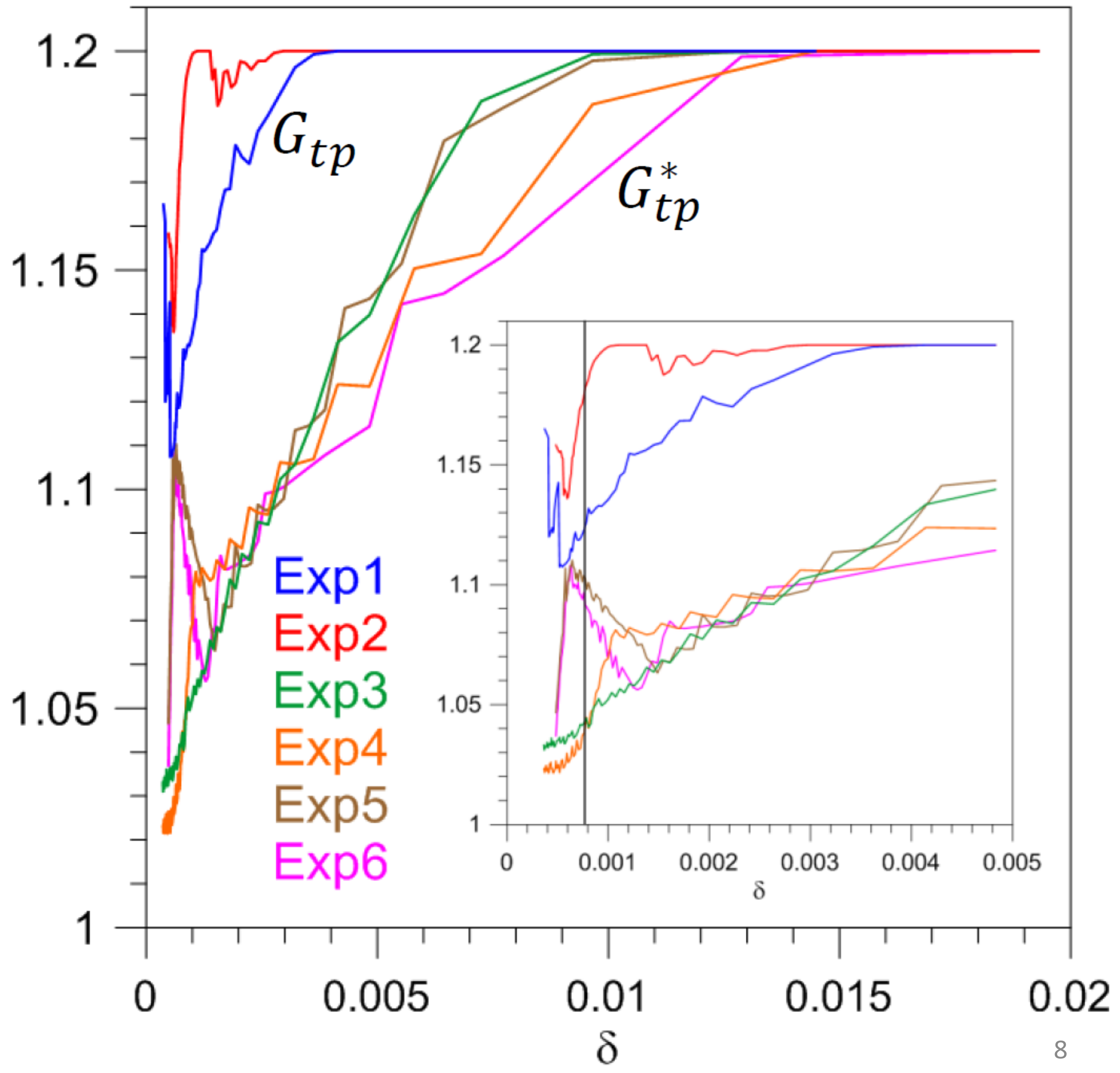
Effects of a Ramp and of a Periodic Perturbation – IV

Impact of **ramp steepness** δ

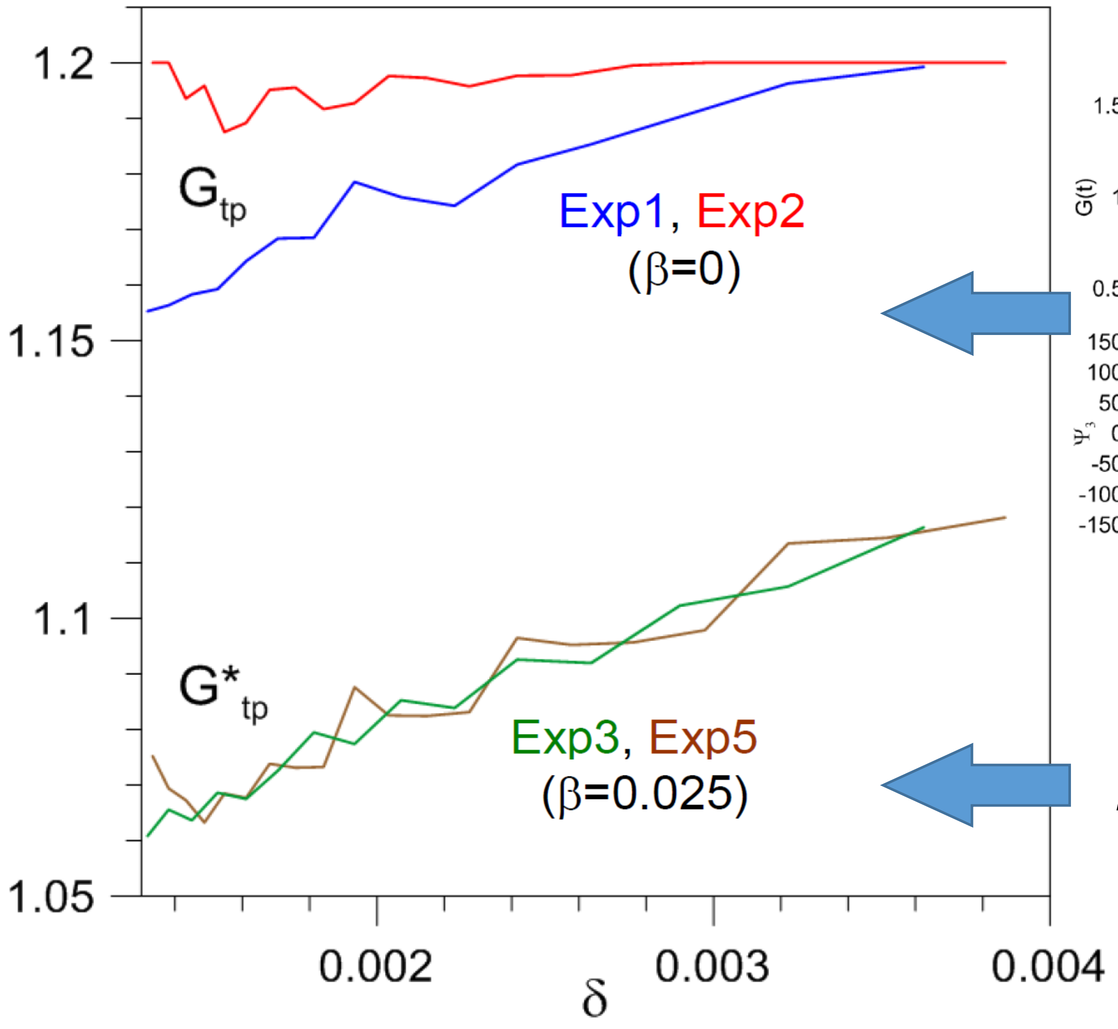
The **ramp steepness** δ of the forcing $G(t)$ can also be thought of as the **drift rate**:

$$\delta = \alpha\, R'_{\tau}\Big|_{t=(t_1+t_2)/2}$$

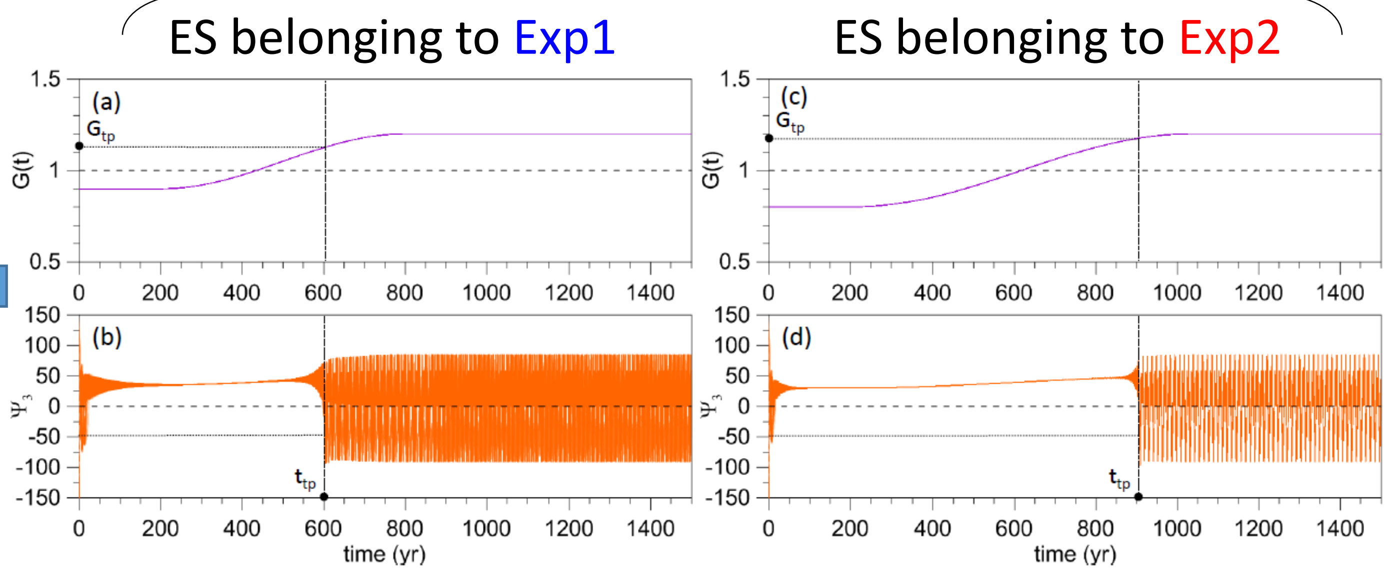
- Forcing level G_{tp} at the TPs **decreases** with ramp length τ and **increases** overall with ramp steepness δ.
- A periodic perturbation lowers the forcing level required to reach a TP: $G^*_{tp} \leq G_{tp}$.
- **Surprises occur where the PBA splits.**



Effects of a Ramp and of a Periodic Perturbation – V



same δ, different TPs



The difference between Exp1 vs. Exp2 and Exp3 vs. Exp5 consists in the values $\gamma = 0.9, \alpha = 0.3$ vs. $\gamma = 0.8, \alpha = 0.4$.

If the forcing is perturbed by a periodic component, i.e. $\beta \neq 0$, the scenario changes drastically: over a wide range of δ-values the forcing value at the TP depends "only" on δ.

This is, therefore, a case of
rate-induced tipping in an excitable system

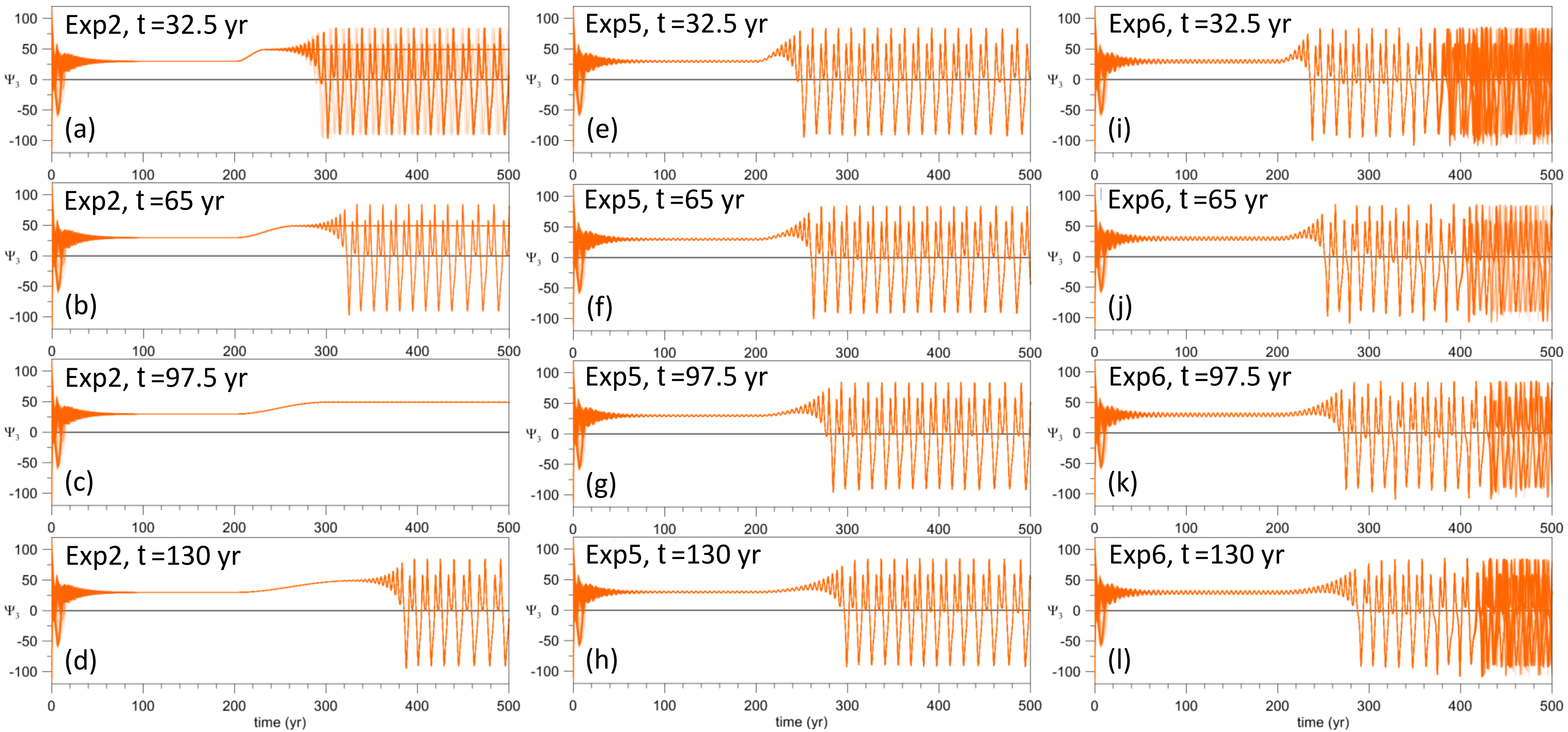
Effects of a Ramp and of a Periodic Perturbation – VI

Multiple local PBAs, Independence on initial data

Exp2 b=0

Exp5 b=0.025

Exp6 b=0.05



EXP2: **Multiple PBAs** appear in (a,b), **total independence on initial data** (**TIID**) is found in (c,d) and no TP arises in (c).

EXP5: With a periodic perturbation ($\beta \neq 0$) these features are lost, except the TIID, with the TP shifting away as τ increases.

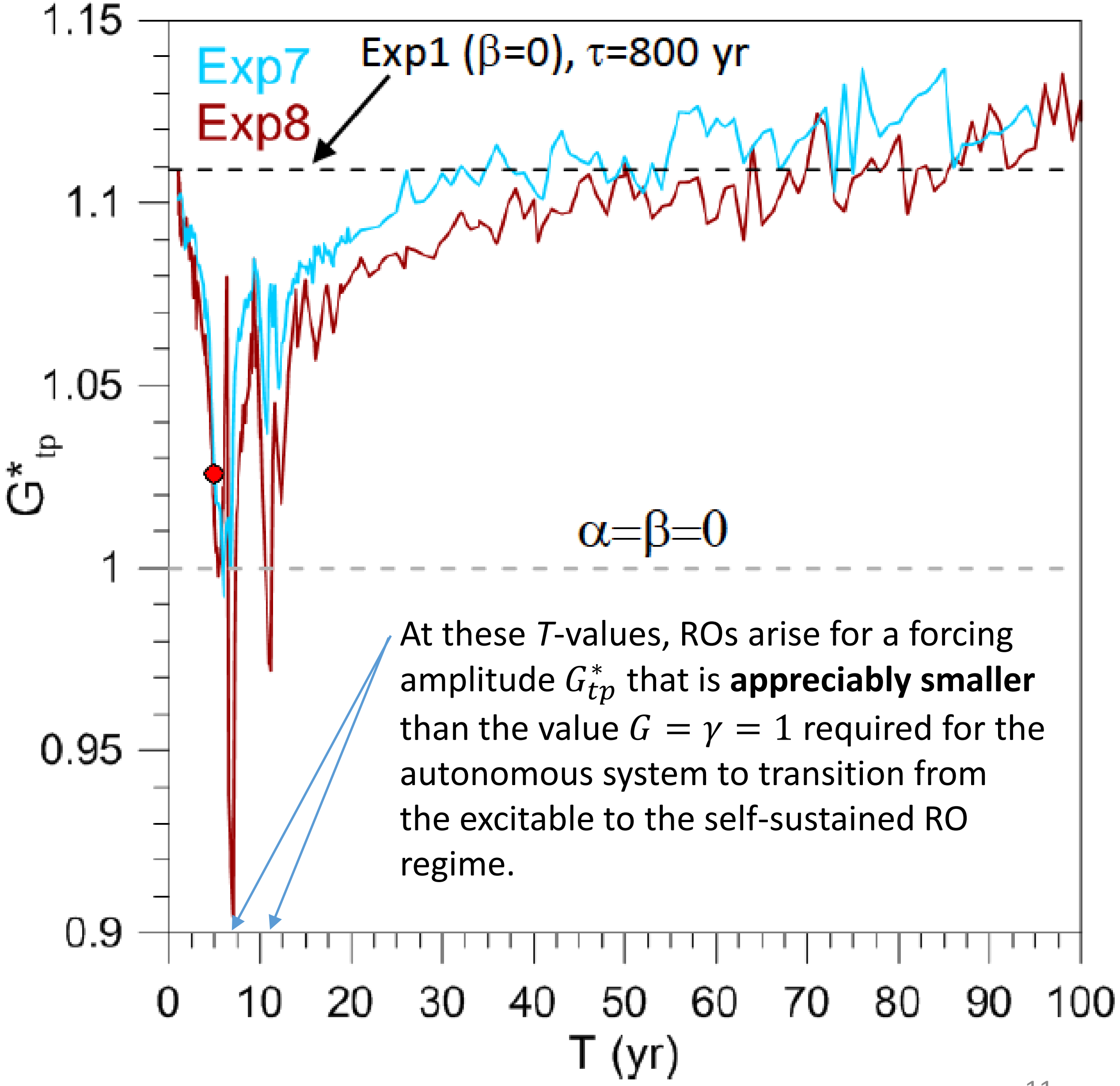
EXP6: When the periodic perturbation is larger (larger β), the TPs stay almost the same but the ROs become irregular.

Effects of a Ramp and of a Periodic Perturbation – VII

Nonlinear resonance

Exp7 & Exp8 differ from EXP1 in that a periodic forcing is now present, with T that varies from 1 to 100 yr, while τ = 800 yr is constant.

The abrupt reduction of G^*_{tp} occurs for periods T that are comparable to the typical time scale of the ROs; thus, **nonlinearly resonant**–like behavior seems to occur.

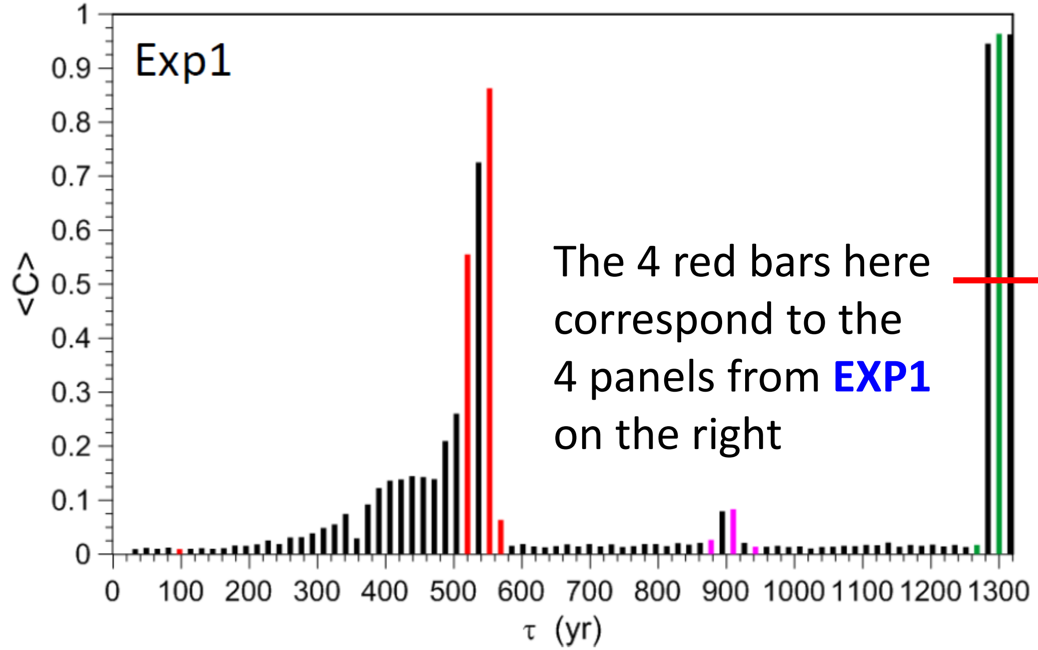


Clustering of Trajectories and Phase Coherence – I

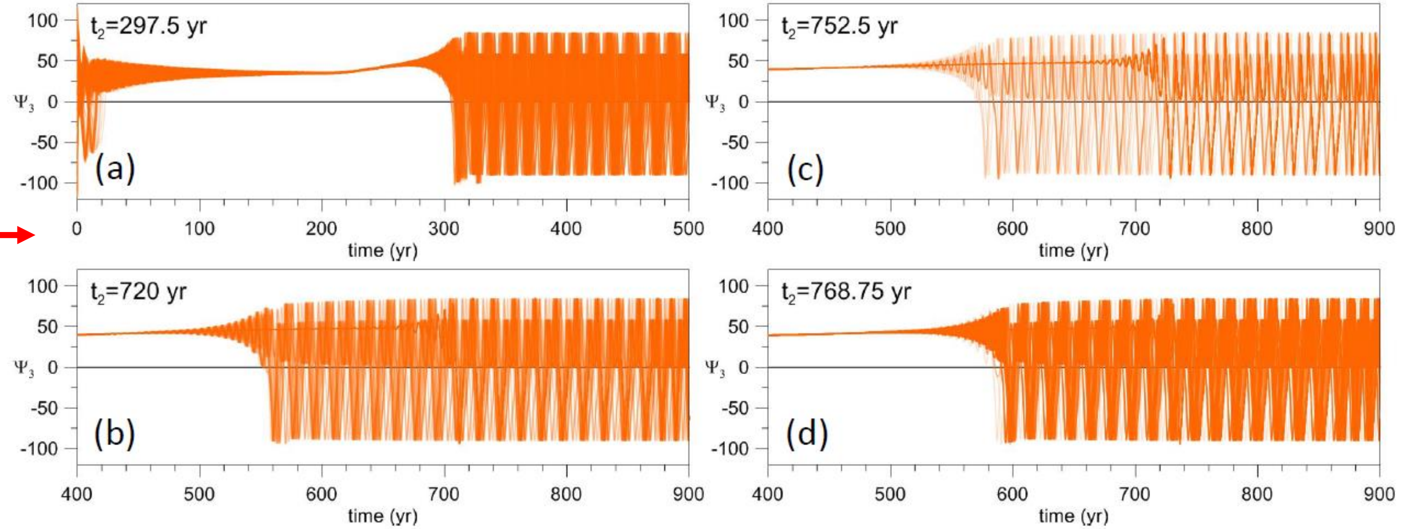
To study these aspects of the model's ROs, Pierini (2014) proposed to use the **parameter $\bar{C}$**, given by

$$\bar{C}(r,\tau) = \frac{1}{T_0}\int_{t_{tp}(\tau)}^{t_{tp}(\tau)+T_0} C(r,t,\tau)dt \quad \text{with} \quad C(r,t,\tau) = \frac{1}{N^2}\sum_{i,j}^{N} H\left[r - \left|\Psi_\tau^{(i)}(t) - \Psi_\tau^{(j)}(t)\right|\right]$$

to obtain information about the **clustering** and **phase dependence** of ensemble members. Here $H(r - x)$ is a Heaviside function that counts proximity of orbit pairs.



The 4 red bars here correspond to the 4 panels from **EXP1** on the right



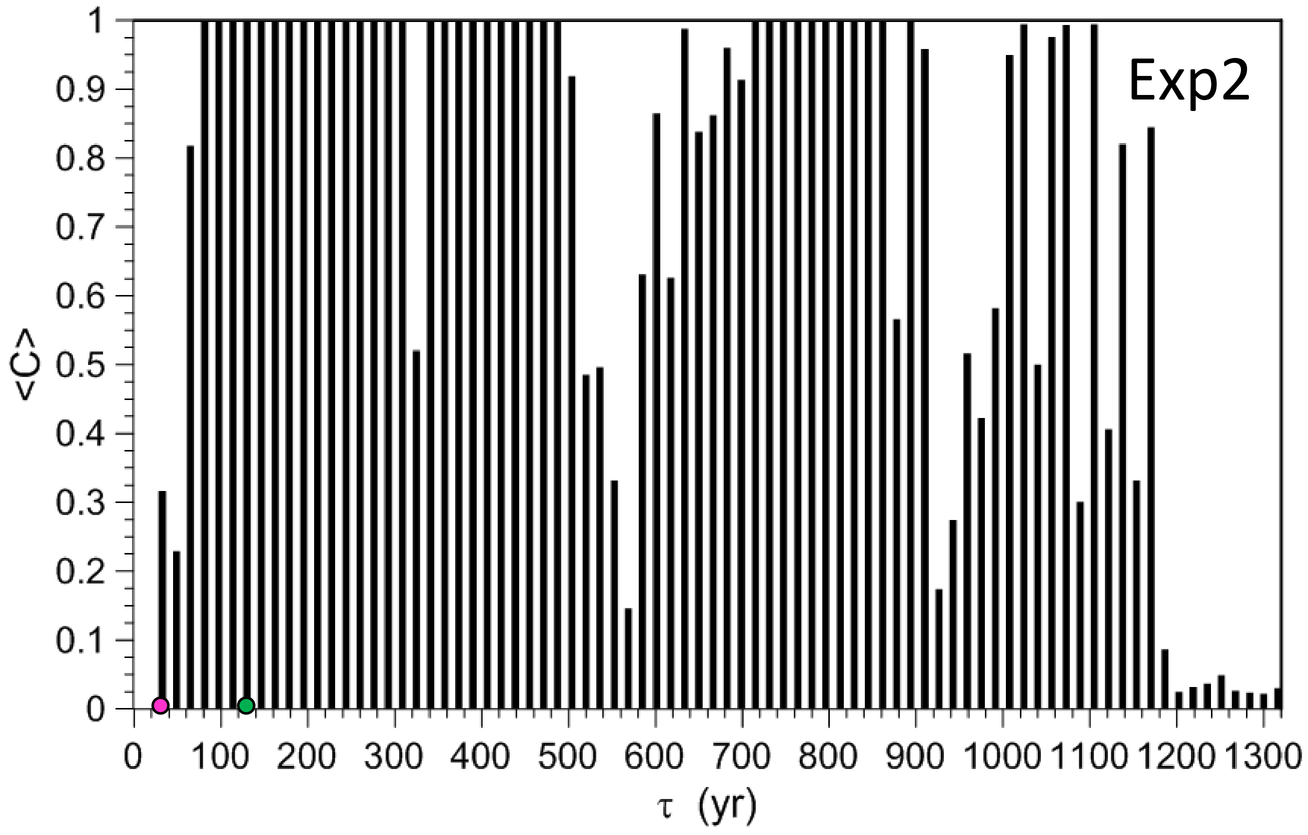
Notice strong contrast between panels (c), with $\bar{C} \cong 1$, and (d).

$(1/n) \leq \bar{C} \leq 1$, where n is the number of clusters defined by the distance r.

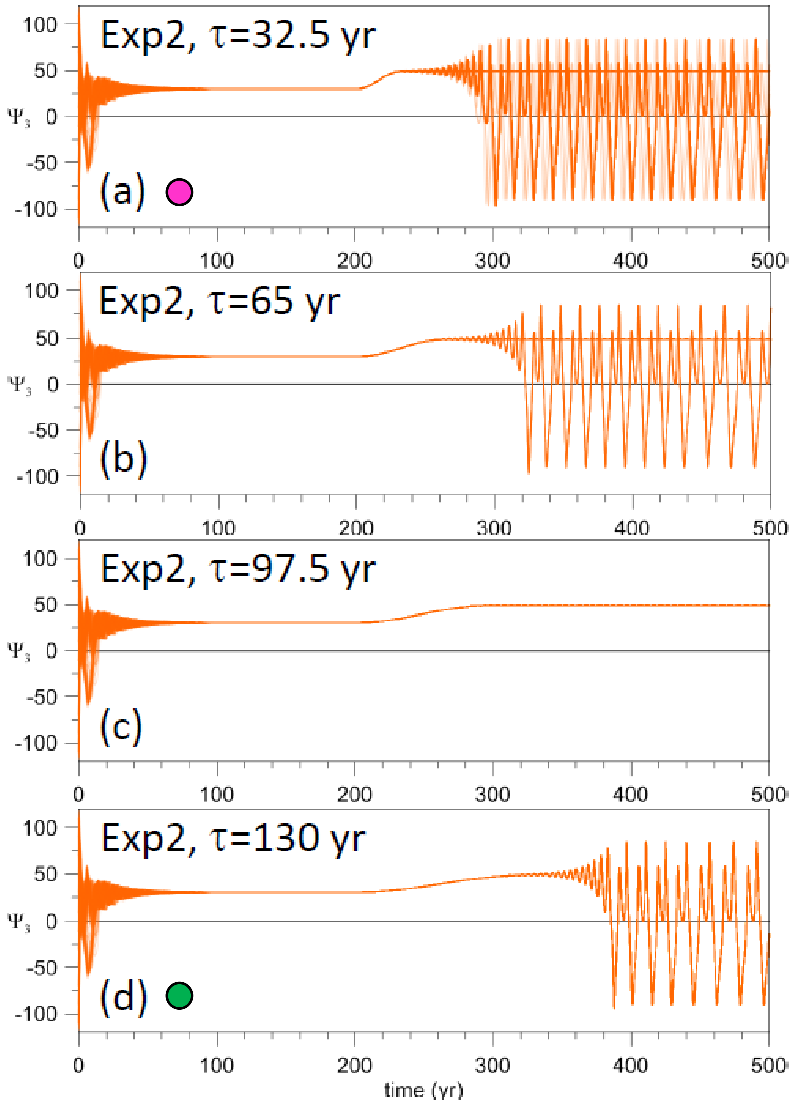
For $\bar{C} = 1$ one has only one cluster and, therefore, **TIID**.

Clustering of Trajectories and Phase Coherence – II

Unlike Exp1 in the previous slide, **Exp2** herein exhibits a **high degree** of clustering and phase coherence.



See again the sharp distinction between panel (a), with τ = 32.5 yr and $\bar{C} \cong 0.3$, and, e.g., panel (d), with τ = 130 yr and $\bar{C}$ = 1.

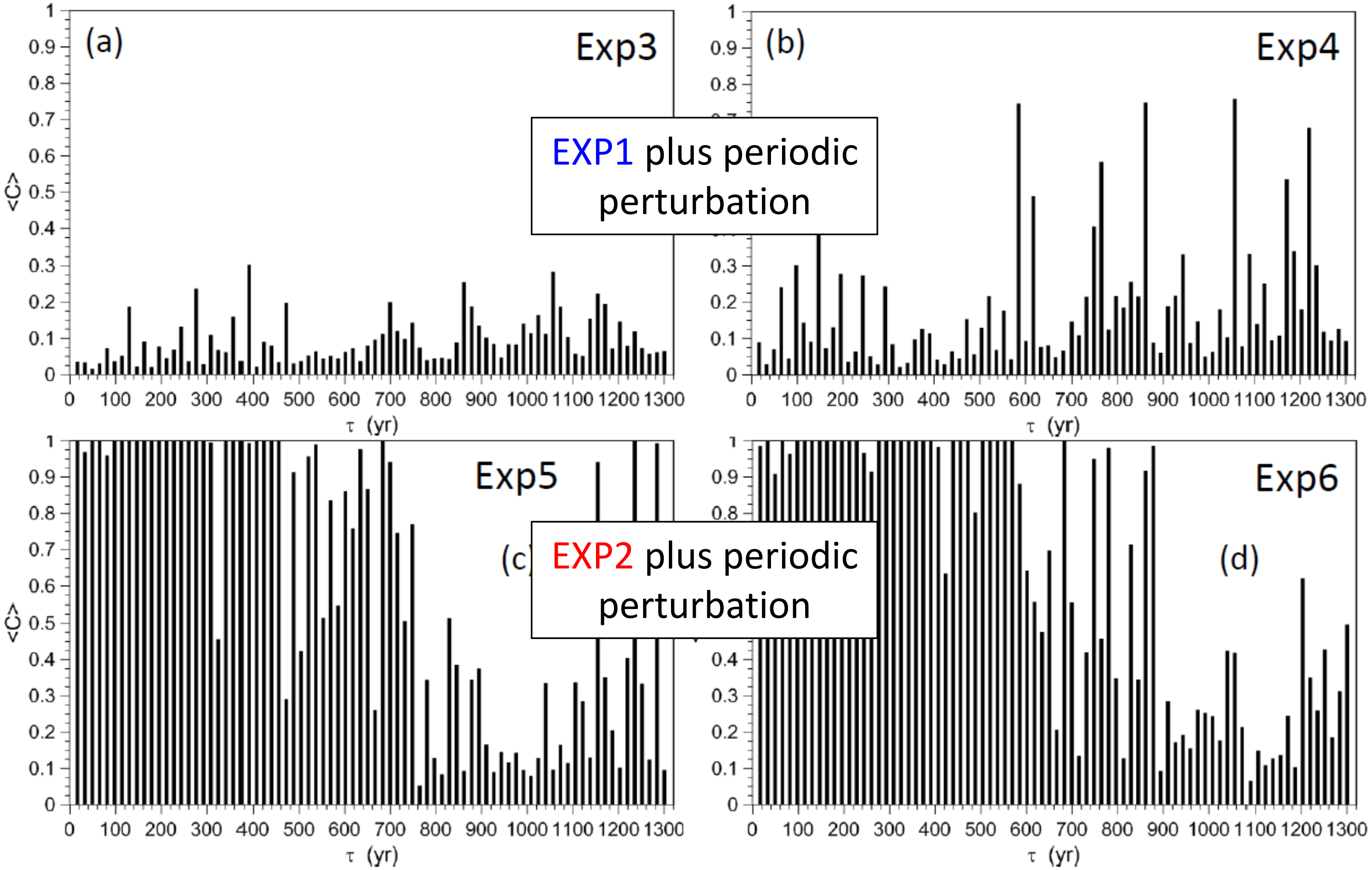


Clustering of Trajectories and Phase Coherence – III

Effect of the periodic perturbation on the clustering

$\gamma = 0.9, \alpha = 0.3,$
$\boldsymbol{\beta = 0.025}$

$\gamma = 0.9, \alpha = 0.3,$
$\boldsymbol{\beta = 0.050}$



$\gamma = 0.8, \alpha = 0.4,$
$\boldsymbol{\beta = 0.025}$

$\gamma = 0.8, \alpha = 0.4,$
$\boldsymbol{\beta = 0.050}$

Comparison with the previous two slides shows that the degree of clustering is basically preserved under periodic perturbation. Thus, TIID appears to be a fairly robust feature.

Concluding Remarks

- The simple, low-order QG model of the wind-driven circulation studied herein provides an excellent illustration of an excitable nonlinear model with relaxation oscillations (ROs).
- The ramp *R*(*t*) in the forcing represents anthropogenic warming or, rather, cooling, while the periodic forcing represents a seasonal effect.
- The time-dependent forcing leads to a remarkable plethora of phenomena that distinguish the model's NDS from the autonomous case.
- Overall, both the timing t_{tp} and the forcing value, G_{tp} or G_{tp}^*, at which the NDS tipping points (TPs) occur are distinct from the bifurcations of the autonomous model.
- The duration τ and the steepness δ of the forcing ramp play a major role in determining the changes in timing and in the forcing threshold of the TPs.
- The fine structure of the model's global PBA changes, with multiple local attractors equipped with distinct stability properties appearing and disappearing as parameters change.
- Periodic perturbations of small amplitude have a major effect on the TPs, as well as on the PBAs fine structure.
- There's much to do to fully understand this rich phenomenology.

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