

Introduction

Nonuniqueness of the underlying source is a well-known issue with potential field problems as they arise in gravimetry and geomagnetism. Here, we collect some old and new results concerning the reconstruction of a magnetization $\mathbf{M}$ in a domain $\Omega \subset \mathbb{R}^3$ from knowledge of the corresponding magnetic field $\mathbf{B} = -\nabla V$ in the exterior of that domain. The magnetic potential V is given in the known form

$$V(x) = \frac{\mu_0}{4\pi} \int_{\Omega} \mathbf{M}(y) \cdot \frac{x-y}{|x-y|^3} d\mu(y).$$

Magnetization on Surfaces: General Case

We assume that the domain Ω is a sphere or a plane. Then it is known (Gubbins et al. 2011, Baratchart et al. 2013, Gerhards 2016) that the magnetization can be decomposed into three contributions

$$\mathbf{M} = \mathbf{M}_+ + \mathbf{M}_- + \mathbf{M}_{df},$$

of which $\mathbf{M}_+$ denotes the gradient of a harmonic function inside of Ω , $\mathbf{M}_-$ the gradient of a harmonic function outside of Ω , and $\mathbf{M}_{df}$ a tangential divergence-free vector field.

We distinguish the following assumptions on the magnetization, that lead to more specific results on the contributions of $\mathbf{M}$ that can be reconstructed uniquely:

- The magnetization is concentrated on a surface (e.g., a sphere or a planar sample).
- The magnetization is of induced type (i.e., it has the form $\mathbf{M} = \chi \mathbf{v}$, where χ denotes a susceptibility and $\mathbf{v}$ an inducing field).
- The magnetization is piecewise constant.

The following assertions hold true concerning uniqueness (cf. Figs. 1 and 2 for illustration):

- $\mathbf{M}_-$ and $\mathbf{M}_{df}$ do not yield a magnetic field (they are called “invisible”). Only $\mathbf{M}_+$ can be reconstructed uniquely from knowledge of the magnetic field (it is called “visible”).
- If it is known that the magnetization is restricted to a subdomain of the surface (as is the case, e.g., in scanning microscopy of rock samples), then both $\mathbf{M}_+$ and $\mathbf{M}_-$ are “visible”.

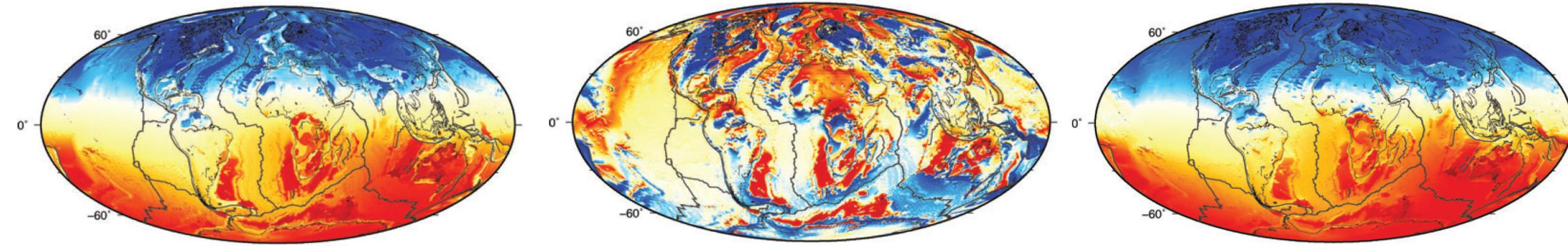


Fig. 1: A geology based model of the Earth's (vertically integrated) magnetization. Left: full magnetization $\mathbf{M}$, Center: visible contribution $\mathbf{M}_+$, Right: invisible contribution $\mathbf{M}_-$. Figure courtesy: Gubbins et al. (2011).

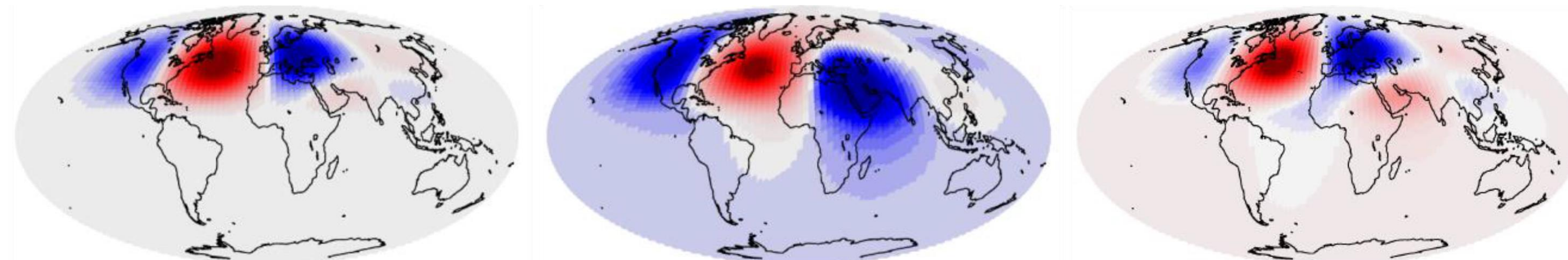


Fig. 2: A toy model of a magnetization that is restricted to the upper hemisphere. Left: full magnetization $\mathbf{M}$, Center: visible contribution $\mathbf{M}_+$, Right: invisible contribution $\mathbf{M}_-$ (which becomes visible due to the restriction to the upper hemisphere). Note that while the full magnetization is spatially localized, this is not necessarily true for its visible and invisible contributions.

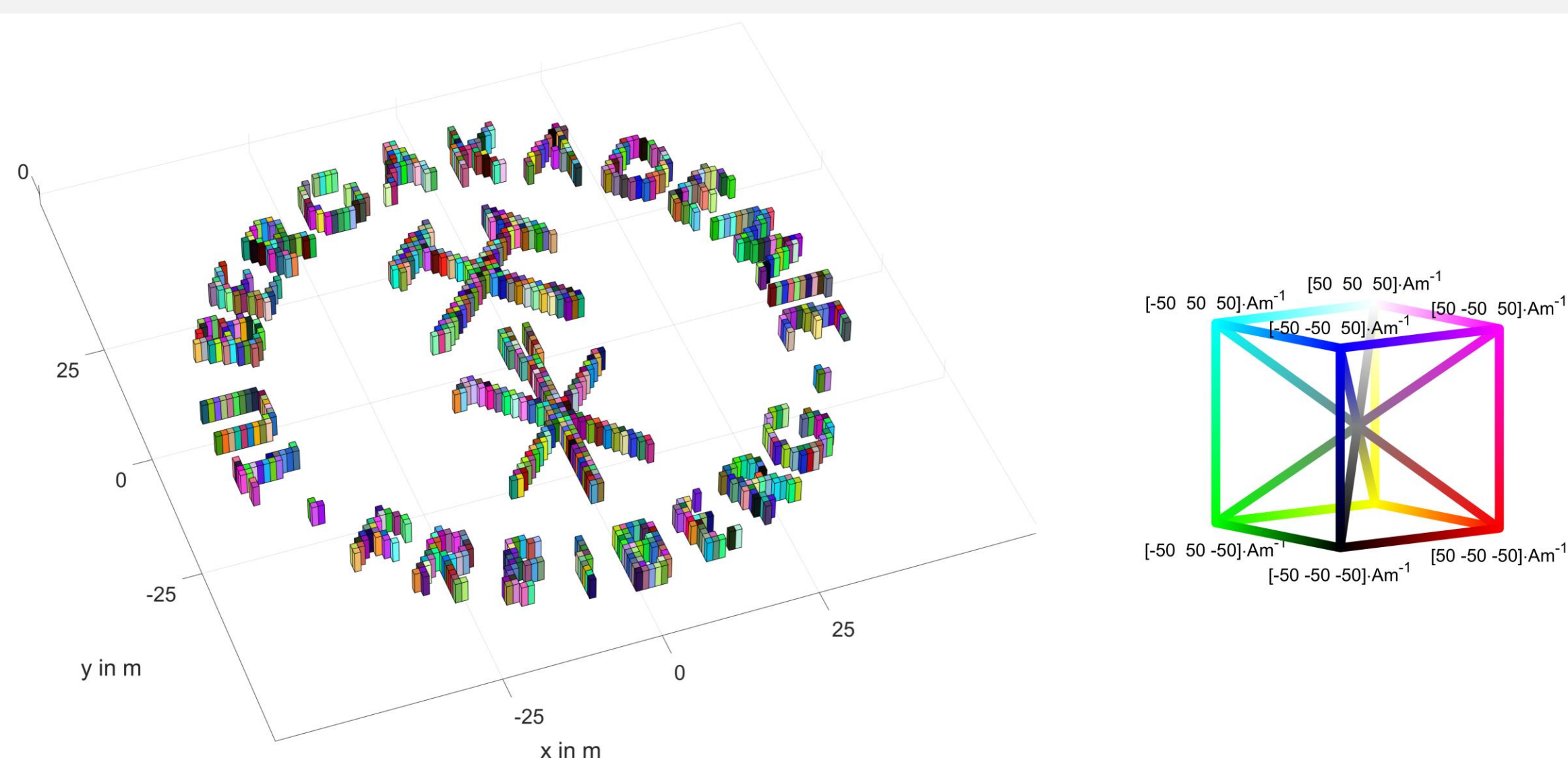


Fig. 3: Left: illustration of a simple (piecewise constant) magnetization with box-shaped subdomains. The magnetization intensity and direction within each box is color-coded based on the color-cube shown above. Such a magnetization is uniquely determined by the corresponding magnetic field.

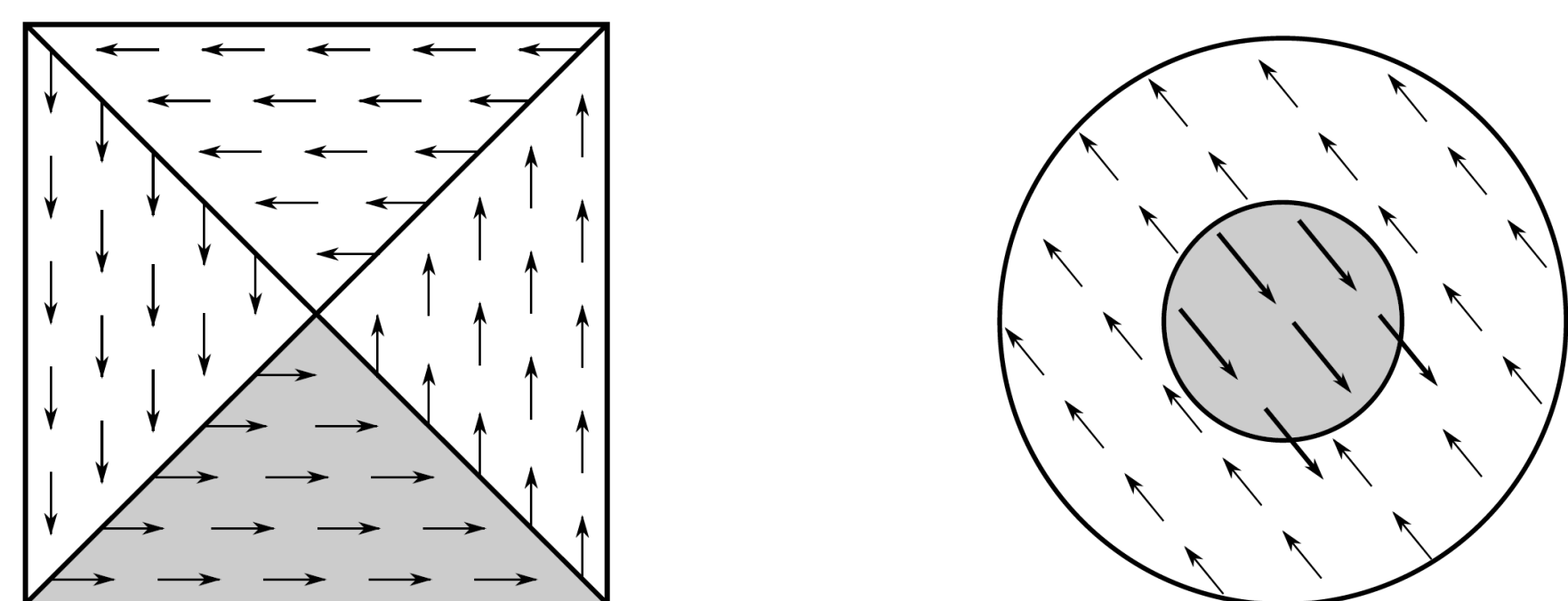


Fig. 4: Illustration of two simple (piecewise constant) magnetizations that are invisible, i.e., that do not produce a magnetic field measurable outside of the objects.

Magnetization in Volumes: General Case

Similar to the case of surfaces, a magnetization in a volumetric domain Ω can be decomposed into three contributions

$$\mathbf{M} = \mathbf{M}_{harm} + \mathbf{M}_{\nabla} + \mathbf{M}_{df},$$

of which $\mathbf{M}_{harm}$ is a harmonic gradient, $\mathbf{M}_{\nabla}$ the gradient of a function with zero boundary values, and $\mathbf{M}_{df}$ a divergence-free vector field which is tangential on the boundary of Ω .

The following assertions hold true concerning uniqueness (Baratchart et al. 2021):

- Only the contribution $\mathbf{M}_{harm}$ can be reconstructed uniquely from knowledge of the corresponding magnetic field outside of Ω . Both $\mathbf{M}_{\nabla}$ and $\mathbf{M}_{df}$ do not produce a magnetic field (they are called “invisible”).
- For several disjoint subdomains (e.g., magnetized grains within a rock sample), each subdomain inherits invisibility from the entire domain (Fabian & de Groot 2019).
- Nothing can be said about the thickness of the magnetization: for every subdomain of Ω one can find a magnetization $\mathbf{M}'$ that generates the same magnetic field as $\mathbf{M}$.

Piecewise Constant Magnetization

While harmonic gradients characterize (in theory) all contributions of a magnetization that can be recovered uniquely, they tend to be geophysically unintuitive and do not allow, e.g., magnetizations that vanish within an open domain.

Alternatively, one can consider simple (piecewise constant) magnetizations of the form

$$\mathbf{M} = \sum_{n=1}^N \mathbf{m}_n \mathbb{I}_{\Omega_n},$$

where $\mathbf{m}_n$ is a magnetization that is constant within each subdomain Ω_n . The subdomains may overlap and may have (a fairly) arbitrary shape. Still, this finite dimensional setup does not necessarily yield uniqueness (cf. Figure 4). Surprisingly, however, if one only allows box-shaped subdomains, such simple magnetizations are determined uniquely by knowledge of the magnetic field (Baratchart et al. 2021; cf. Figure 3).

Induced Magnetization

On Surfaces

In various cases, it can be reasonable to assume that the magnetization is of induced type, e.g., in continental regions or in small rock samples. Then the question is not necessarily to recover the full magnetization $\mathbf{M}$ but rather to recover, e.g., the (vertically integrated) susceptibility χ , provided that the inducing field $\mathbf{v}$ is known (being, e.g., the Earth's core magnetic field). In paleomagnetism, the inducing field $\mathbf{v}$ itself might be unknown and represents the quantity of interest. In particular its orientation is sought after, e.g., in order to infer orientations of a possible past core magnetic field.

The assumption that the entire magnetization is concentrated on a surface is rather restrictive but reasonable, e.g., for slices of rock samples or large scale lithospheric studies where the lateral extent dominates the thickness of the lithosphere.

In Volumes

In the case that Ω actually describes a volumetric domain, not much can be said about uniqueness of the susceptibility in general, even if $\mathbf{v}$ is known. However, opposed to the surface

The following assertions hold true concerning uniqueness:

- In case $\mathbf{v}$ is known but no further assumptions are made, the susceptibility cannot be determined uniquely (Maus & Haak 2003).
- In case $\mathbf{v}$ is known and the susceptibility is assumed to be restricted to a subdomain of Ω , then the susceptibility is determined uniquely (Gerhards 2016).
- In case $\mathbf{v}$ is not known but it is known to be constant in the study region, then candidates for $\mathbf{v}$ can be computed (Lesur & Vervelidou 2020).
- In case $\mathbf{v}$ is an unknown dipole field and the susceptibility is known to be restricted to a subdomain of Ω , then candidates for the dipole direction can be computed (Gerhards 2019; cf. Figure 5).

case, the following holds true for the inducing field: if $\mathbf{v}$ is unknown but constant and if the susceptibility has nonvanishing net moment, then $\mathbf{v}$ is determined uniquely (Baratchart et al. 2021).

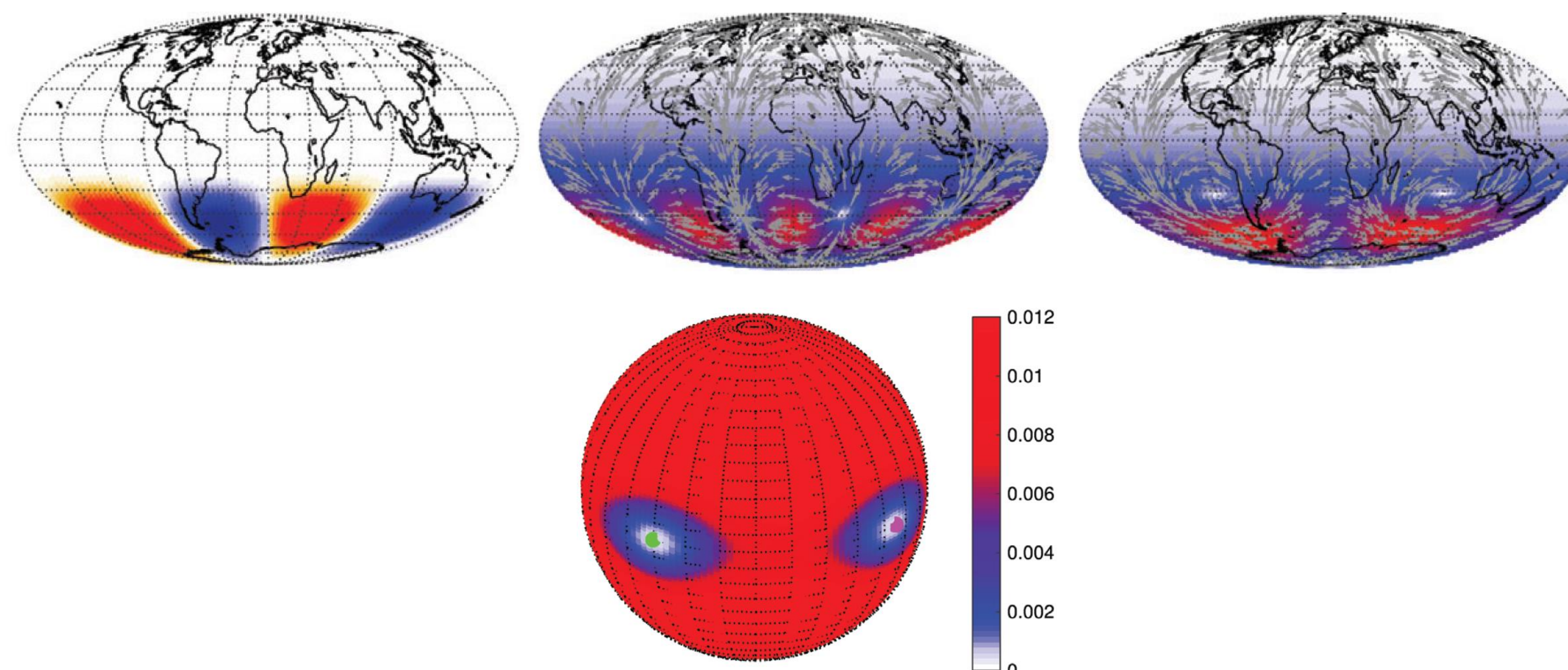


Fig. 5: Illustration of a toy magnetization of induced type: Top left: radial contribution, Top center: curl-free contribution, Top right: divergence-free contribution. The bottom row shows a polynomial (which can be derived from the radial and curl-free contribution) whose zeros indicate the candidates for the dipole direction of the inducing field magnetic field $\mathbf{v}$. As can be seen, this particular setup allows for two dipole directions which could lead to the same measured magnetic field.

Conclusions

We gave an overview on various general and specific uniqueness and nonuniqueness results for inverse magnetization problems concerning surface as well as volumetric magnetizations. These results obviously do not resolve the

general issues with geophysical potential fields, but they may help to understand which a priori assumptions on the magnetization affect the outcome of inversion procedures and in which way they do so.