

Wave-like solutions in Cosserat micropolar elasticity

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Quantities of Cosserat elasticity

- deformation $\varphi : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$
- deformation gradient tensor $F = \nabla \varphi = \mathbb{1} + \nabla \mathbf{u}$
- classical polar decomposition $F = R U$, U positive definite and symmetric, $R = \text{polar}(F)$ polar part of F
- $\bar{R} = \exp(\bar{A})$ rotation field, $\bar{A}$ skew-symmetric matrix
- dislocation density tensor $\bar{K} = \bar{R}^T \text{Curl } \bar{R}$

Our model – elastic + micropolar

$$V_{\text{elastic}}(F, \bar{R}) = \mu \| \text{sym}(\bar{R}^T F) - \mathbb{1} \|^2 + \frac{\lambda}{2} \text{tr}(\text{sym}(\bar{R}^T F) - \mathbb{1})^2$$

$$V_{\text{micropolar}}(\bar{R}) = \kappa_1 \| \text{dev sym}(\bar{R}^T \text{Curl } \bar{R}) \|^2 + \kappa_2 \| \text{skew}(\bar{R}^T \text{Curl } \bar{R}) \|^2 \\ + \kappa_3 \text{tr}(\bar{R}^T \text{Curl } \bar{R})^2$$

$$V_{\text{kinetic}} = \frac{1}{2} \rho \| \dot{\mathbf{u}} \|^2 + \rho_{\text{rot}} \| \dot{\bar{R}} \|^2$$

μ, λ Lamé parameters, κ_i Cosserat/micropolar parameters or (micro)elastic constants

Variations of the energy functional

Equations of motion are derived by computing variations of the energy functional with respect to the dynamical variables. For the elastic part we work with the displacement $\mathbf{u}$.

When considering small variations $\delta \bar{R}$, then $\bar{R} + \delta \bar{R}$ is not necessarily orthogonal. Hence, there are two options for the micro-rotations:

- (i) Lagrange multiplier in the variational functional $\xi(\bar{R}^T \bar{R} - \mathbb{1})$.
- (ii) More elegant: $\bar{A} + \delta \bar{A}$ is always skew-symmetric and therefore $\exp(\bar{A} + \delta \bar{A})$ is orthogonal. So, we vary with respect to $\bar{A}$.

Towards analytical solutions

The equations of motion of our system are still very complicated. So we seek solutions of a particular form. Rotations about z-axis only.

$$\bar{A} = \begin{pmatrix} 0 & -\phi & 0 \\ \phi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{R} = \exp(\bar{A}) = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

With $\phi = \phi(z, t)$, longitudinal waves only.

Displacements must occur along the axis of rotation. Therefore, $u_1 = 0$, $u_2 = 0$, and $u_3 = \psi(z, t)$, also longitudinal waves $\psi = \psi(z, t)$.

Transversal waves can be studied similarly.

The equations of motion I

elastic part

$$\rho \partial_{tt} \psi + 2\lambda \sin \phi \partial_z \phi - (\lambda + 2\mu) \partial_{zz} \psi = 0 \quad (1)$$

(micro)rotational part

$$\begin{aligned} \rho_{\text{rot}} \partial_{tt} \phi - (\kappa_1 + 2\kappa_3) \partial_{zz} \phi \\ + (\lambda + \mu)(1 - \cos \phi) \sin \phi - \frac{\lambda}{2} \sin \phi \partial_z \psi = 0 \end{aligned} \quad (2)$$

The equations of motion II

elastic part – assuming small displacements

$$\rho \partial_{tt} \psi - (\lambda + 2\mu) \partial_{zz} \psi = 0 \quad (3)$$

$$\text{longitudinal wave speed} \quad v_l = \sqrt{(\lambda + 2\mu)/\rho} \quad (4)$$

(micro)rotational part – assuming small rotations

$$\rho_{\text{rot}} \partial_{tt} \phi - (\kappa_1 + 2\kappa_3) \partial_{zz} \phi = 0$$

$$\text{longitudinal wave speed} \quad v_{\text{long}} = \sqrt{(\kappa_1 + 2\kappa_3)/\rho_{\text{rot}}} \quad (5)$$

Next we seek general solutions of the form $\phi = f(z - vt) = f(s)$ and $\psi = g(z - vt) = g(s)$, $s = z - vt$, no smallness assumptions!

The equations of motion III

Equation (1) can be integrated once to find an expression for ψ' in terms of ϕ . Then the second equation becomes

$$\rho_{\text{rot}} v^2 f'' - (\kappa_1 + 2\kappa_3) f'' + \left[(\lambda + \mu) - \frac{\lambda}{2} C_1 \right] \sin(f) - \frac{1}{2} \left[(\lambda + \mu) + \frac{\lambda^2}{\rho v^2 - (\lambda + 2\mu)} \right] \sin(2f) = 0. \quad (6)$$

Double Sine-Gordon type equation which has solitonic solutions.

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