

On the physical constraints for the exceeding probability of rogue waves in deep water

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Basic definitions

Standing out in comparison with surrounding waves in height, steepness and vertical asymmetry, rogue waves are defined as:

$$H(\text{wave height}) = \mathcal{Z}_c + \mathcal{Z}_t > 2.0H_{1/3}(\text{significant wave height})$$

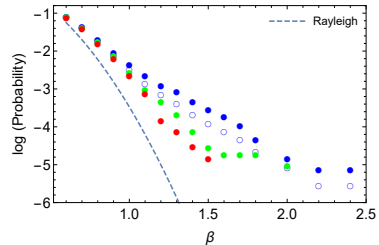
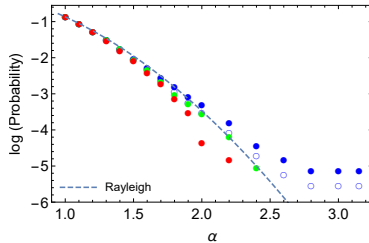
$$\mathcal{Z}_c(\text{crest height}) > 1.2H_{1/3}(\text{significant wave height})$$

In Gaussian seas, the Rayleigh distribution of waves reads:

$$\mathcal{R}(H > \alpha H_{1/3}) = e^{-2\alpha^2} ; \mathcal{R}(\mathcal{Z}_c > \beta H_{1/3}) = e^{-8\beta^2}$$



Probability stratification



Group	$\mathcal{N}_{\beta=1.2}$	$\langle H_{1/3} \rangle$	$\langle T_z \rangle$	$\langle \lambda_{1/3} \rangle$	$\eta_{1/3}$
(I)	782	6.06	8.28	161.6	1.237
(II)	1,450	5.71	8.14	166.2	1.178
(III)	2,184	5.29	7.77	161.2	1.158
(IV)	7,697	6.27	8.80	185.4	1.155

Figure: Empirical probabilities for groups of storms, where I, III and IV are depicted by blue, green and red dots respectively and circles depict the overall set (II). $\mathcal{N}_{\beta=1.2}$ is the inverse of the probability when $\beta = 1.2$.

Ensemble constraints

$$S_{Eff,i} = \frac{H_i}{\langle \langle \lambda_2 \rangle \rangle}; s_s = \frac{\langle H_{1/3} \rangle}{\langle \lambda_2 \rangle}; \varepsilon = \frac{\langle H_{1/3} \rangle}{\langle \lambda_{1/3} \rangle}; \eta_{1/3} \equiv \left(\frac{\langle Z_c \rangle}{\langle Z_t \rangle} \right)_{\alpha > 1}$$

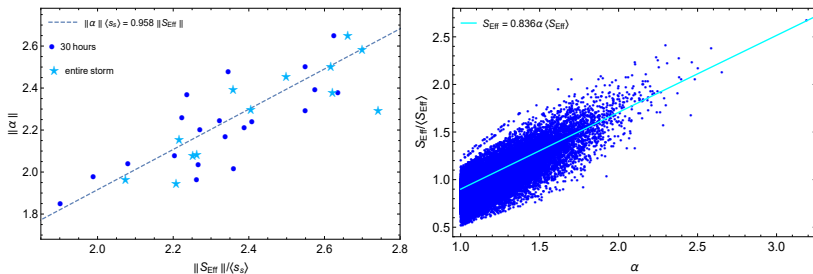


Figure: (Left) The relationship between maximum normalised wave heights, maximum effective steepness and average significant steepness and (Right) The ratio $S_{Eff}/\langle S_{Eff} \rangle$ as a function of normalised height.

Ensemble constraints - steepness limit

$$S_{Eff} \leq \frac{22}{105} \tanh \left(\frac{30 \pi D}{22 \langle \langle \lambda_2 \rangle \rangle} \right)$$

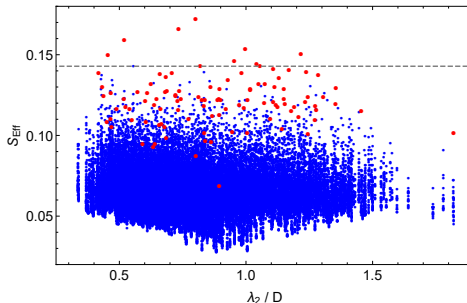


Figure: Effective steepness of all waves taller than $H_{1/3}$ (blue dots) and rogue waves (red dots). Dashed line conveys the deep water limit $\varepsilon < 1/7$ and λ_2 denotes the zero-crossing period.

Ensemble constraints - relative height

$$\epsilon\alpha = \frac{\alpha H_{1/3}}{D} = \frac{H}{D} \leq \frac{16\pi}{55} \left[1 + \frac{4D}{\lambda_2} \tanh\left(\frac{2\pi D}{\lambda_2}\right) \right]^{-1} \equiv \|\epsilon\alpha\|_{\infty}$$

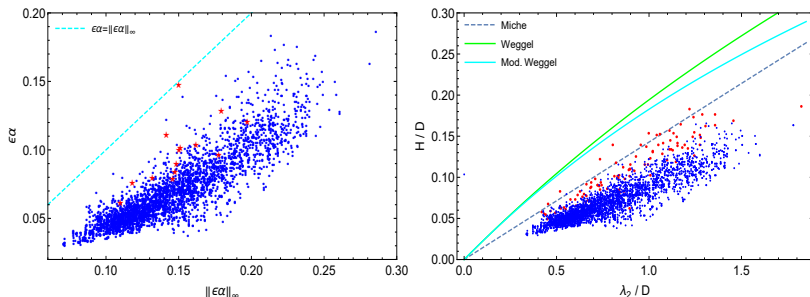


Figure: (Left) Observed H/D ratio versus the theory and (Right) Miche (1944) and Weggel (1972) models versus data, with blue dots for $1 < \alpha \leq 2$, red dots for $\alpha > 2$ and red stars for storm averages.

Ensemble constraints - likelihood pattern

The significant steepness obeys $\varepsilon < 1/8$, whereas the ratio between height and water depth obeys $H_{1/3} < D$ and $\eta_{1/3}$ measures the vertical asymmetry and *nonlinearity* of the system, with no known upper bound, though typically $\eta_{1/3} < 2$ even in rough seas.

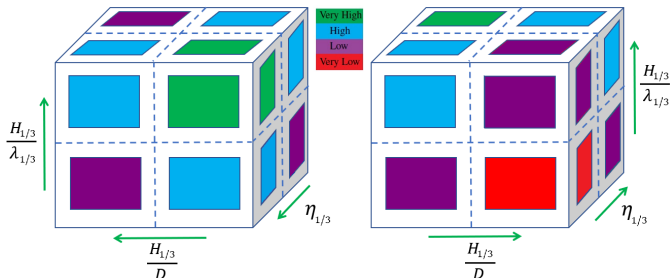


Figure: Qualitative depiction for the likelihood of rogue wave formation. Green arrows point to the direction of increase.

Expected Maximum

Model expectation:

$$\mathbb{E}(\|\alpha\|) = \alpha_{\star} \left[1 - \langle \delta_2 \rangle \right], \quad \alpha_{\star} = \frac{22}{105 \langle s_s \rangle} \tanh \left(\frac{30\pi D}{22 \langle \lambda_2 \rangle} \right),$$

and the sea parameters are combined through,

$$\langle \delta_2 \rangle = \sqrt{\frac{3\pi\tau \langle \lambda_{1/3} \rangle}{875 D}} \cdot \frac{\langle T_z \rangle^{1/4}}{\eta_{1/3}^4},$$

where τ is a measure of the curvature of the $H_{1/3}(t)$ evolution.
The Rayleigh Expectation is computed by the Gumbel (1958) method (N is the number of waves in the time series):

$$\mathbb{E}_{\mathcal{R}}(\alpha) = \sqrt{\frac{1}{2} \ln N} \left[1 + \frac{(1 + \sqrt{5})}{4 \ln N} \right]$$

Expected Maximum - storm geometry

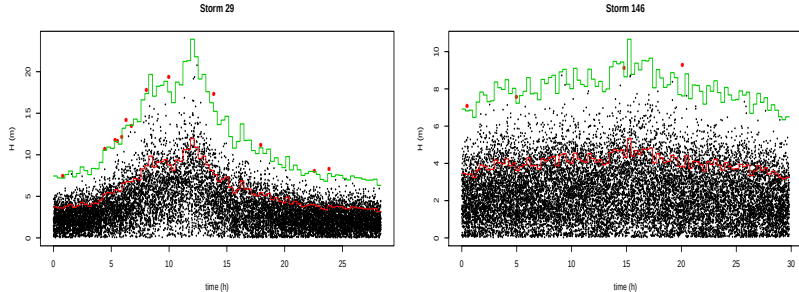


Figure: Significant wave height time series (red solid lines) of selected storms and rogue wave threshold (green), with red dots depicting rogue waves and black dots the remaining waves. Storm parameters are estimated to be at a maximum $\langle \tau_{29} \rangle \approx 15$ and at a minimum $\langle \tau_{146} \rangle \approx 6$.

Expected Maximum Height

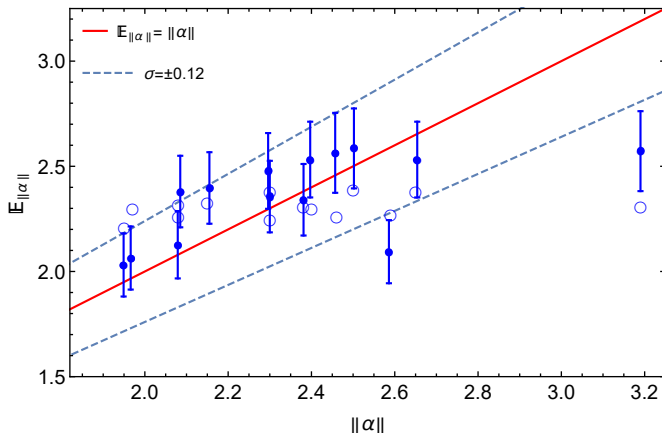


Figure: Expected maximum normalised heights (blue dots and error bars) and Rayleigh derived ones (blue circles).

Upper bounds

$$\alpha < \alpha_{\infty} = \alpha_{\star} [1 + \langle \delta_2 \rangle] ; \langle s_s \rangle < \langle s_s \rangle_{\infty} = \frac{11}{105} [1 + \langle \delta_2 \rangle] \tanh \left(\frac{30\pi D}{22\langle \lambda_2 \rangle} \right)$$

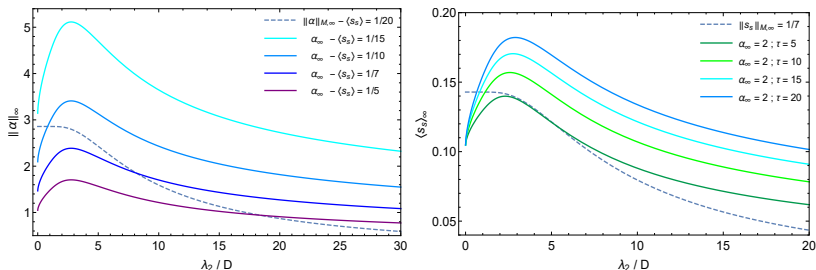


Figure: (Left) Bound on normalised height with fixed depth ($D = 100\text{m}$), nonlinearity ($\eta_{1/3} = 1.2$) and $\tau = 15$. (Right) Bound on the steepness with fixed depth ($D = 100\text{m}$) and nonlinearity ($\eta_{1/3} = 1.2$).

Wave crest heights

$$\left\langle \frac{\beta}{\alpha} \right\rangle \approx \frac{\eta_{1/3}}{1 + \eta_{1/3}} \left[1 + \frac{2\eta_{1/3} \sqrt{\alpha - 1}}{7 + 2\sqrt{\alpha - 1}} \right], \quad \forall \alpha > 1.$$

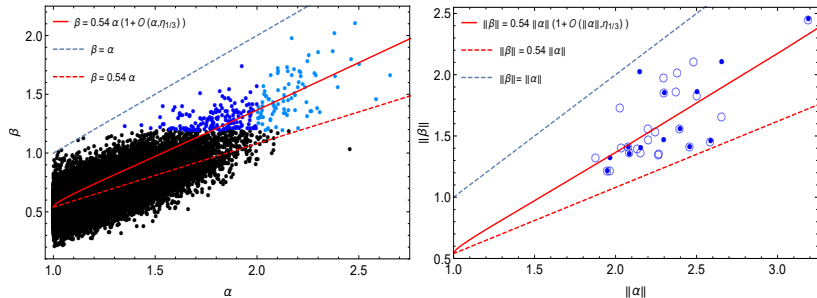


Figure: (Left) Ratio between crest and wave height and (Right) Ratio between maxima.

Wave crest heights - Expected maximum model

$$\mathbb{E}_\beta \cong \mathbb{E}_\alpha \left(1 + \frac{4\eta_{1/3}}{7\mathbb{E}_\alpha} \right) \left(\frac{\eta_{1/3}}{1 + \eta_{1/3}} \right)$$

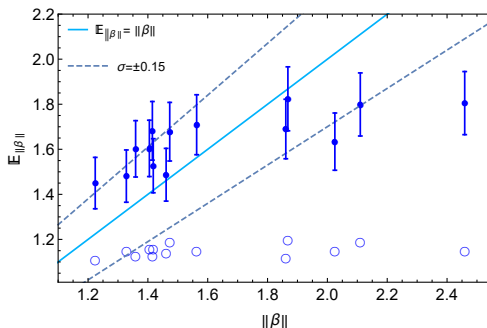


Figure: Expected maximum normalised crest heights (blue dots and error bars) and Rayleigh derived ones (blue circles).

Conclusions - Novelties

- ❑ Ensemble relations among physical variables
- ❑ Bounds for heights and significant steepness
- ❑ Finite depth predictions
- ❑ Homogeneity window through τ
- ❑ Meaningful definition of an effective steepness
- ❑ Breakdown of rogue wave formation into sea parameters
- ❑ Maximum height in terms of sea parameters
- ❑ Accurate model for height-crest conversion
- ❑ Corroboration of height constraints for crests
- ❑ Improved maximum crest height model

References

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Thank you!

