

Interaction of Fully-Nonlinear Internal Solitary Waves of Opposite Polarity

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Overview

- Internal Solitary Waves (ISWs) are known to exist in three-layer and double pycnocline stratifications provided certain conditions are satisfied. In particular, the upper and lower pycnoclines must be 'close' enough to the upper and lower boundaries in some sense.
- Conjugate flow solutions are used to determine the regions in parameter space where ISWs of both polarities exist.
- In these regions the interaction of two ISWs of equal or opposite polarity are investigated.
- Simulations done using a fully nonlinear numerical model. Initial solitary waves are provided by solutions of the DJL equation which are exact ISW solutions of the governing equations.

DIMENSIONLESS GOVERNING EQUATIONS

$$u_t + uu_x + wu_z = -p_x$$

$$w_t + uw_x + ww_z = -p_z - \rho$$

$$\rho_t + u\rho_x + w\rho_z = 0$$

$$u_x + w_z = 0$$

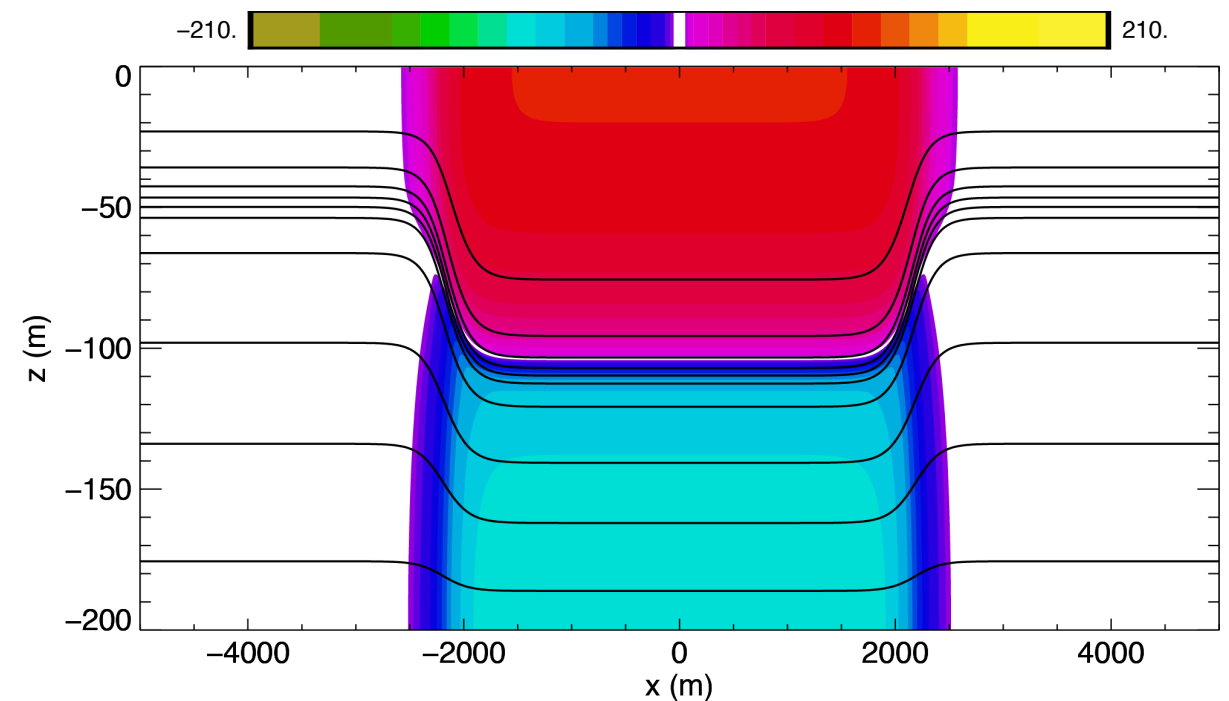
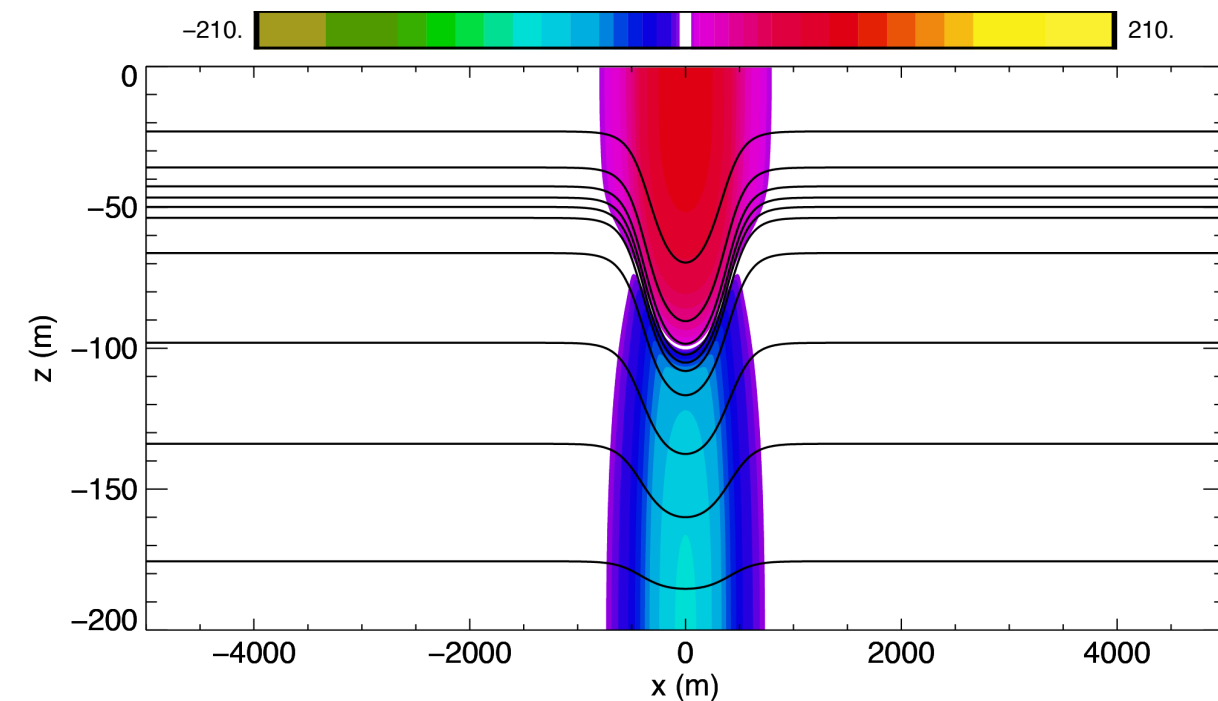
nondimensionalized so that the density variation ρ varies between approximately -0.5 and 0.5 .

- Nonhydrostatic
- Boussinesq approximation
- two-dimensional model:
independent of y
- incompressible
- no rotation
- rigid lid

Initial ISWs obtained from solutions of the Dubreil-Jacotin Long (DJL) equation

(Turkington et al 1991, Lamb & Wan 1998, Stastna & Lamb 2002)

For many stratifications waves have a limiting amplitude becoming flat-crested and horizontally uniform in the centre



Conjugate Flows

In the centre of a flat-crested ISW the flow becomes horizontally uniform. This flow is called a *conjugate flow* (relative to the flow outside the wave) and is given by solutions of

$$\begin{aligned}\eta'' + \frac{N^2(z - \eta)}{c^2} \eta &= 0 \\ \eta(-H) &= \eta(0) = 0 \\ \eta'(0) &= s\end{aligned}$$

where $\eta(z)$ is the vertical displacement of isopycnals/streamlines.

The value of $s \neq 0$ is chosen to satisfy an *auxiliary condition* (Benjamin 1966; Lamb & Wan 1998)

$$T(s) = \int_{-H}^0 \eta'^3(z; s) \, dz = 0$$

where the prime denotes differentiation with respect to z .

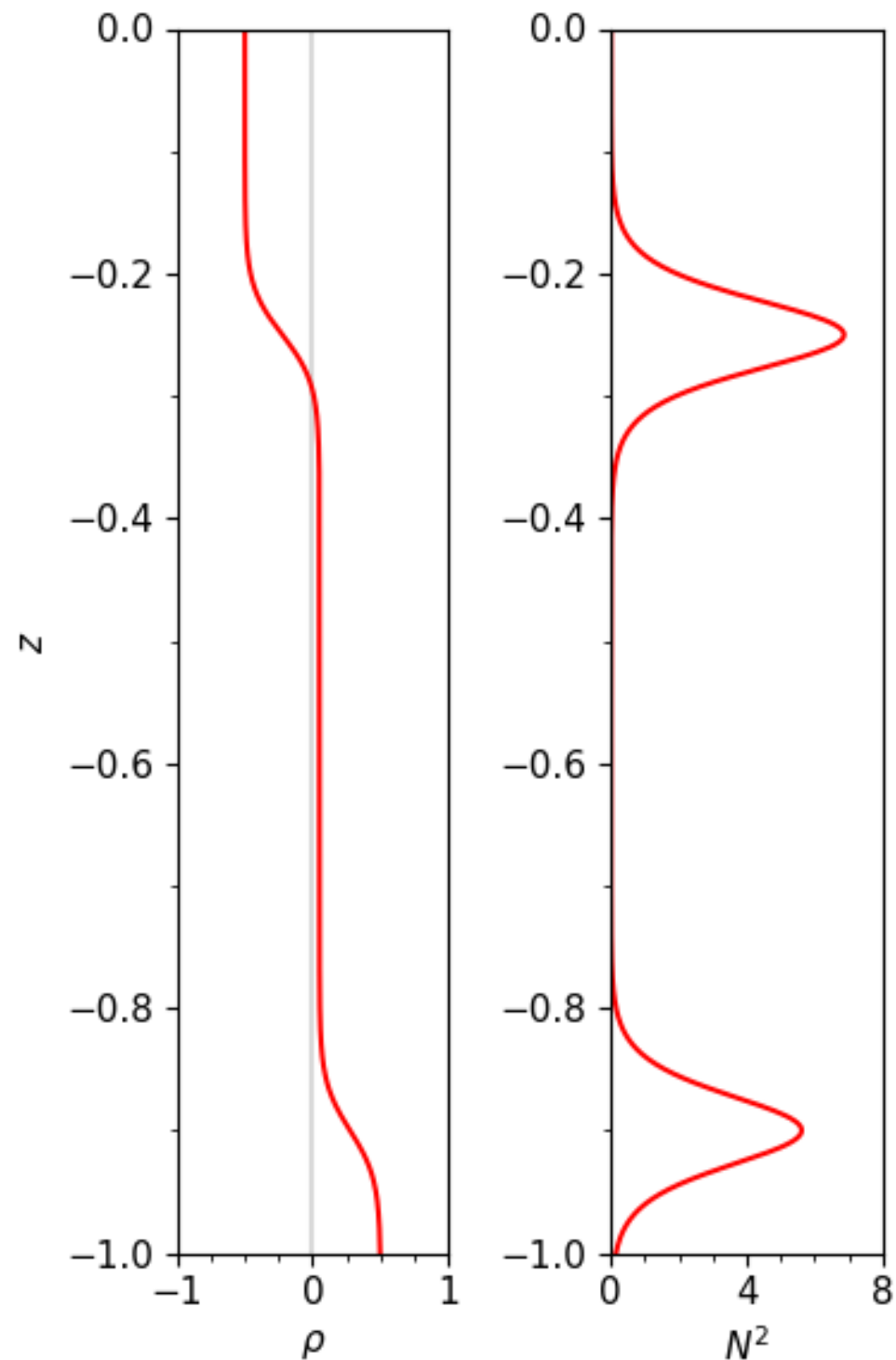
Double-Pycnocline Stratifications

$$\bar{\rho}(z) = -\frac{1}{2}\Delta\rho_1 \tanh \frac{z - z_1}{d_1} - \frac{1}{2}(1 - \Delta\rho_1) \tanh \frac{z - z_2}{d_2}.$$

- $-1 < z_2 < z_1 < 0$ so the upper pycnocline is centred at z_1 and the lower pycnocline is centred at z_2 ;
- $\Delta\rho_1 + \Delta\rho_2 = 1$: the total density jump from well below the lower pycnocline to far above it is equal to 1;
- $d_1 = d_2 = 0.04$.

For stratifications with pycnoclines close to the top or bottom boundaries the bottom to surface density difference may be less than 1.

Example Stratification



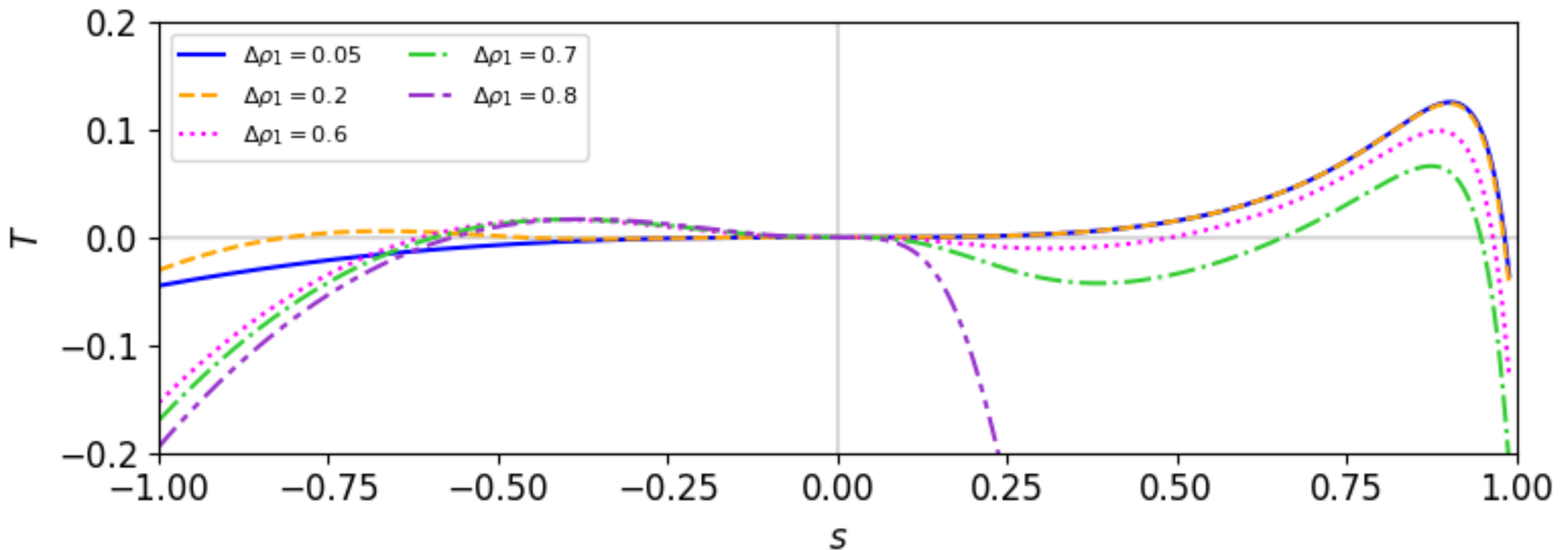
$$(z_1, \Delta\rho_1) = (-0.25, 0.55)$$

$$(z_2, \Delta\rho_2) = (-0.9, 0.45)$$

Asymmetric about the mid-depth:
upper pycnocline slightly stronger
and closer to the mid-depth

Conjugate flow auxiliary condition: example $T(s)$ curves

$$(z_1, z_2) = (-0.25, -0.9)$$



Negative roots

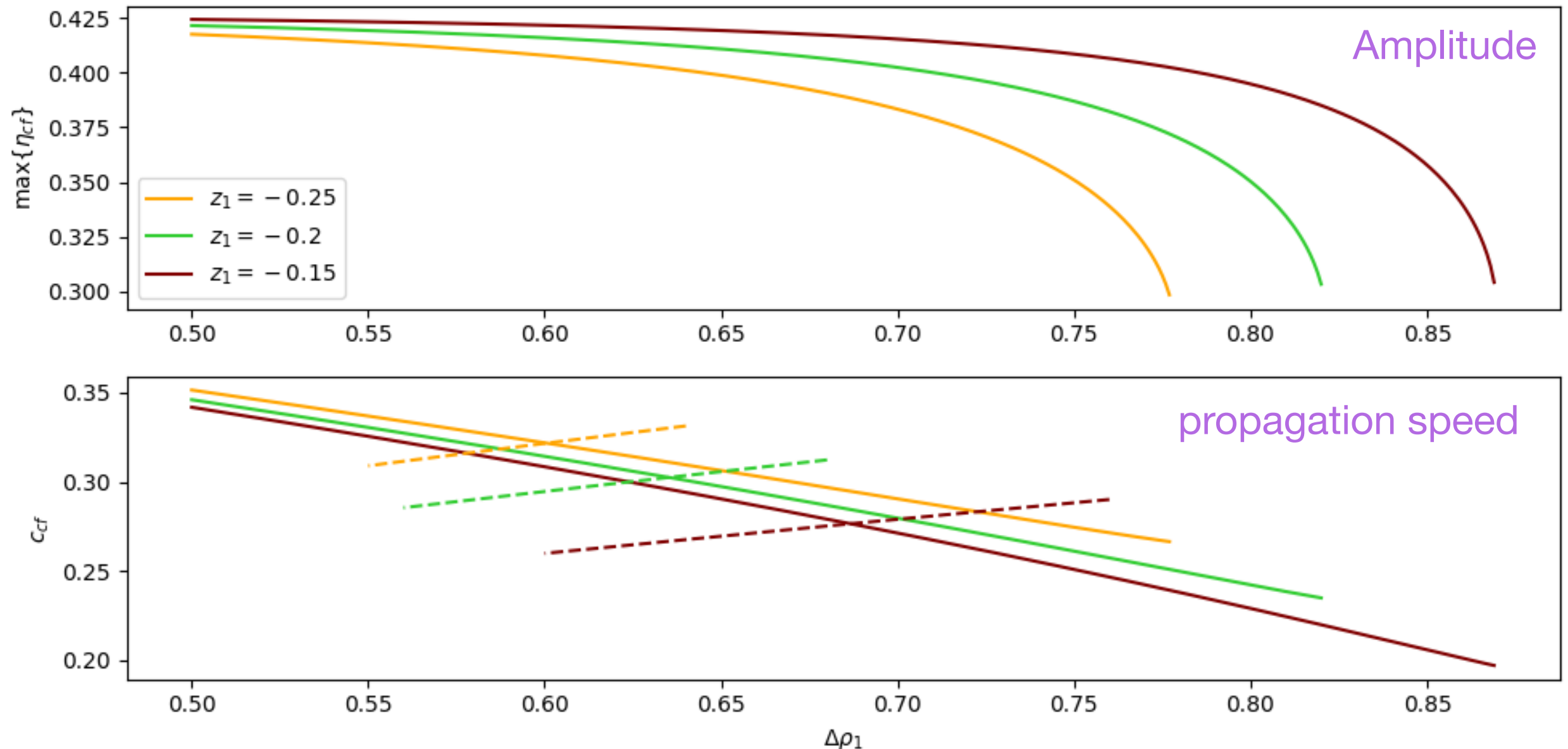
=> conjugate flow of depression

Positive roots

=> conjugate flow of elevation

Can have multiple solutions: for ISWs of both polarities to exist need at least one depression and at least one elevation conjugate flow

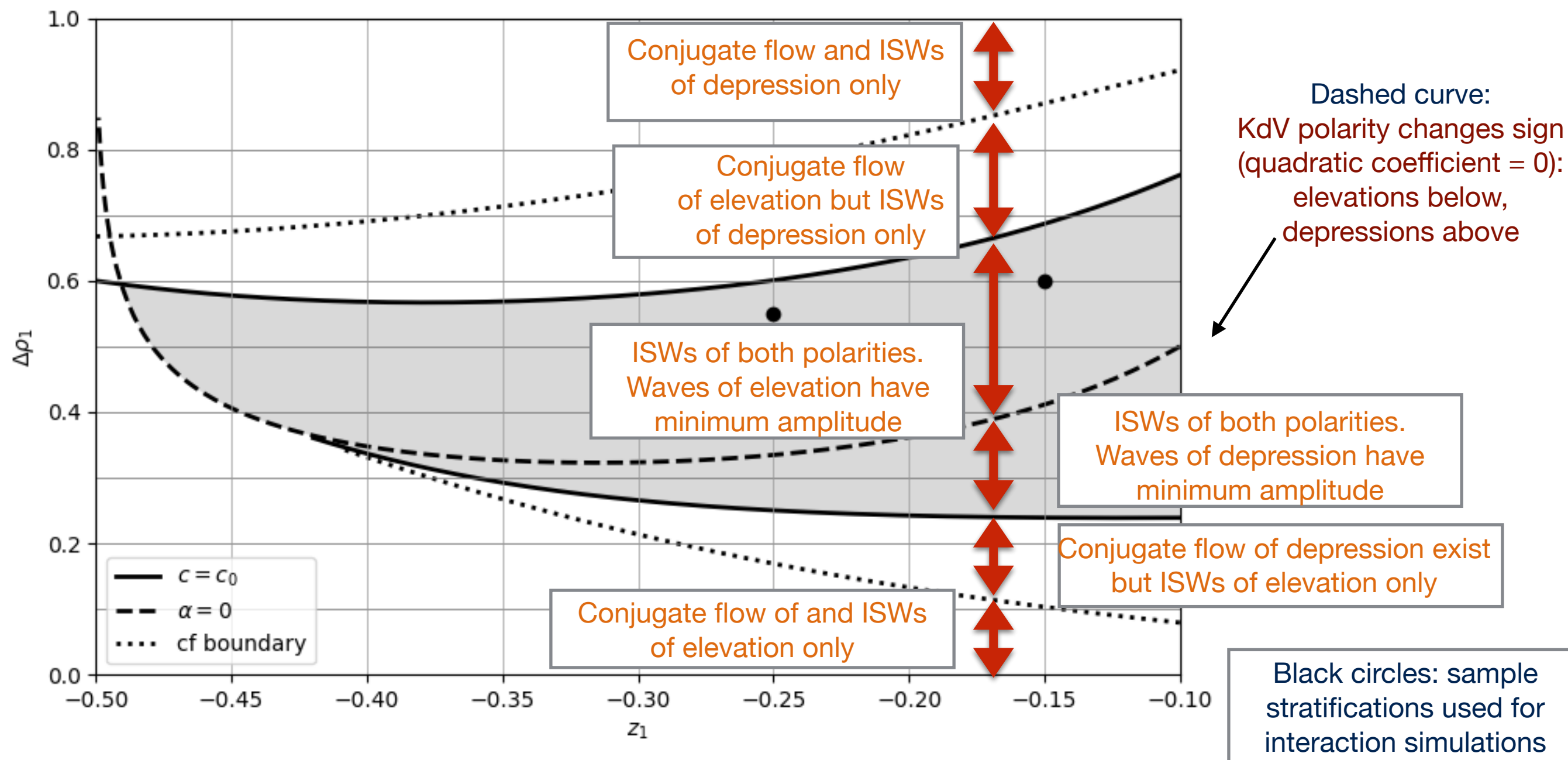
Conjugate flow solutions of elevation for different z_1 as functions of $\Delta\rho_1$ for $z_2 = -0.9$



- Solid curves: conjugate flows of elevation. For these stratifications KdV theory predicts ISWs of depression.
- Dashed curves: linear long wave propagation speed
- Conjugate flow provides limiting ISW amplitude only if its propagation speed is greater than the linear long wave speed: to the left of where the solid and dashed curves intersect

Region where ISWs of both polarities exist

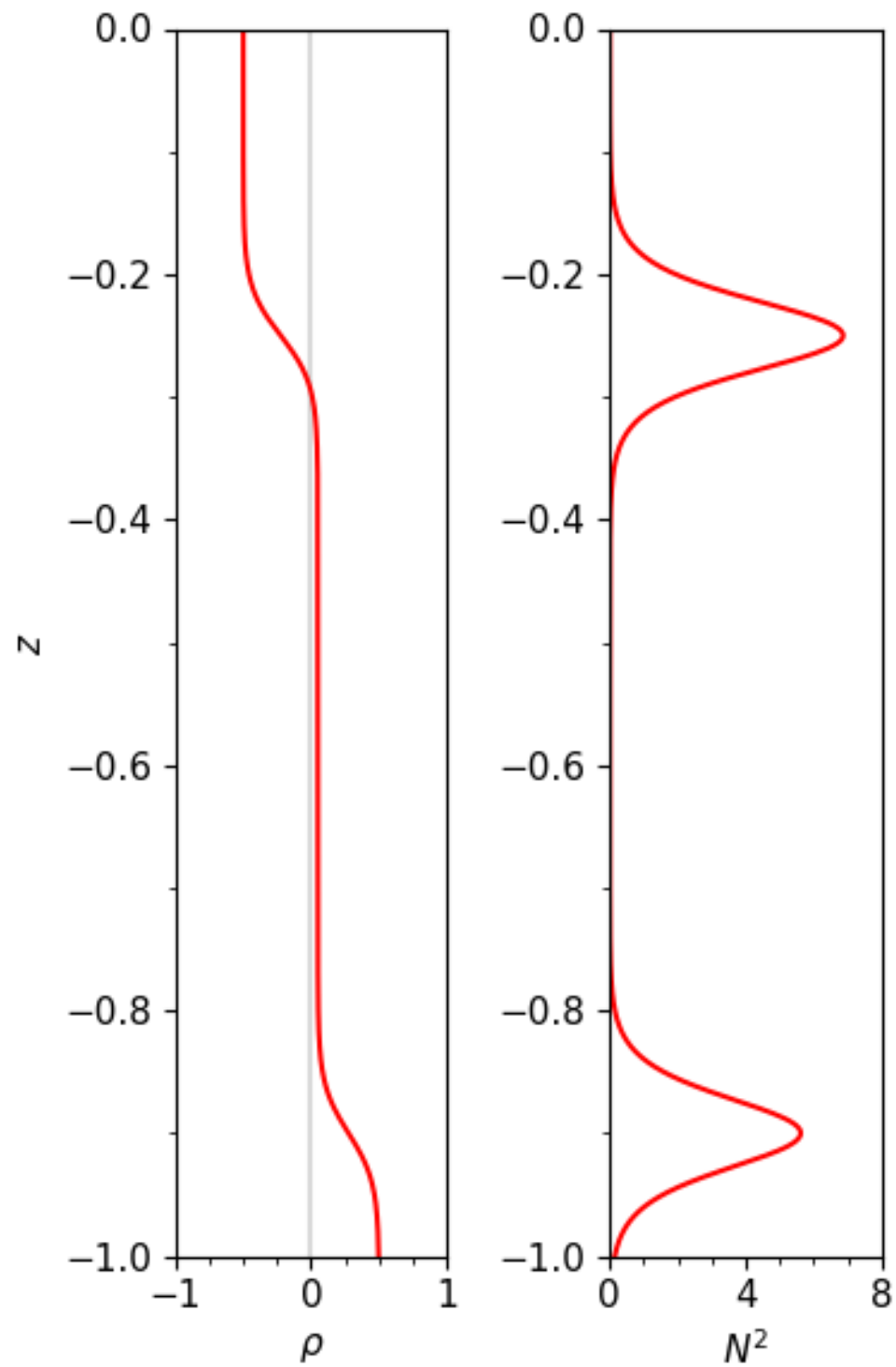
- Location of lower pycnocline fixed at $z_2 = -0.9$;
- Vary location of upper pycnocline (z_1) and density jump across it ($\Delta\rho_1$);
- Grey region on z_1 - $\Delta\rho_1$ plane shows where ISWs of both polarities exist;
- Bounding solid curves: conjugate flow propagation speed equals the linear long wave propagation speed.



Interaction Simulations

- All ISWs propagate to the right with respect to the fluid
- Simulations done in reference frame moving with the average propagation speed of the two ISWs
- In this frame the faster propagating wave moves to the right and the slower wave moves to the left

Stratification 1

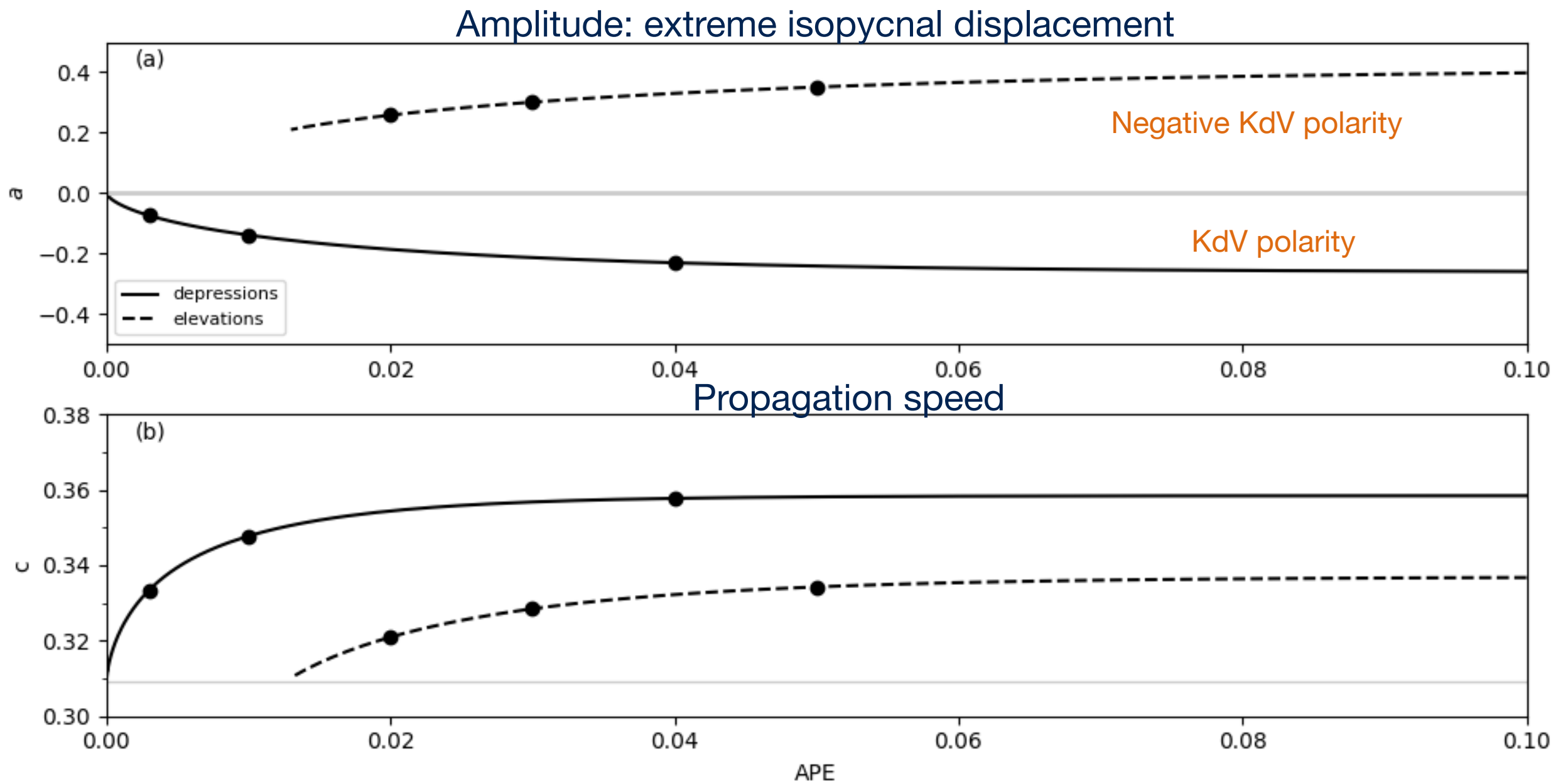


$$(z_1, \Delta\rho_1) = (-0.25, 0.55)$$

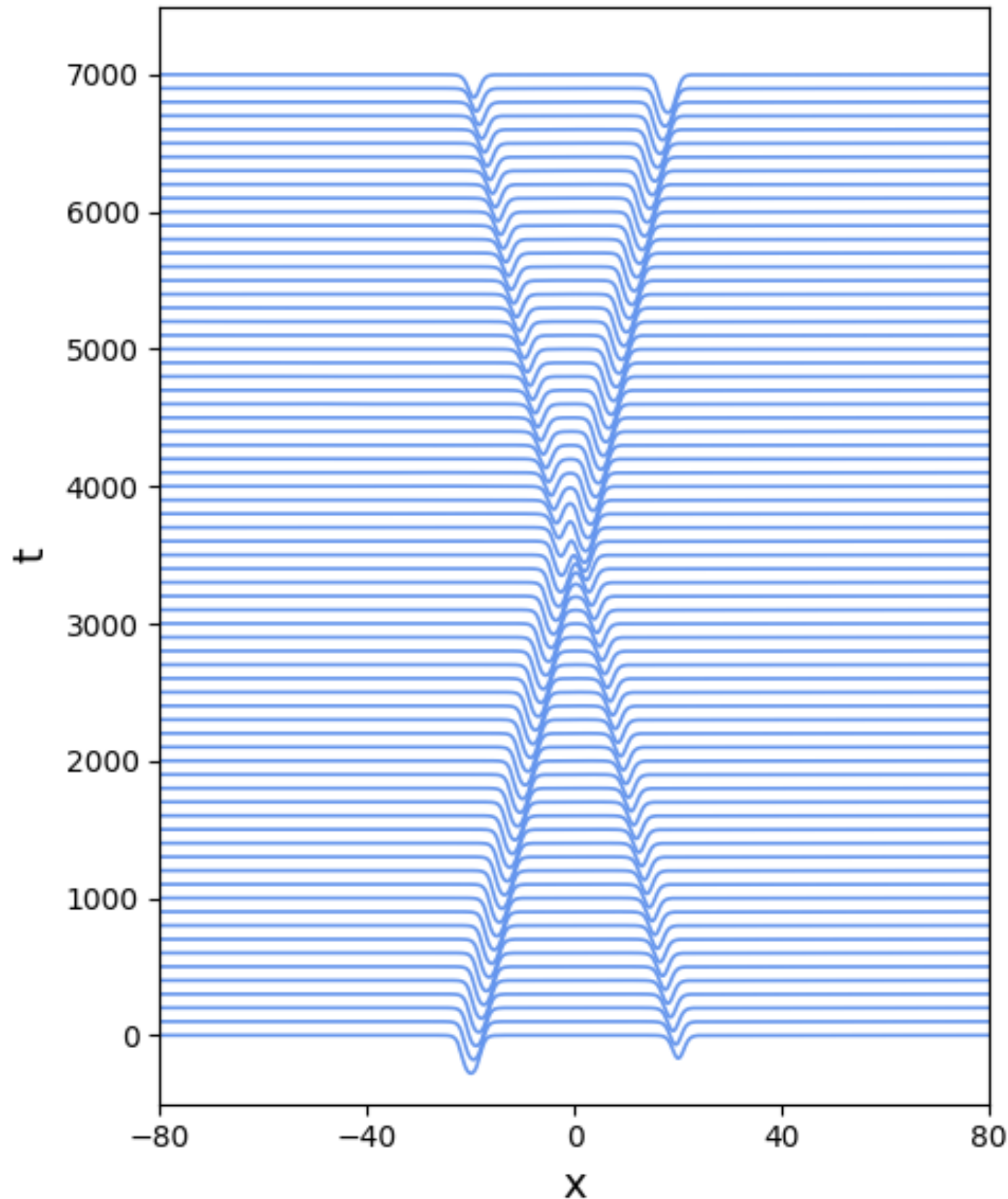
$$(z_2, \Delta\rho_2) = (-0.9, 0.45)$$

Stratification 1: ISW properties as a function of available potential energy (APE)

- KdV theory => ISWs of depression
- Solid: KdV polarity ISWs.
- Dashed: negative KdV polarity ISWs.
Minimum amplitude 0.210

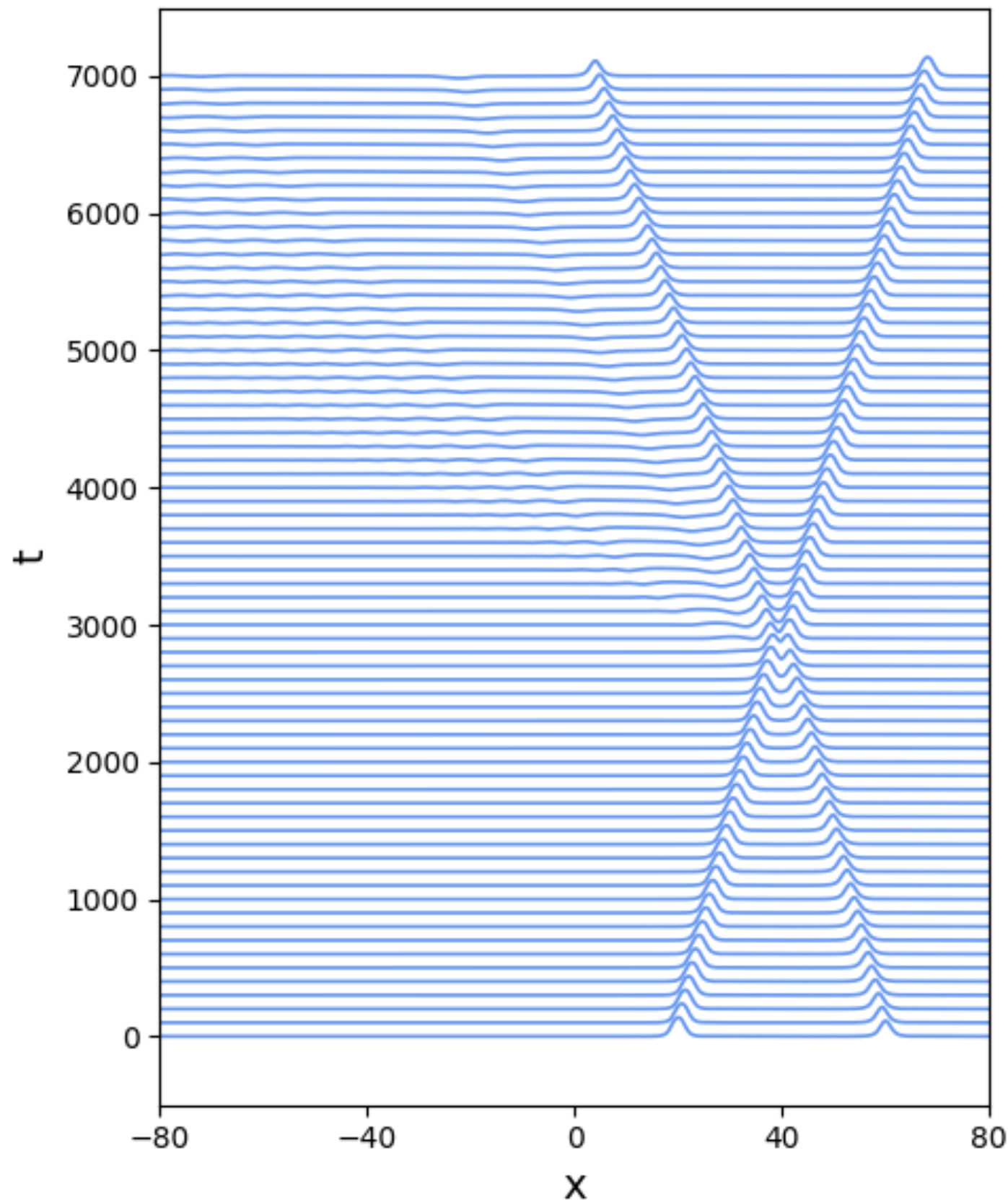


Interaction of two KdV polarity ISWs



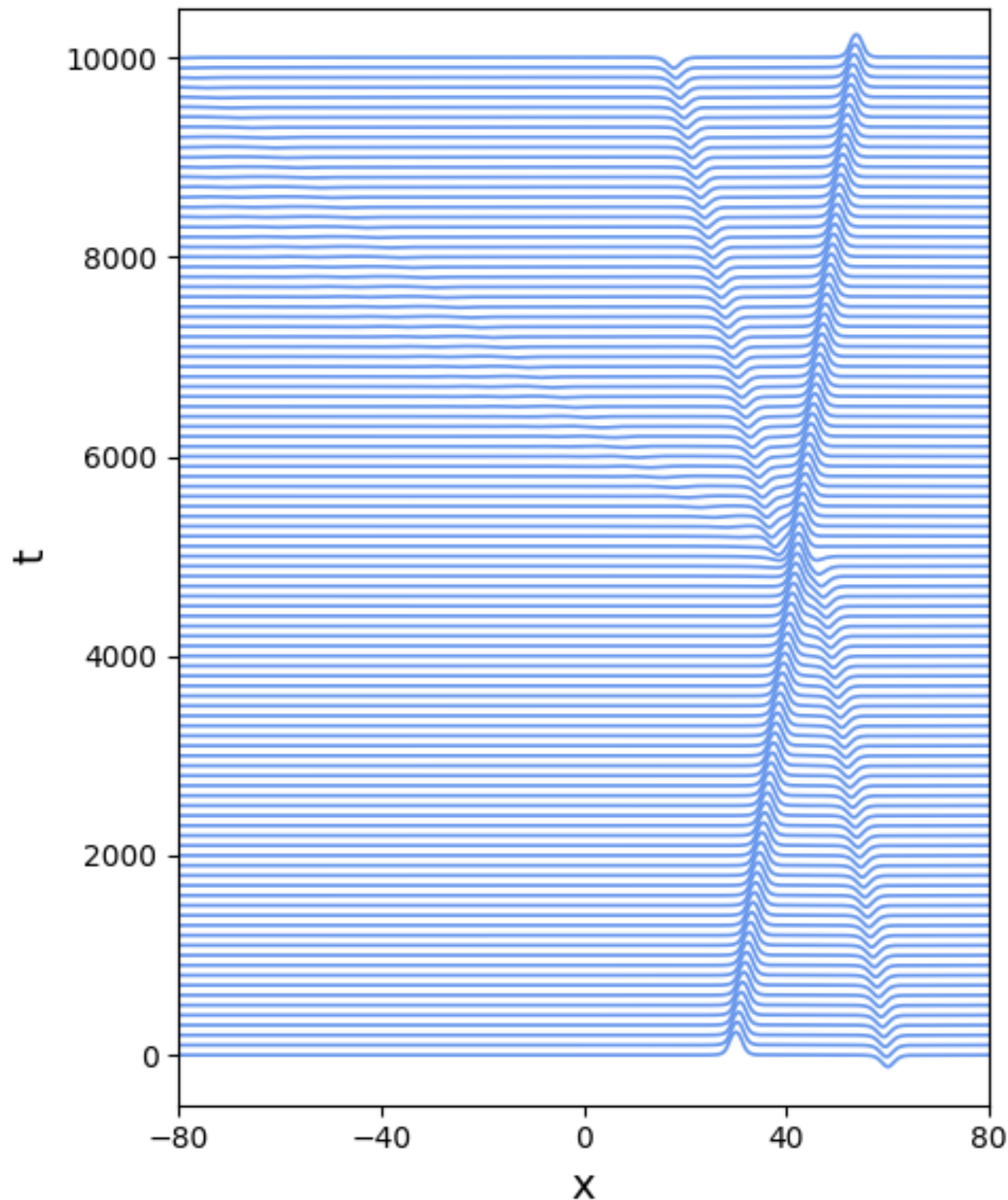
- Isopycnal at centre of upper pycnocline is plotted.
 - Waves of depression have largest displacements in the upper pycnocline
 - ISWs of elevation have largest displacements in the lower pycnocline
 - small amplitude linear waves have largest displacements in the upper pycnocline
- Initial APEs: 0.04 and 0.01 (see earlier slide for amplitude and propagation speed)
- Very small waves generated by the interaction with amplitude 0.6% that of the initial smaller ISW
- The large and small ISWs decrease in amplitude by 0.4% and 0.7% after the interaction
- VERY SOLITON LIKE

Interaction of two negative KdV polarity ISWs



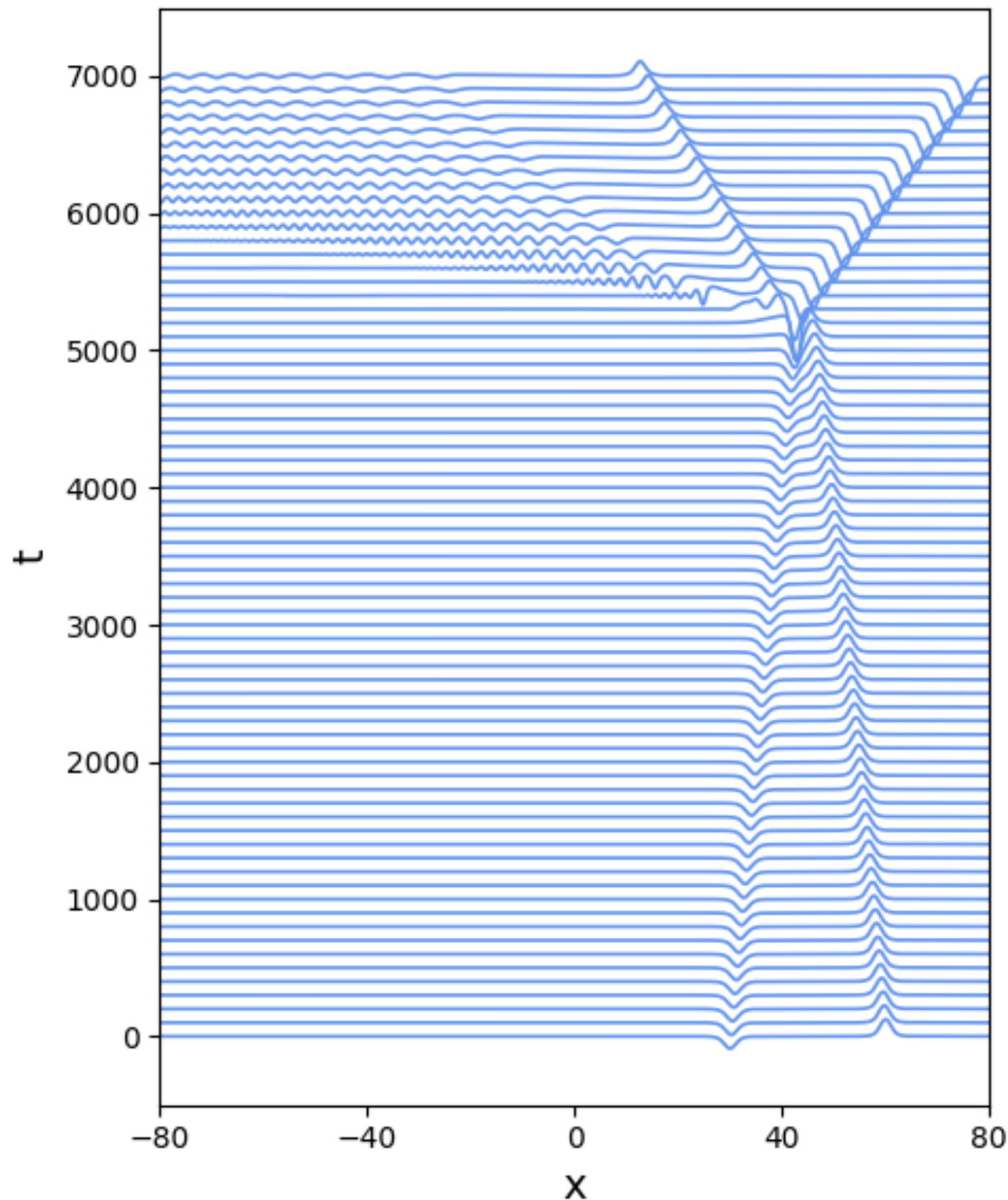
- Isopycnal at centre of upper pycnocline is plotted;
- Initial APEs: 0.02 and 0.05;
- Waves generated by the interaction much larger now but still small;
- The large and small ISWs decrease in amplitude (of this isopycnal) by 0.4% and 1.9% after the interaction;
- SOLITON LIKE

Interaction of negative KdV polarity ISW initially behind a KdV polarity ISW



- Isopycnal at centre of upper pycnocline is plotted;
- Initial APEs: 0.002 and 0.05;
- small waves generated by interaction. Minor change in amplitude of the two ISWs
- SOLITON-LIKE

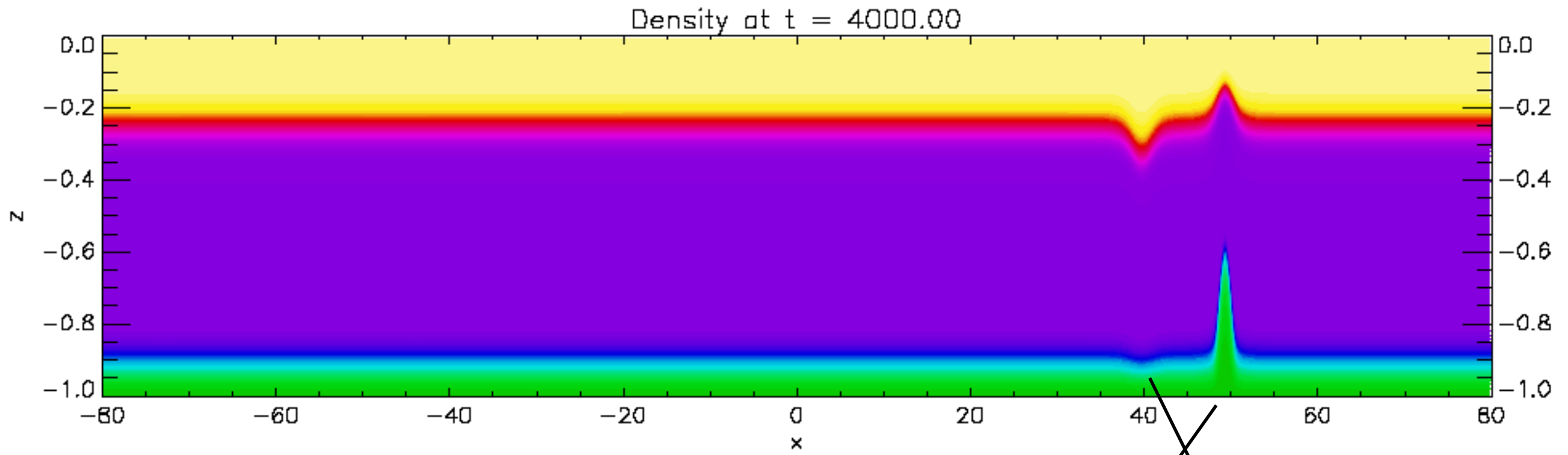
Interaction of KdV polarity ISW initially behind a negative KdV polarity ISW



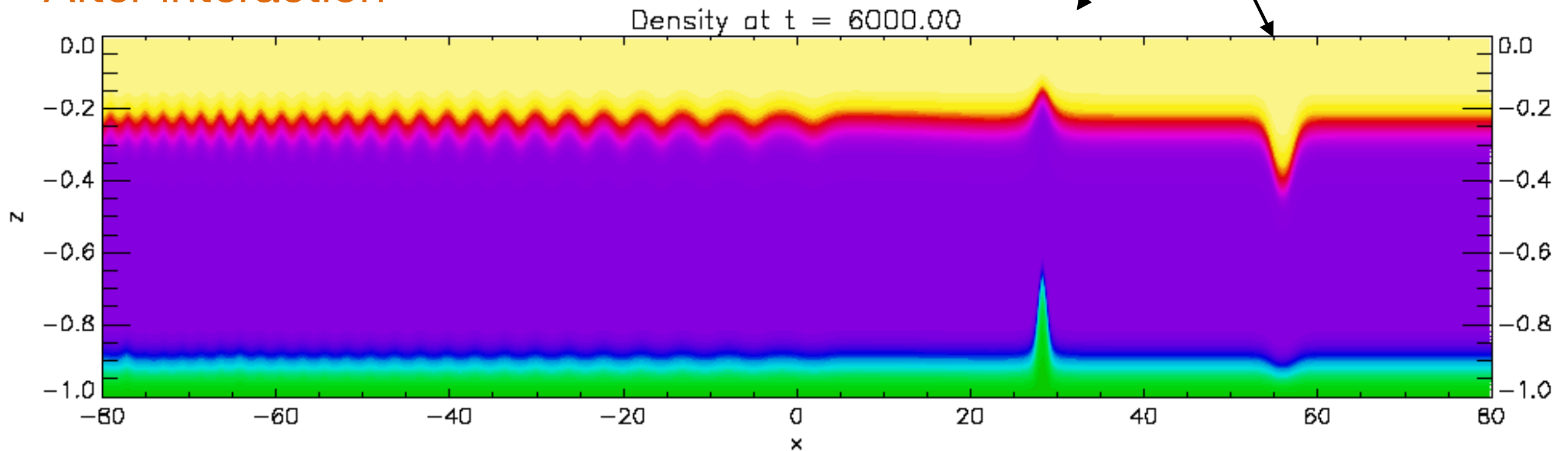
- Isopycnal at centre of upper pycnocline is plotted;
- Initial APEs: 0.02 and 0.05;
- Waves generated by the interaction much larger now but still small;
- The amplitude of the KdV depression increases by 113.9% after the interaction => more than doubling its initial amplitude!
- The amplitude of the negative KdV polarity elevation decreases by 14.0%
- The energy of the KdV ISW increases by a factor of 4.4
- The energy of the negative KdV polarity ISW is reduced to about half its initial energy.
- FAR FROM SOLITON-LIKE

Density field for previous case

Before interaction



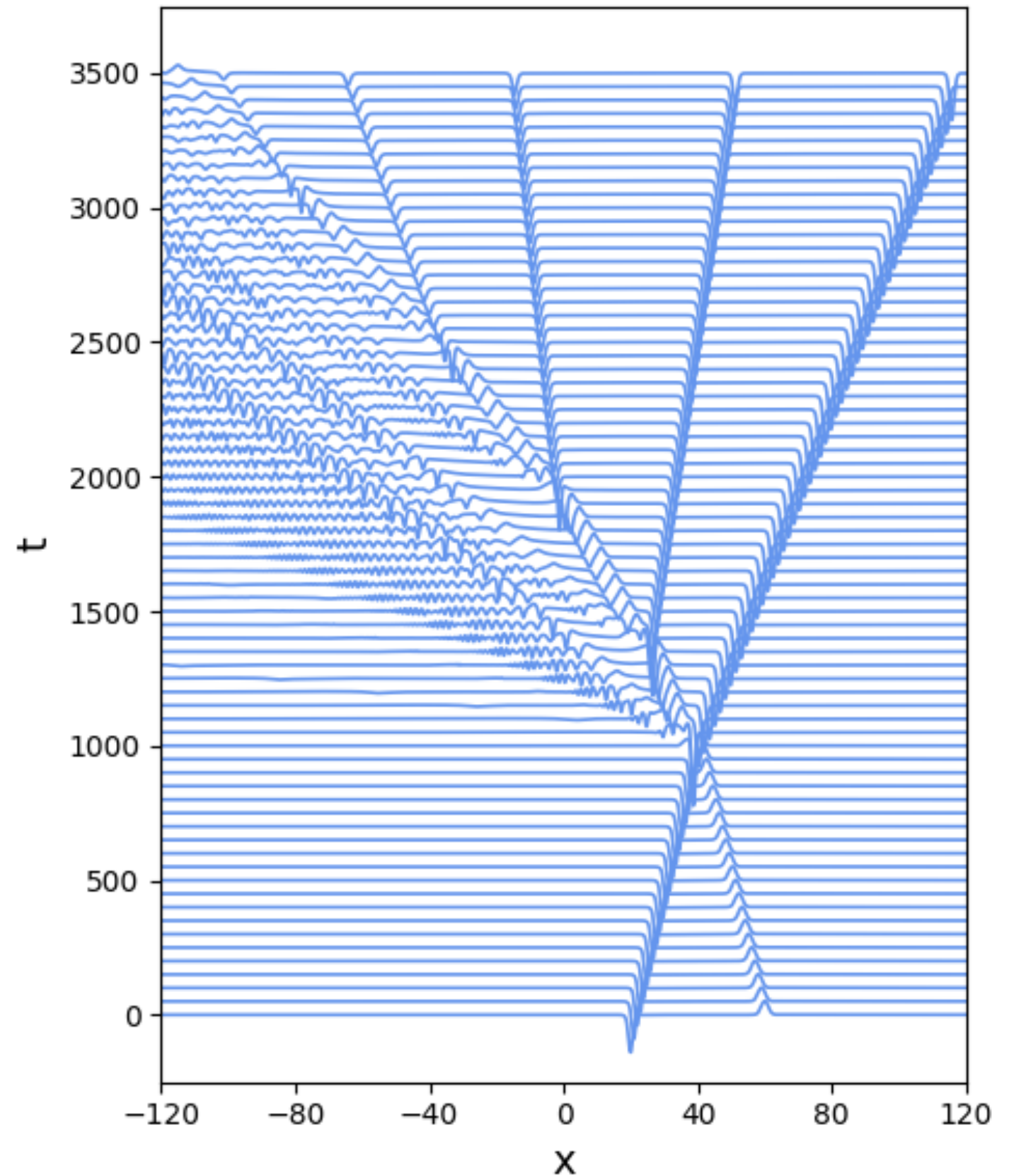
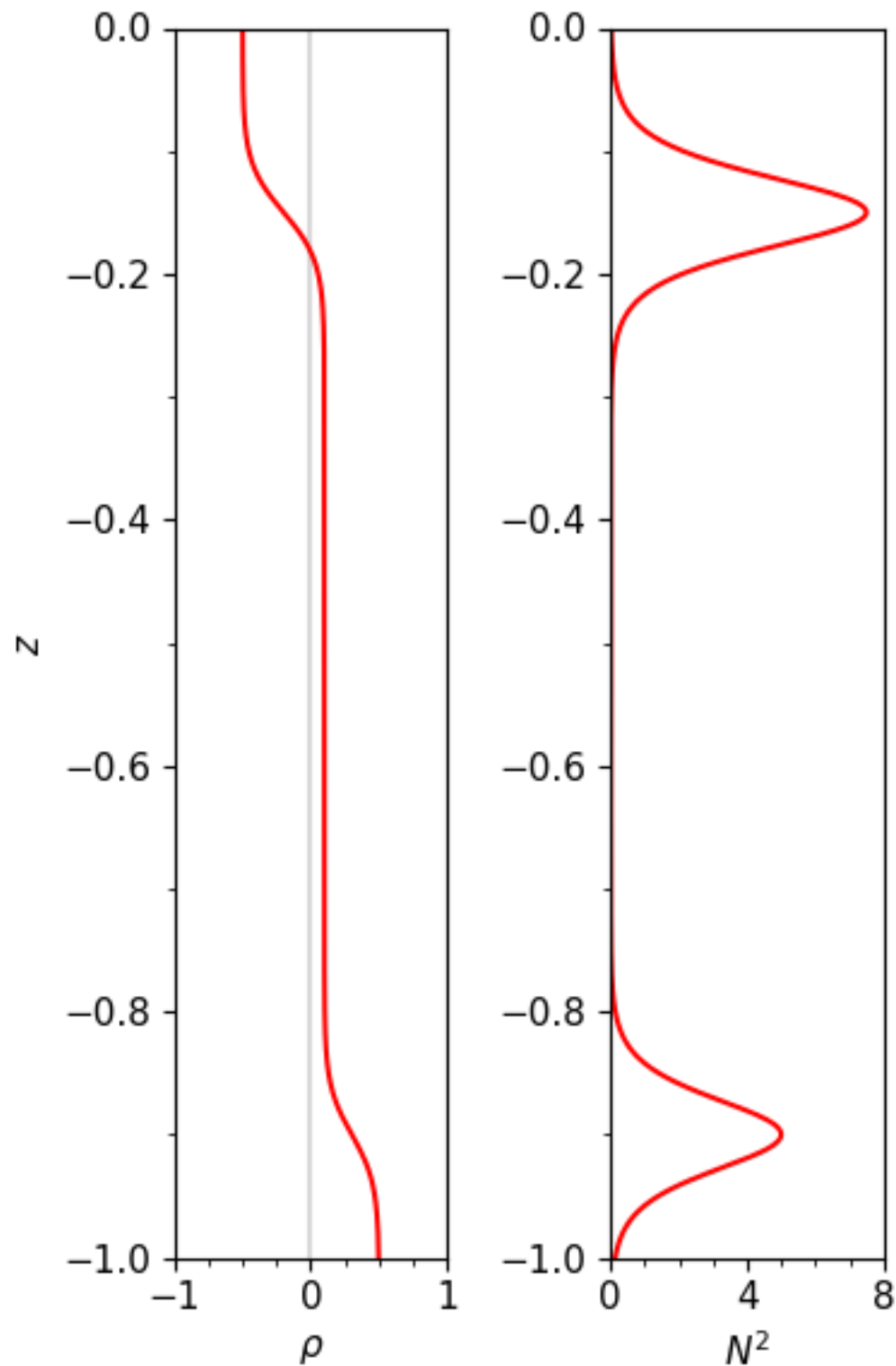
After interaction

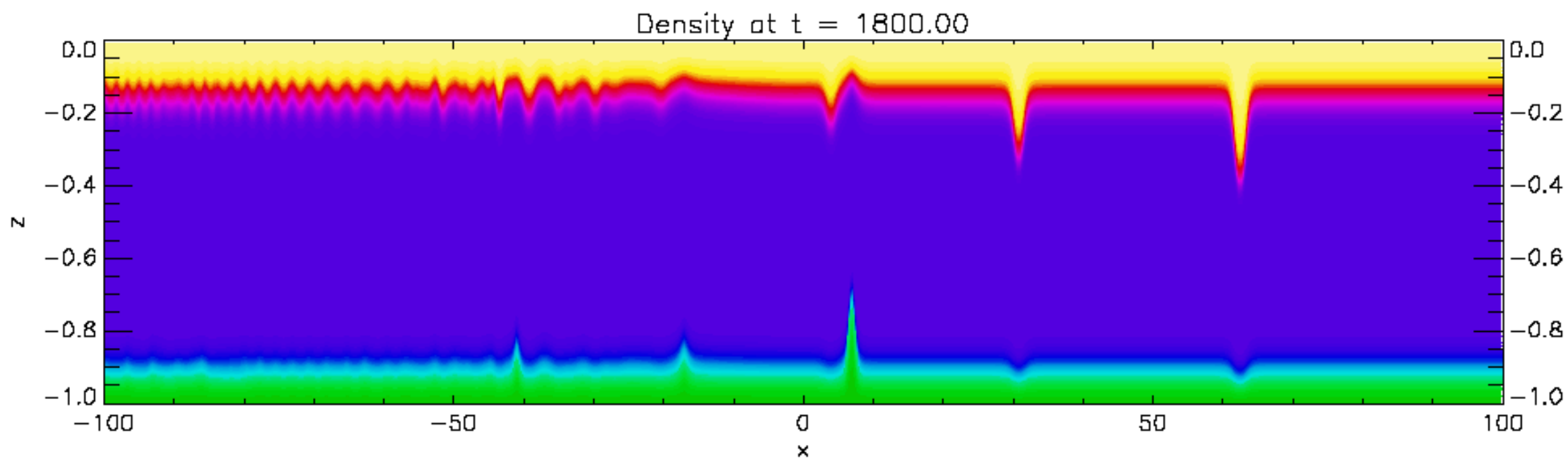
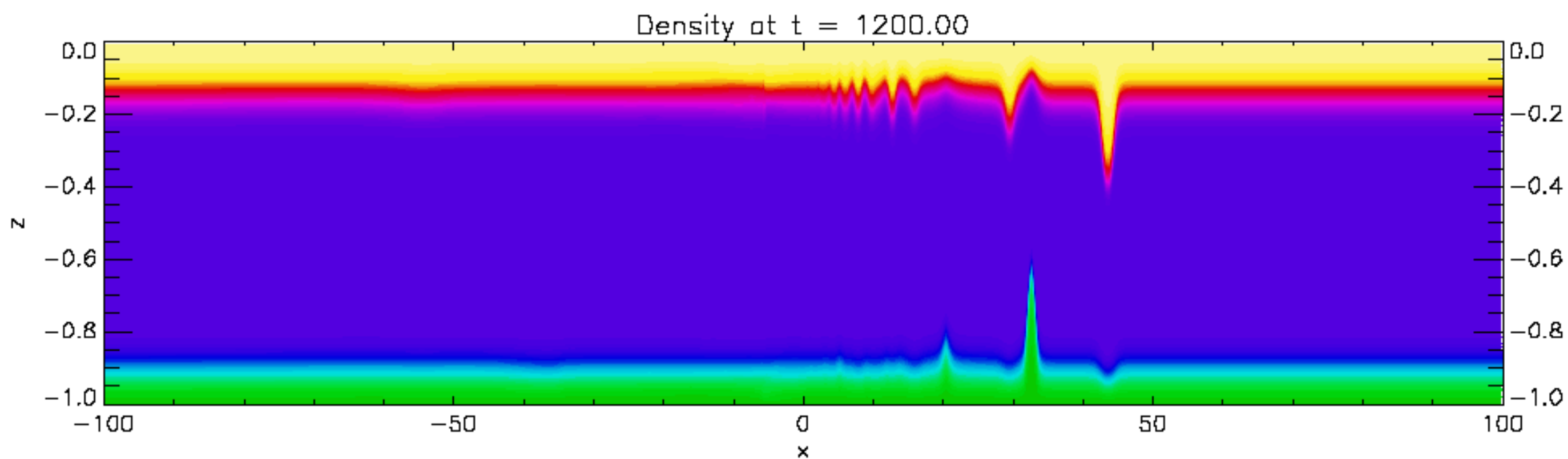
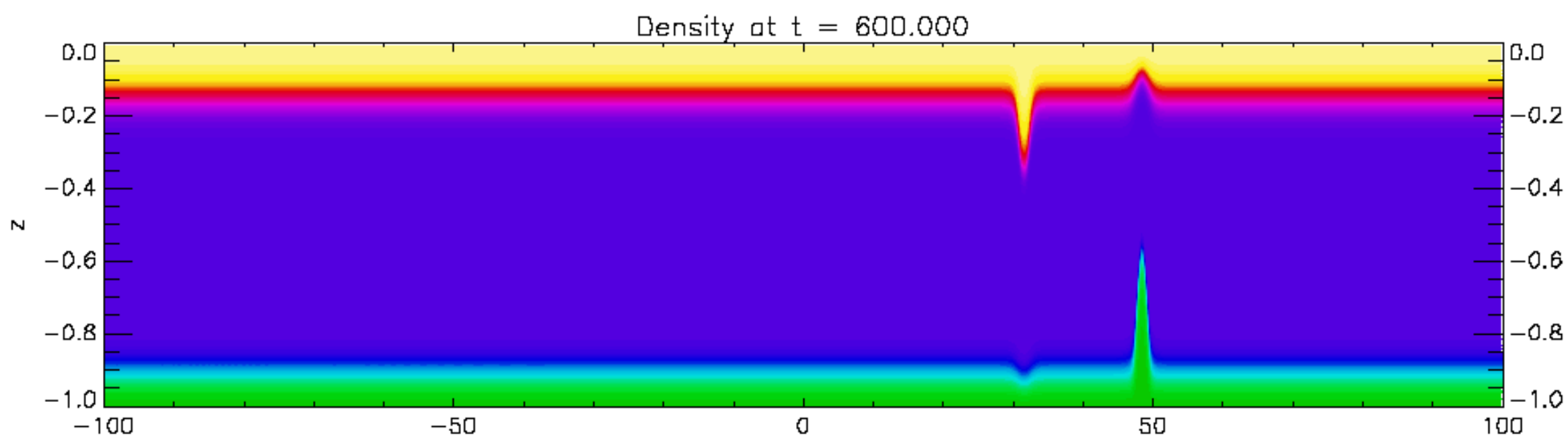


Stratification 2: slightly higher, stronger upper pycnocline

$$(z_1, \Delta\rho_1) = (-0.15, 0.6)$$

$$(z_2, \Delta\rho_2) = (-0.9, 0.4)$$





Density field
for previous
case at three
different times

Conclusions

- The interaction of two ISWs of KdV polarity are very soliton like. The interaction results in a small decrease in amplitude of the two ISWs and a small amplitude wave train is generated;
- The interaction of two ISWs of negative KdV polarity generates larger trailing waves but is still soliton-like;
- The interaction of two ISWs of opposite polarity, with the trailing faster wave having KdV polarity, results in:
 - significant changes to the amplitudes of the two ISWs;
 - a large loss of energy ($>10\%$) to a trailing internal wave train;
 - the generation of multiple large solitary waves of KdV polarity under some conditions;
- For the interaction of waves of opposite polarity with the trailing faster wave having negative KdV polarity the interaction is more soliton-like though this has not been investigated in much depth yet;
- In stark contrast, solitary wave solutions of the weakly-nonlinear Gardner equation (KdV + cubic nonlinearity) of both polarities are solitons ;
- For symmetric stratifications the interaction of ISWs of opposite polarity is also very soliton-like.