Stable finite element method for solving the oblique derivative boundary value problems in geodesy

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Abstract

- We presents local gravity field modelling in spatial domain using the finite element method (FEM).
- For such a problem we derive a new numerical scheme where the oblique derivative BC are considered directly at computational nodes on the discretized Earth's topography.
- The developed FEM approach is tested by a reconstruction of a known harmonic function (EGM2008) above the extremely complicated Earth's topography in the Himalaya.
- A main numerical experiment is focused on very detailed local gravity field modelling in Slovakia using terrestrial gravity data.

Formulation of the FGBVPs

- In the bounded domain Ω, we consider the following BVP

$$\Delta T(\mathrm{x}) = 0, \quad \mathrm{x} \in \Omega \subset \mathrm{R}^3, \tag{1}$$

$$\nabla T(\mathrm{x}) \cdot \mathrm{s}(\mathrm{x}) = -\delta g(\mathrm{x}), \quad \mathrm{x} \in \Gamma \subset \partial\Omega, \tag{2}$$

$$T(\mathrm{x}) = T_{SAT}(\mathrm{x}), \quad \mathrm{x} \in \partial\Omega - \Gamma, \tag{3}$$

where $\Gamma \subset \partial\Omega$ represents the part of the Earth's topography, $\partial\Omega - \Gamma$ represents the top boundary together with side boundaries, and T_{SAT} is the disturbing potential generated from any GRACE/GOCE-based satellite-only geopotential model.

Formulation of the FGBVPs

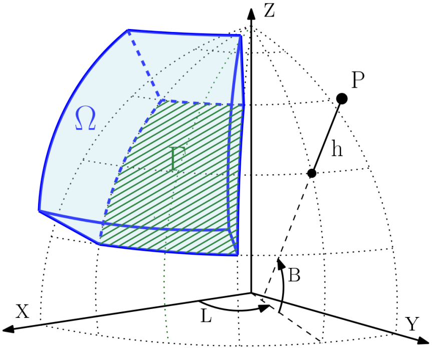

Figure 1: The computational domain Ω. The domain Ω is delimited by blue edges; the bottom surface $\Gamma \subset \partial\Omega$ hatched by green colour represents a chosen part of the Earth's surface; B, L, h coordinates denote ellipsoidal latitude, longitude and height, respectively.

Solution of the GBVP by the FEM on non uniform mesh

- In our approach, we follow the fundamental principles of FEM published in [1].
- We multiply the equation (1) by $w \in V$ and using Green's identity we get

$$\int_{\Omega} \nabla T \cdot \nabla w \,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_{\partial\Omega} \nabla T \cdot \mathbf{n}\, w \,\mathrm{d}\sigma, \quad w \in V. \tag{4}$$

Now we split the oblique vector **s** into one normal and two tangential components

$$\mathbf{s} = c_1 \mathbf{n} + c_2 \mathbf{t}_1 + c_3 \mathbf{t}_2, \tag{5}$$

where n is the normal vector and $\mathbf{t}_1$, $\mathbf{t}_2$ are tangent vectors to $\Gamma \subset \partial\Omega \subset R^3$. These three vectors together form an orthonormal basis.

[1] J.N. Reddy, An Introduction to the Finite Element Method, 3rd Edition, McGraw-Hill Education, New York, ISBN: 9780072466850 (2006)

Solution of the GBVP by the FEM on non uniform mesh

- Then we put (5) into (2) to obtain

$$\nabla T \cdot \mathrm{s} = c_1 \nabla T \cdot \mathbf{n} + c_2 \nabla T \cdot \mathbf{t}_1 + c_3 \nabla T \cdot \mathbf{t}_2 = -\delta g. \tag{6}$$

- From (6) we express the normal derivative

$$\nabla T \cdot \mathbf{n} = \frac{-\delta g}{c_1} - \frac{c_2}{c_1}\frac{\partial T}{\partial \mathbf{t}_1} - \frac{c_3}{c_1}\frac{\partial T}{\partial \mathbf{t}_2} \tag{7}$$

and we insert it to (4) to get

$$\int_{\Omega^e} \nabla T \cdot \nabla w \,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_{\Gamma^e} \left(\frac{-\delta g}{c_1} - \frac{c_2}{c_1}\frac{\partial T}{\partial \mathbf{t}_1} - \frac{c_3}{c_1}\frac{\partial T}{\partial \mathbf{t}_2} \right) w \,\mathrm{d}\sigma + \int_{\partial\Omega^e \setminus \Gamma^e} \nabla T \cdot \mathbf{n}\, w \,\mathrm{d}\sigma. \tag{8}$$

Solution of the GBVP by the FEM on non uniform mesh

- After some rearrangement, we have

$$\int_{\Omega^e} \nabla T \cdot \nabla w \,\mathrm{d}x\mathrm{d}y\mathrm{d}z + \frac{c_2}{c_1} \int_{\Gamma^e} \frac{\partial T}{\partial \mathbf{t}_1} w \,\mathrm{d}\sigma + \frac{c_3}{c_1} \int_{\Gamma^e} \frac{\partial T}{\partial \mathbf{t}_2} w \,\mathrm{d}\sigma =$$

$$= \int_{\Gamma^e} \frac{-\delta g}{c_1} w \,\mathrm{d}\sigma + \int_{\partial\Omega^e \setminus \Gamma^e} \nabla T \cdot \mathbf{n}\, w \,\mathrm{d}\sigma. \tag{9}$$

- In this way, we have obtained the weak formulation (4) or (9) of the BVP (1)-(3) on every element Ω^e of our finite element discretization. The study of weak solution of the oblique derivative BVP is included in the book by Lieberman[2].

[2] G.M. Lieberman, Oblique Derivative Problems for Elliptic Equations, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, ISBN: 978-981-4452-32-8 (2013)

Solution of the GBVP by the FEM on non uniform mesh

- To evaluate boundary integrals over a boundary Γ^e in Eq. (9), which include tangential derivatives, we approximate derivatives in the tangential direction like in the finite difference method. So using values of basis functions at nodes N_j^e of element Ω^e we have

$$\frac{\partial \psi_j}{\partial \mathbf{t}_{j,1}} \approx \frac{\psi_j(N_{j+1}^e) - \psi_j(N_j^e)}{d(N_j^e, N_{j+1}^e)}, \tag{10}$$

$$\frac{\partial \psi_j}{\partial \mathbf{t}_{j,2}} \approx \frac{\psi_j(N_{j-1}^e) - \psi_j(N_j^e)}{d(N_j^e, N_{j-1}^e)}, \tag{11}$$

where d denotes the distance between two neighbouring nodes that corresponds to the length of edge of an element Ω^e.

Solution of the GBVP by the FEM on non uniform mesh

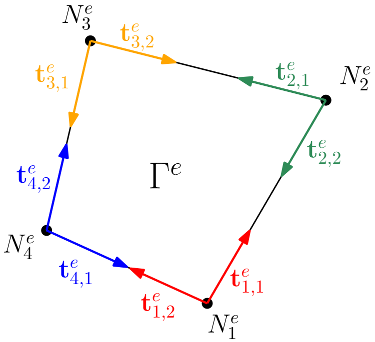

Figure 2: Illustration of tangent vectors $t_{i,j}^e$ to bottom boundary Γ^e at nodes N_i^e of an element Ω^e.

Reconstruction of EGM2008 over the Himalayas

- Numerical experiments aim to demonstrate how precise we are able to reconstruct a harmonic function above the extremely complicated Earth's topography in the Himalayas and Tibetan Plateau.
- The EGM2008 geopotential model up to degree 2160[3] has been used.
- The upper boundary has been chosen at the altitude of 230 km above the reference ellipsoid corresponding to an average altitude of the GOCE satellite orbits. The bottom boundary has been given by grid points located on the Earth's surface. Their spacing in horizontal directions has been uniform. Their heights have been interpolated from the SRTM30 PLUS topography model[4].

[3] N.K. Pavlis, S.A. Holmes, S.C. Kenyon, J.K. Factor, The development and evaluation of the Earth Gravitational Model 2008 (EGM2008). Journal of Geophysical Research, 117, B04406,(2012)

[4] J.J. Becker, et al., Global Bathymetry and Elevation Data at 30 Arc Seconds Resolution: SRTM30 PLUS, Marine Geodesy, 32,4, 355-371, (2009)

Reconstruction of EGM2008 over the Himalayas

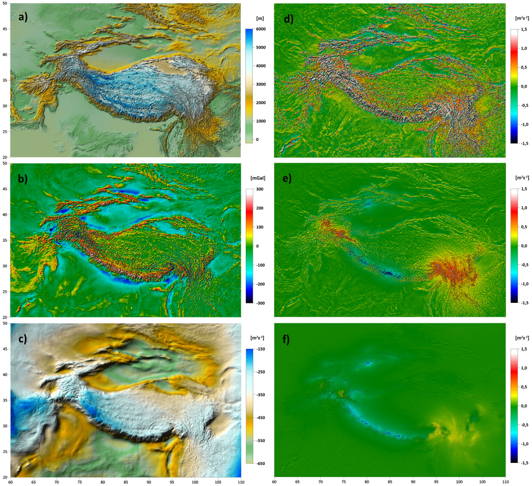

Figure 3: **a)** The Earth's surface topography in the Himalayas and Tibetan Plateau (the bottom boundary), **b)** gravity disturbances (the oblique derivative BC), **c)** the disturbing potential generated from EGM2008 on the Earth's surface (a reconstructed harmonic function), and (**d**, **e**, **f**) residuals between the FEM solutions and EGM2008 for different discretizations of the computational domain: **d)** 501 × 301 × 25, **e)** 1001 × 601 × 49, and **f)** 2001 × 1201 × 97.

Reconstruction of EGM2008 over the Himalayas

Table 1: Statistics of residuals between our FEM solution and EGM2008 on the bottom boundary in the Himalayas (units: $m^2 s^{-2}$)

No. of nodes	$501 \times 301 \times 25$	$1001 \times 601 \times 49$	$2001 \times 1201 \times 97$
Min. value	-4.26	-5.69	-3.07
Mean value	0.17	0.01	-0.02
Max. value	7.19	2.12	0.76
St. deviation	0.61	0.17	0.09

Reconstruction of EGM2008 over the Himalayas

Table 2: Statistics of residuals between our FEM solution and EGM2008 in the whole 3D computational domain above the Himalayas (units: $m^2 s^{-2}$)

No. of nodes	$501 \times 301 \times 25$	$1001 \times 601 \times 49$	$2001 \times 1201 \times 97$
Min. value	-2.80	-1.48	-0.75
Mean value	0.95	0.41	0.18
Max. value	20.90	10.98	3.25
St. deviation	1.45	0.65	0.29

Local gravity field modelling in Slovakia

- Input data
  - gravity disturbances: generated from the detailed map of the Complete Bouguer Anomalies[5] using the CBA2G software[6]
  - disturbing potential on top and 4 side boundaries: EIGEN-6C4 geopotential model up to d/o 2160 [7]
  - terrestrial data: Earth's topography with the horizontal resolution 100 x 100 m
- Computational domain
  - number of elements: 4,700 x 2,500 x 450 (longitude x latitude x height)

[5] R. Pašteka, et al., High resolution Slovak Bouguer gravity anomaly map and its enhanced derivative transformations: New possibilities for interpretation of anomalous gravity fields. Contributions to Geophysics and Geodesy, 47 (2), pp. 81-94. DOI: 10.1515/congeo-2017-0006 (2017)

[6] I. Marušiak, et al., CBA2G (Complete Bouguer Anomaly To Gravity), program for calculation of the gravity acceleration from complete Bouguer anomaly, program guide. Manuscript, G-trend Ltd, (2015)

[7] Ch. Förste, et al., EIGEN-6C4 The latest combined global gravity field model including GOCE data up to degree and order 2190 of GFZ Potsdam and GRGS Toulouse. GFZ Data Services. (2014)

Local quasigeoid model in Slovakia

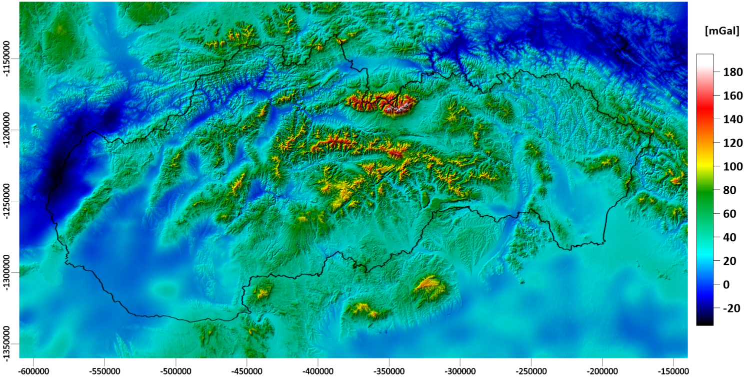

Figure 4: Gravity disturbances on the Earth's topography as the oblique derivative BC on the bottom boundary

Local gravity field modelling in Slovakia

- To get a local quasigeoid model, the disturbing potential T obtained on the bottom boundary, i.e. at points directly on the Earth's surface, has been transformed into the quasigeoidal heights ξ.
- To validate its precision, the GNSS/levelling test has been performed at 396 benchmarks. The local quasigeoid model has been also compared with DVRM05 (Digital Vertical Reference Model), which is currently "*an official model in Slovakia to transform ellipsoidal heights (determined by GNSS in the ETRS89 system) into sea level heights, namely into the normal heights in the Bpv vertical system*" (www.geoportal.sk).

Local quasigeoid model in Slovakia

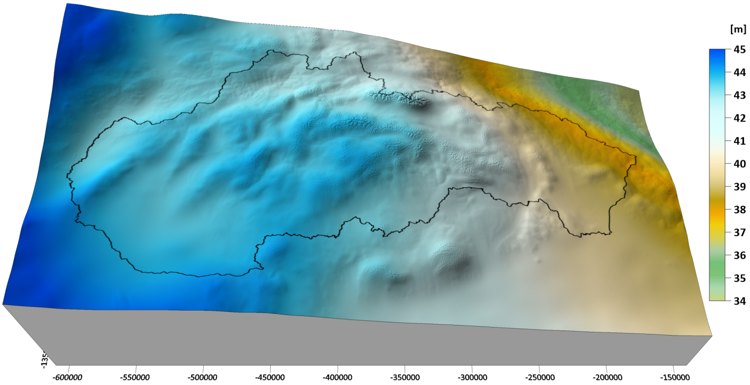

Figure 5: Local quasigeoid model in Slovakia as the FEM numerical solution of FGBVP (the horizontal resolution: 100 m x 100 m)

Local quasigeoid model in Slovakia

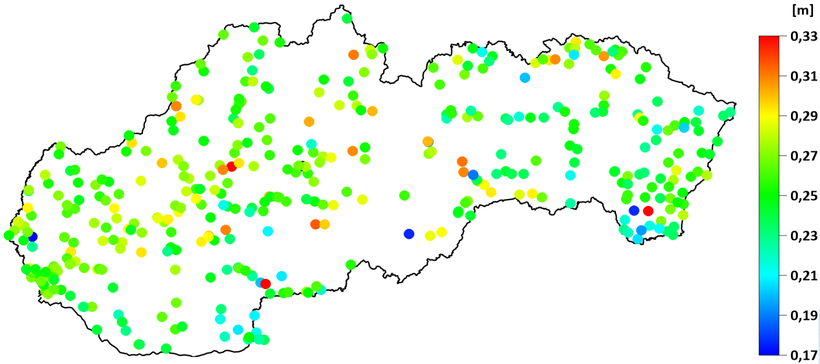

Figure 6: GNSS/levelling benchmarks with differences between the obtained local quasigeoid model and DVRM05.

Local quasigeoid model in Slovakia

Table 3: Statistics of the GNSS/levelling test of the recent quasigeoid models at 396 GNSS-levelling benchmarks.

Local quasigeoid model	The FEM approach	The FVM approach[8]	Spherical Radial Basis Functions [9]
Range	19.3 cm	20.5 cm	18.8 cm
Mean	22.3 cm	23.1 cm	-53.7* cm
St. dev.	2.54 cm	2.63 cm	2.46 cm

* the mean value differs due to different transformation of the disturbing potential and related to the GRS-80 reference ellipsoid

[8] Čunderlík, M. Medľa, K. Mikula, Local quasigeoid modelling in Slovakia using the finite volume method on the discretized Earth's topography. Contributions to Geophysics and Geodesy, 50(3), 287-302.

[9] B. Bucha, J. Janák, J. Papčo, A. Bezděk, High-resolution regional gravity field modelling in a mountainous area from terrestrial gravity data, Geophysical Journal International, Volume 207, Issue 2, 1 November 2016, 949–966, (2016)

Conclusions

- We have presented an numerical scheme to approximate the solution of the Laplace equation with an oblique derivative boundary condition by the finite element method.
- Reconstruction of EGM2008 as a harmonic function over the Himalayas and Tibetan Plateau has shown that with a sufficient refinement of the discretization we are able to achieve "cm-level" accuracy, even on such extremely complicated Earth's surface.
- Our approach based on the local gravity field modelling in spatial domain using FEM on the unstructured 3D mesh about the real Earth's topography has resulted in the quasigeoid model whose accuracy is about 2.54 cm.