

Steady radiating baroclinic vortices in vertically sheared flows

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Purpose

Understand how a single vortex in a vertical shear flow drifts and generates Rossby waves

Applications

- oceanic vortices in subtropical gyres
- mid-latitude storms in the atmosphere

2-layer quasi-geostrophic equations

$$\frac{\partial \Pi_1}{\partial t} + J(\Psi_1, \Pi_1) = 0$$

$$\frac{\partial \Pi_2}{\partial t} + J(\Psi_2, \Pi_2) = 0$$

Potential vorticity (PV):

$$\Pi_1 = \nabla^2 \Psi_1 + \frac{f_0^2}{g'H_1} (\Psi_2 - \Psi_1) + \beta_0 y$$

$$\Pi_2 = \nabla^2 \Psi_2 + \frac{f_0^2}{g'H_2} (\Psi_1 - \Psi_2) + \beta_0 y$$

Key elements

- **Dispersion relation:** if the zonal vortex drift velocity coincides with the phase velocity of Rossby waves, waves will be generated
- **Conservation of zonal momentum:** determines the meridional vortex drift in response to the wave generation
- **PV conservation in the vortex core:** determines the response of the vortex amplitude to the meridional drift

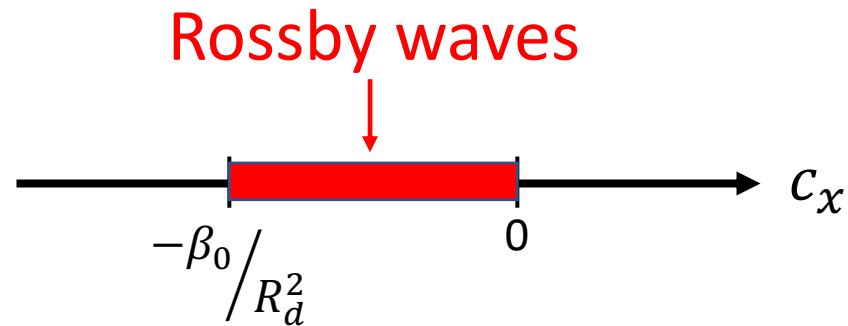
Prelude: single layer, no background flow

$$\frac{\partial \Pi}{\partial t} + J(\Psi, \Pi) = 0$$

$$\Pi = \nabla^2 \Psi - \frac{\Psi}{R_d^2} + \beta_0 y, \quad R_d^2 = \frac{g' H}{f_0^2}$$

Dispersion relation:

$$\omega = \frac{-\beta_0 k}{k^2 + l^2 + R_d^2}$$



Simulation results

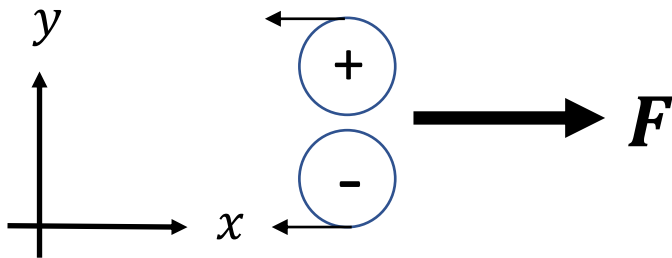
- Westward drift, slightly slower than $-\beta_0/R_d^2$
 $\Rightarrow$ Rossby waves are generated
- Wave radiation causes anticyclone to drift towards the equator
- Conservation of $\nabla^2\psi - \frac{\psi}{R_d^2} + \beta_0 y$ in the vortex core weakens the PV-anomaly

Zonal momentum

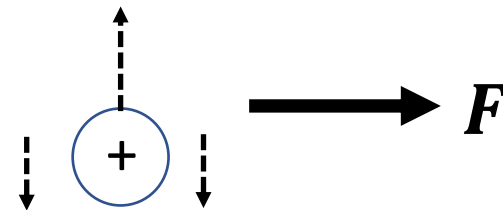
Conserved fluid impulse:

$$I_x = \int \Pi y \, dx dy, \quad \Pi = \nabla^2 \Psi - \frac{\psi}{R_d^2} + \beta_0 y$$

Result of localized external force $\mathbf{F}$:

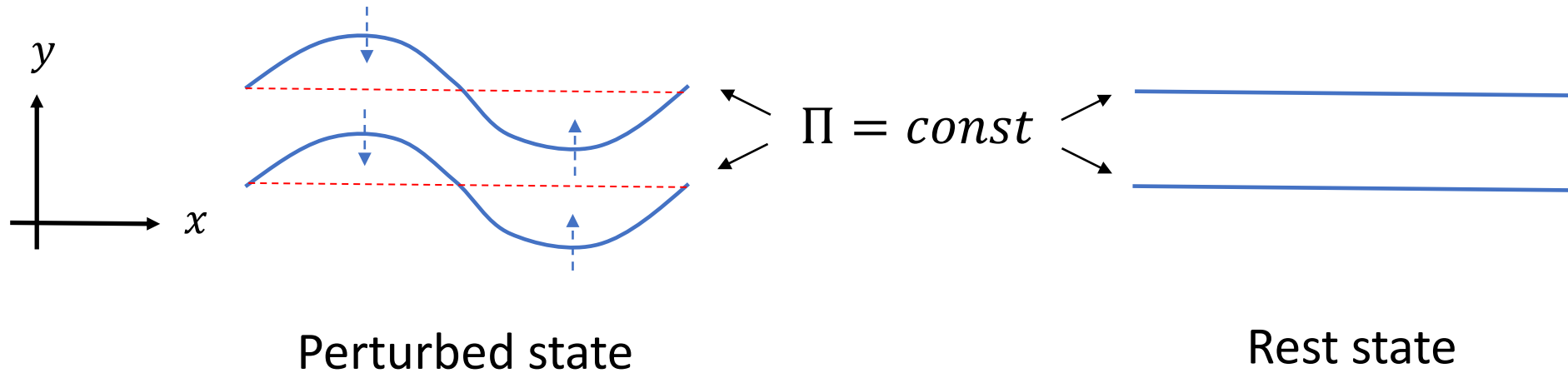


Creation of dipole vortex



Displacement of existing vortex (PV conserved)

For a Rossby wave, subtract the impulse of the isovortical rest state:



The difference to lowest nonvanishing order is the pseudomomentum (better: available momentum, as APE)

$$M = -\frac{1}{\beta} \int \frac{q^2}{2} dx dy, \quad q = \nabla^2 \psi - \frac{\psi}{R_d^2}$$

M negative definite $\Rightarrow$ all Rossby waves carry westward momentum.

Thus, generating the wake of Rossby waves gives the vortex eastward momentum ("wave drag"), which pushes a cyclone poleward and an anticyclone equatorward.

2-layer model with uniform background flow

Nondimensional units:

$$\frac{\partial q_1}{\partial t} + F_1 J(\psi_1, q_1) + \beta_1 \frac{\partial \psi_1}{\partial x} + u_1 \frac{\partial q_1}{\partial x} = 0$$

$$\frac{\partial q_2}{\partial t} + F_2 J(\psi_2, q_2) + \beta_2 \frac{\partial \psi_2}{\partial x} + u_2 \frac{\partial q_2}{\partial x} = 0$$

Background PV gradients:

$$\beta_1 = \beta + (u_1 - u_2)F_1$$

$$\beta_2 = \beta - (u_1 - u_2)F_2$$

PV anomaly:

$$q_1 = \nabla^2 \psi_1 - F_1 \psi_1 + F_2 \psi_2$$

$$q_2 = \nabla^2 \psi_2 + F_1 \psi_1 - F_2 \psi_2$$

Conserved fluid impulse:

$$I_x = \int (\pi_1 y + \pi_2 y) dx dy, \quad \pi_i = q_i + (\beta_i / F_i) y$$

Pseudomomentum:

$$M = -\frac{1}{2} \int \left(\frac{F_1 q_1^2}{\beta_1} + \frac{F_2 q_2^2}{\beta_2} \right) dx dy$$

- If $\beta_1, \beta_2 \geq 0 \Rightarrow$ all waves carry westward momentum, the flow is stable
- If β_1 and β_2 have different sign $\Rightarrow$ baroclinic instability

Special case:

- $u_2 = 0, \quad u_1 = u < 0$
- $\beta_1 = 0$ (marginally stable vertical shear)

We get

$$\begin{aligned}\pi_1 &= q_1 \\ \pi_2 &= q_2 + \frac{\beta_2}{F_2} y = q_2 + \left(\frac{\beta}{F_2} - u \right) y\end{aligned}$$

The PV-anomaly q_1 is a Lagrangian invariant, and must vanish in a wave.

Now M is singular; there are no background PV-contours in layer 1.
 Use instead a mixed momentum invariant:

$$M^* = \int \left(\underset{\substack{\uparrow \\ \text{vortex}}}{q_1} y - \frac{F_2 \underset{\substack{\uparrow \\ \text{wave}}}{q_2^2}}{2\beta_2} \right) dx dy$$

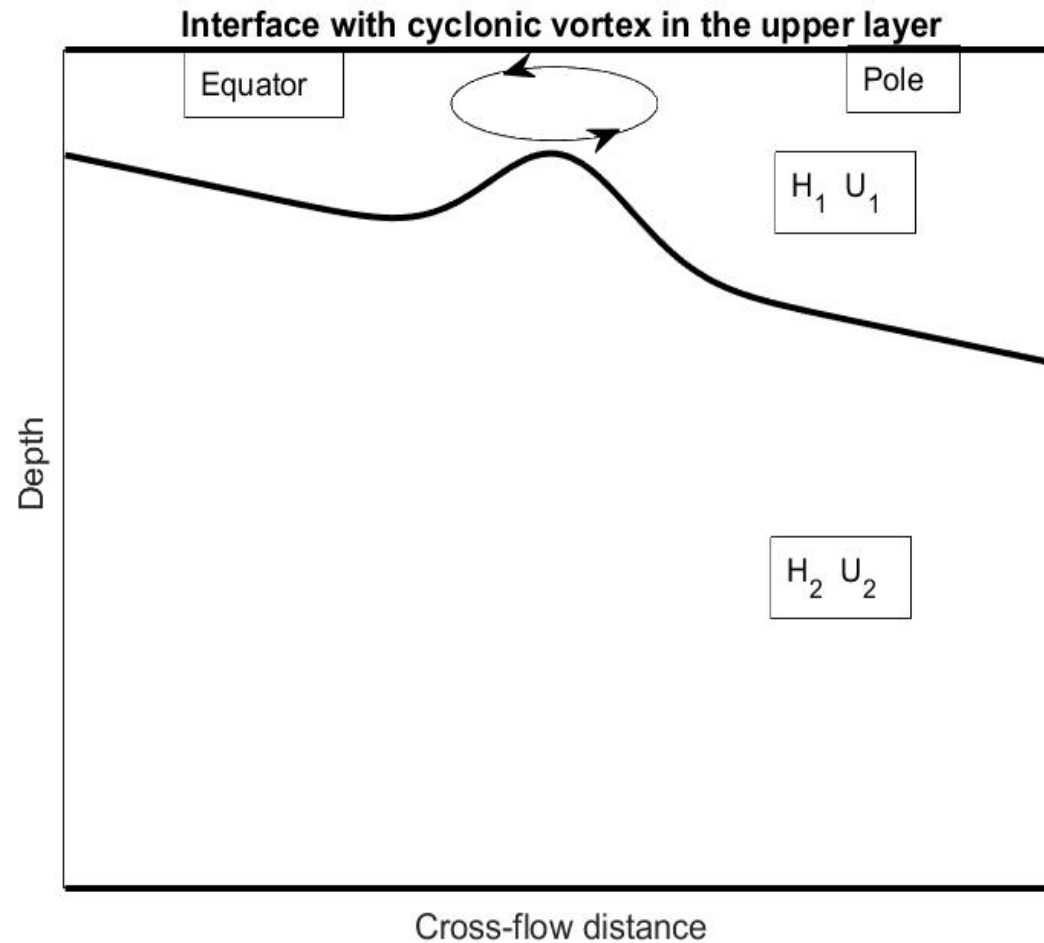
Dispersion relation, two branches:

$$\omega = uk \quad (\text{advection of } q_1)$$

$$\omega = - \frac{\beta_2 k (k^2 + l^2 + F_1)}{(k^2 + l^2)(k^2 + l^2 + F_1 + F_1)}$$

Range of phase velocity: $-\infty < \omega/k < 0$

Radiating vortex



- Put cyclonic circular PV patch in layer 1
- Westward drift with velocity $\cong u$; trapped vortex core in layer 1, open streamlines in layer 2
- Rossby lee waves are generated
- To conserve M^* , the cyclone drifts poleward
- The vortex amplitude remains constant, since $\pi_1 = q_1$

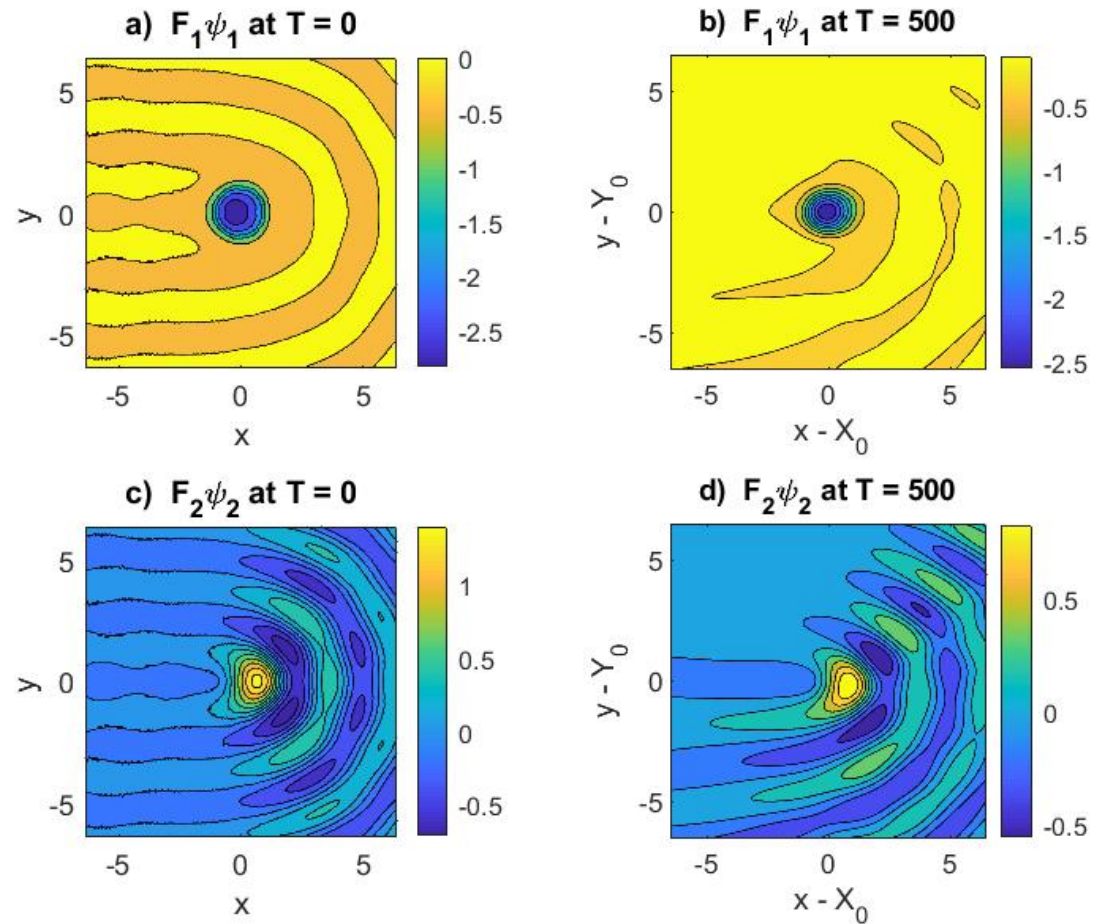
Analytic solution

- Explicit stationary solution for vortex and wave field
- Momentum transport by waves is calculated, gives poleward drift

Numerical simulation

- Pseudospectral code
- Doubly periodic boundary conditions
- Analytic solution as initial condition

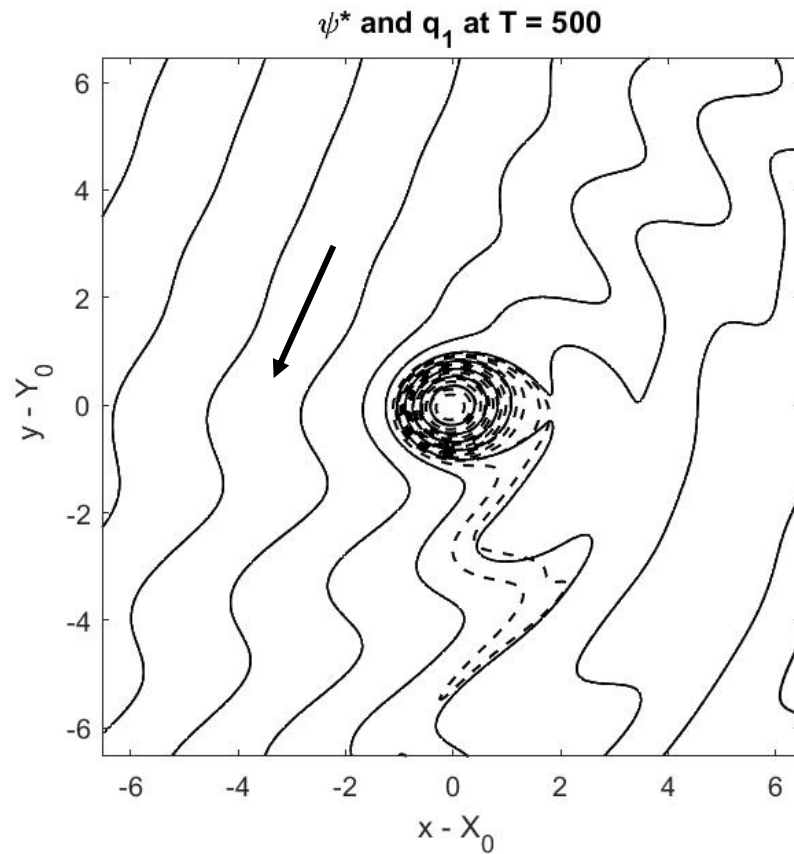
Streamfunctions



Analytic

Simulation, large time

Flow in layer 1



— Streamfunction in co-moving frame
- - - PV

Energy

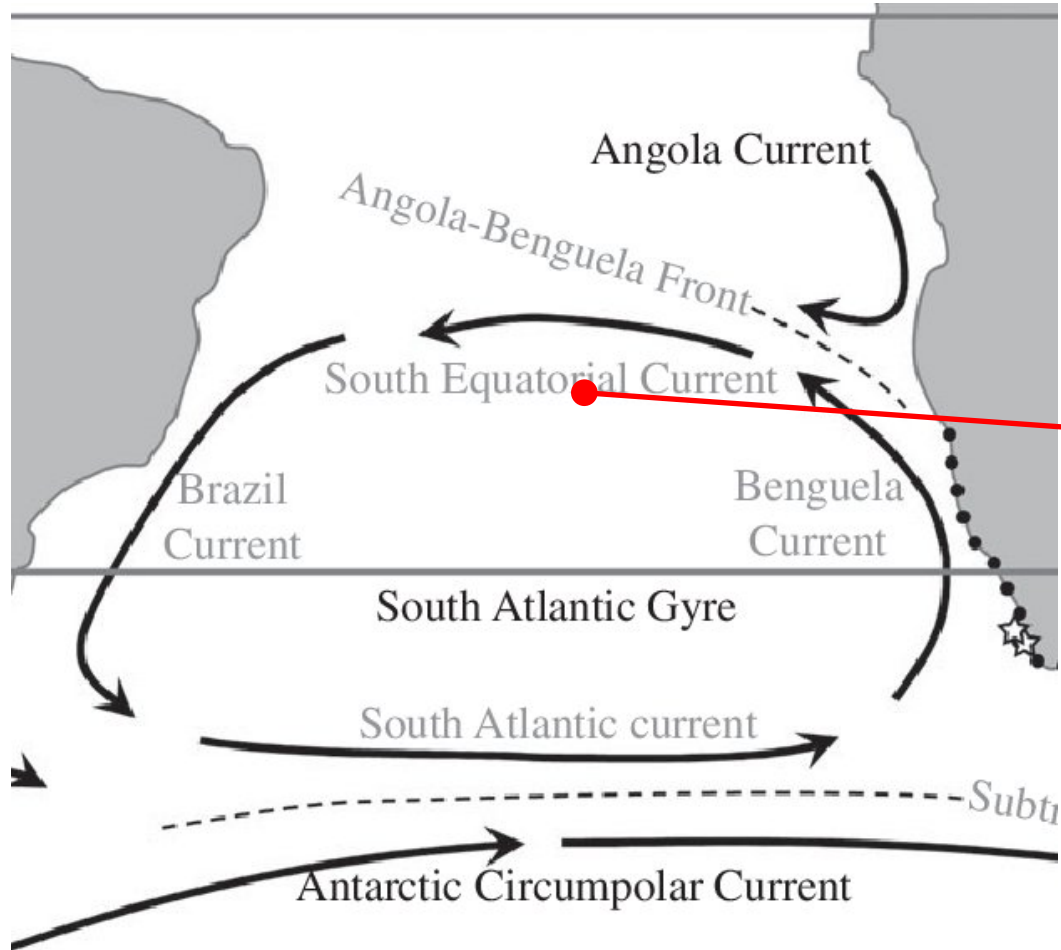
Conserved perturbation energy:

$$E = \int \left(\frac{F_1}{2} (\nabla \psi_1)^2 + \frac{F_2}{2} (\nabla \psi_2)^2 + \frac{h^2}{2} + u q_1 y \right) dx dy$$

where $h = F_1 \psi_1 + F_2 \psi_2$

Thus, the waves (in which $q_1 = 0$) transport away energy, which is compensated by the poleward vortex drift. The ultimate energy source is the background potential energy.

Ocean



Long-lived vortices in the subtropical gyres

Atmosphere

- $u_2 = 0, \quad u_1 = u > 0$ (mid-latitude westerlies)
- $\beta_2 = 0$ (marginally stable)
- $\beta_1 = \frac{\beta}{F_1} + u$ (enhanced PV gradient in upper troposphere)

Cyclone in lower troposphere drifts poleward and generates lee waves in the upper troposphere while keeping its strength.

Conclusion

- The meridional vortex drift is determined by the momentum transported away by lee waves.
- A drifting vortex can release the background potential energy even if there is no baroclinic instability. The PV-anomaly acts as a “catalyst”.