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On strongly nonlinear gravity waves in a vertically sheared atmosphere

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vEGU 2021, 26th April 2021

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DE GRUYTER Math. Clim. Weather Forecast. 2020; 6:63–74

Research Article **Open Access**

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On strongly nonlinear gravity waves in a vertically sheared atmosphere

Part I: Spectral stability of the refracted wave

https://doi.org/10.1515/mcwf-2020-0103
Received October 21, 2020; accepted December 14, 2020

Abstract: We investigate strongly nonlinear stationary gravity waves which experience refraction due to a thin vertical shear layer of horizontal background wind. The velocity amplitude of the waves is of the same order of magnitude as the background flow and hence the self-induced mean flow alters the modulation properties to leading order. In this theoretical study, we show that the stability of such a refracted wave depends on the classical modulation stability criterion for each individual layer, above and below the shearing. Additionally, the stability is conditioned by novel instability criteria providing bounds on the mean-flow horizontal wind and the amplitude of the wave. A necessary condition for instability is that the mean-flow horizontal wind in the upper layer is stronger than the wind in the lower layer.

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Model equations and assumptions

- use Grimshaw's modulation equations in 2 dimensions

$$\text{vert. wave number} \qquad 0 = \frac{\partial k_z}{\partial t} + \frac{\partial \omega}{\partial z} \tag{1a}$$

$$\text{wave action density} \qquad 0 = \rho \frac{\partial a}{\partial t} + \frac{\partial}{\partial z}(\hat{\omega}' \rho a) \tag{1b}$$

$$\text{horiz. mean flow} \qquad 0 = \rho \frac{\partial u}{\partial t} + \frac{\partial}{\partial z}(\hat{\omega}' k_x \rho a) \tag{1c}$$

- this is a multiscale approach, here (z, t) are scaled "slow" coordinates
- consider Boussinesq approximation
- with constants

$$\rho, k_x, N = \text{const.}$$

- valid for quasi-monochromatic internal wave
- no assumption on wave amplitude $\rightarrow$ strongly non-linear theory

Refracted wave solutions I

- consider non-hydrostatic waves with dispersion relation

$$\hat{\omega}^2 = \frac{N^2 k_x^2}{|\boldsymbol{k}|^2} \qquad |\boldsymbol{k}|^2 = k_x^2 + k_z^2 \tag{2}$$

- with vertical group velocity

$$\hat{\omega}' = \frac{\partial \omega}{\partial k_z} = -\frac{N k_x k_z}{|\boldsymbol{k}|^3} \tag{3}$$

- **seek stationary wave solutions**

$$0 = \omega = \hat{\omega} + k_x U \qquad k_z(z,t) = K_z(z) = -\sqrt{\frac{N^2}{U^2} - k_x^2} \tag{4}$$

Refracted wave solutions II

- choose a step like jet structure

$$u(z,t) = U(z) = \begin{cases} U_1, & z < 0 \\ U_2, & z > 0 \end{cases} \tag{5}$$

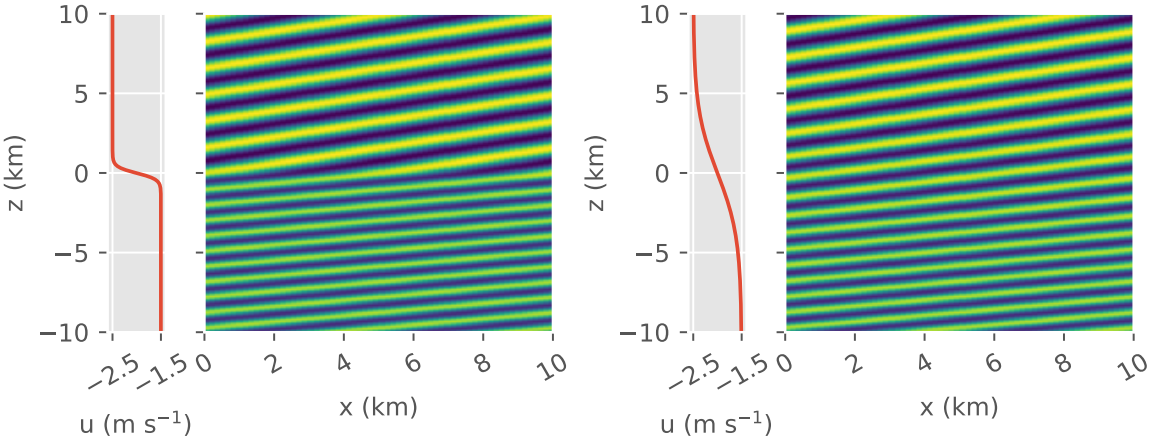
- with boundary condition at the jump

$$\hat{\omega}'(K_{z,j})A_j = \text{const.} \qquad j \in \{1,2\} \tag{6}$$

- divergence free pseudo-momentum flux

Refracted wave solutions II

- examples for differently smooth jet edges



Asymptotic validity of the wave solutions

- show validity using LES
- set a smooth jet with

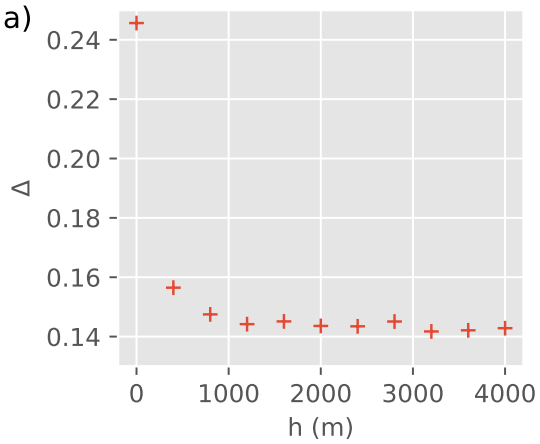
$$U_0(z) = u_1 + \frac{1}{2}(u_2 - u_1)\tanh\left(\frac{z}{h}\right) \tag{7}$$

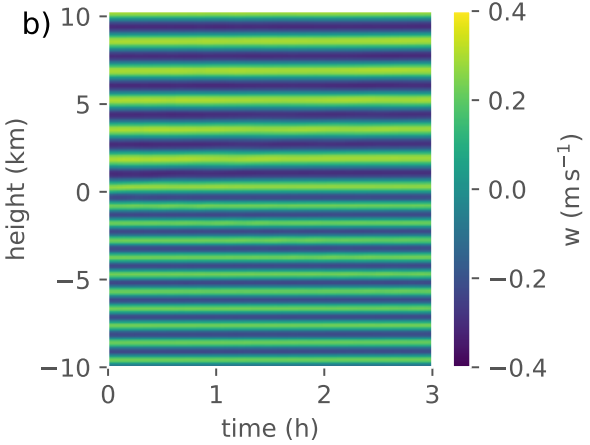
- for the asymptotic limit let $H \to \lambda_z = 2\pi/k_z$
- measure the norm deviation from the initial conditions

$$\Delta(x,t) = \frac{||w(x,z,t) - w(x,z,0)||_2}{||w(x,z,0)||_2} \qquad ||w(z)||_2 = \int_{-\zeta}^{\zeta} w(z)\,dz \tag{8}$$

Asymptotic validity of the wave solutions

- solution is valid for $H \to \lambda_z = 2\pi/k_z$





Linear instability analysis

- linearize Eqs. (1a) and (1c) around the refracted wave solution
- use an exponential ansatz for the perturbation

$$\boldsymbol{p}_j(z,t) = \begin{pmatrix} k_z \\ a \end{pmatrix} = \tilde{\boldsymbol{p}}_j(z) e^{\lambda t} \tag{7}$$

- rewrite the resulting eigenvalue problem in terms on an ODE

$$\frac{d\tilde{\boldsymbol{p}}_j}{dz} = B(z,\lambda)\tilde{\boldsymbol{p}}_j(z) \tag{8}$$

- find set of λ for which coefficient matrix B is not invertible
- temporal eigenvalues $\rightarrow$ instability growth rates
- also: spatial eigenvalues $\rightarrow$ local decay rates of the perturbation

Modulational instability criteria

- classical non-evanescence condition

$$J_j = \frac{N^2}{K_x^2 U_j^2} > 1 \tag{9}$$

- classical modulation instability criterion

$$J_j = \frac{N^2}{K_x^2 U_j^2} > \frac{3}{2} \tag{10}$$

- condition for new type instability

$$J_1 > 2 \qquad |U_1| < |U_2| \qquad \alpha_1^2 > \frac{2}{J_1}\frac{(J_1-1)^2}{2J_1-3} \geq \frac{8}{9} \tag{11}$$

with the normalized wave amplitude α_1

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Assumptions and underlying equations

→ Grimshaw's modulation equations (Boussinesq)

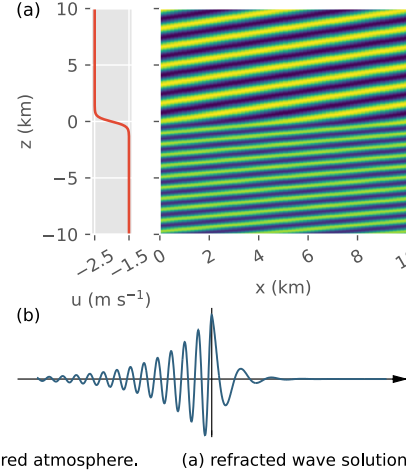
$$0 = \frac{\partial k_z}{\partial t} + \frac{\partial \omega}{\partial z} \qquad 0 = \rho \frac{\partial a}{\partial t} + \frac{\partial}{\partial z}(\hat{\omega}' \rho a) \qquad 0 = \rho \frac{\partial u}{\partial t} + \frac{\partial}{\partial z}(\hat{\omega}' k_x \rho a)$$

→ stationary solution of refracted wave at jet edge → ρ, N, k_x = const.

Modulational instability conditions

→ instability occurs on lower jet edge for large amplitudes

$$|U_1| < |U_2| \qquad \alpha_1^2 > \frac{2}{J_1} \frac{(J_1 - 1)^2}{2J_1 - 3} \qquad J_1 = \frac{N^2}{k_x^2 U_1^2}$$



Schlutow, M. & Voelker, G. (2020). On strongly nonlinear gravity waves in a vertically sheared atmosphere. Mathematics of Climate and Weather Forecasting, 6(1), 63-74. https://doi.org/10.1515/mcwf-2020-0103

(a) refracted wave solution
(b) sketch of instability