

On the inertial instability of the Kolmogorov flow in a rotating stratified fluid

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1. Introduction

Linear and non-linear inertial stability of a Kolmogorov flow in a rotating viscous fluid of uniform density is investigated using the method of continued fractions and the low-order Galerkin approximations. A necessary condition for instability is the violation of the criterion of inviscid inertial stability, and the sufficient condition of instability is formulated in terms of the Reynolds criterion. The existence of stable secondary stationary regimes in the problem is shown, developing in a context of loss of stability of the main flow and having the form of rolls (cloud streets in the atmosphere) oriented along it. Stable density stratification is taken into account within the same low-order model framework when the direction of gravity coincides with the direction of rotation of the fluid. In this case, the necessary condition for the inertial instability of the main flow remains the same, but the critical Reynolds number for the instability depends on two additional dimensionless parameters that appear in the problem: the stratification parameter and the Prandtl number. The case of Prandtl numbers less than or equal to unity has been studied in greater detail, when there exists a secondary stationary regime, which may be unstable – in contrast to the case of a fluid that is uniform in density – and stable density stratification is a destabilizing factor.

2. Formulation of the problem

We proceed from the system of equations for a rotating, uniform in density, incompressible viscous fluid

$$\frac{\partial}{\partial t} \nabla^2 \psi + J(\psi, \nabla^2 \psi) = -f \frac{\partial v}{\partial z} + \nu \nabla^4 \psi, \quad (1)$$

$$\frac{\partial}{\partial t} v + J(\psi, v) = f \frac{\partial \psi}{\partial z} + \nu \nabla^2 v + F(x). \quad (2)$$

The velocity vector has three components (u, v, w) corresponding to the local Cartesian coordinates (x, y, z) . These velocity components depend on two coordinates (x, z) , where the x -axis is directed radially and the vertical z -axis is along the axis of the general rotation of the fluid with a constant angular velocity $f/2$. There is no dependence on the y -coordinate directed azimuthally (zonally). Equations (1), (2) are solved in an infinite domain with periodic boundary conditions in (x, z) -directions. In equations (1), (2), $J(a, b) = (\partial a / \partial x)(\partial b / \partial z) - (\partial a / \partial z)(\partial b / \partial x)$ is the Jacobian, ∇^2 the two-dimensional Laplace operator, ν the coefficient of kinematic viscosity, $F(x)$ the external body force that depends

only on the x -coordinate. The force $F(x)$ is formally necessary to support fluid motion against viscous dissipation.

Let us transform equations (1), (2) to a form more convenient for analysis by introducing the Coriolis acceleration x -component (with a minus sign), $\xi = f v$, and using the notation $G(x) = f F(x)$. It is assumed that the external force periodically depends on the x -coordinate, viz. $G(x) = -G_0 \sin k_0 x$. We have

$$\frac{\partial}{\partial t} \nabla^2 \psi + J(\psi, \nabla^2 \psi) = -\frac{\partial \xi}{\partial z} + \nu \nabla^4 \psi, \quad (3)$$

$$\frac{\partial}{\partial t} \xi + J(\psi, \xi) = f^2 \frac{\partial \psi}{\partial z} + \nu \nabla^2 \xi - G_0 \sin k_0 x. \quad (4)$$

Equations (3), (4) have an exact solution

$$\bar{\psi} = 0, \quad \bar{\xi} = -G_0 k_0^{-2} \nu^{-1} \sin k_0 x \quad (5)$$

describing a stationary spatially-periodic flow corresponding in this formulation of the problem to the well-known Kolmogorov flow.

To bring equations (3), (4) to a dimensionless form, we introduce length scale $L = k_0^{-1}$; the scale of the Coriolis acceleration $\Xi = G_0 k_0^{-2} \nu^{-1}$ (see (5)); time scale $T = L^{1/2} \Xi^{-1/2} = G_0^{-1/2} k_0^{1/2} \nu^{1/2}$; the scale of the stream function $\Psi = L^2 T^{-1} = G_0^{1/2} k_0^{-5/2} \nu^{-1/2}$. In dimensionless variables, equations (3), (4) are written in the form

$$\frac{\partial}{\partial t} \nabla^2 \psi + J(\psi, \nabla^2 \psi) = -\frac{\partial \xi}{\partial z} + \frac{1}{\text{Re}} \nabla^4 \psi, \quad (6)$$

$$\frac{\partial}{\partial t} \xi + J(\psi, \xi) = Q \frac{\partial \psi}{\partial z} + \frac{1}{\text{Re}} \nabla^2 \xi - \frac{1}{\text{Re}} \sin x. \quad (7)$$

Here, $\text{Re} = G_0^{1/2} k_0^{-5/2} \nu^{-3/2}$ is the Reynolds number determined via the external force and $Q = f^2 G_0^{-1} k_0 \nu$ is a dimensionless parameter equal to the ratio of the square of the inertial frequency f to the square of the frequency determined by the external force. If, based on the definition $\xi = f v$, to introduce the velocity scale $V_0 = \Xi f^{-1} = G_0 k_0^{-2} \nu^{-1} f^{-1}$, then $Q = f V_0^{-1} k_0^{-1} \equiv \text{Ro}^{-1}$, where Ro is the Rossby number. Similarly, it is readily shown that $\text{Re}^2 = G_0 k_0^{-5} \nu^{-3} = (V_0^2 k_0^{-2} \nu^{-2}) Q \equiv \text{Re}_*^2 Q = \text{Re}_*^2 \text{Ro}^{-1}$, where $\text{Re}_* = V_0 k_0^{-1} \nu^{-1}$ is the intrinsic Reynolds number.

In a two-dimensional, in the horizontal plane (x, y) , problem of the motion of a uniform in density, incompressible viscous fluid under the action of an external force periodic in space, $F(x) = -F_0 \sin k_0 x$, the equation for vertical vorticity in terms of a stream function Ψ such that $u = -\partial \Psi / \partial y$, $v = \partial \Psi / \partial x$, in dimensionless variables would have the form

$$\frac{\partial}{\partial t} \nabla^2 \Psi + J(\Psi, \nabla^2 \Psi) = \frac{1}{\text{Re}_*} \nabla^4 \Psi - \frac{1}{\text{Re}_*} \cos x,$$

where $\text{Re}_* = F_0 k_0^{-3} \nu^{-2} \equiv G_0 f^{-1} k_0^{-3} \nu^{-2}$ is now the Reynolds number, or alternatively $\text{Re}_* = V_0 k_0^{-1} \nu^{-1}$, with $V_0 = F_0 k_0^{-2} \nu^{-1} \equiv G_0 f^{-1} k_0^{-2} \nu^{-1}$, and $J(\Psi, \nabla^2 \Psi) = (\partial \Psi / \partial x)(\partial \nabla^2 \Psi / \partial y) - (\partial \nabla^2 \Psi / \partial x)(\partial \Psi / \partial y)$. The exact stationary solution $\bar{\Psi} = \cos x$ of this equation, such as $\bar{v} = -\sin x$ is called the Kolmogorov flow (sinusoidal profile of the flow velocity of a two-dimensional incompressible homogeneous viscous fluid) and is unstable if $\text{Re}_* > \sqrt{2}$ (Meshalkin and Sinai 1961). It is useful to compare the two introduced Reynolds numbers. It is obvious, also as has been shown above, that $\text{Re}^2 = \text{Re}_*^2 Q$. This result will be used later on.

3. Linear stability problem

Consider the linear problem of stability of a stationary solution $\bar{\psi} = 0$, $\bar{\xi} = -\sin x$ corresponding to (5). Assuming $\psi = \bar{\psi} + \psi' = \psi'$, $\xi = \bar{\xi} + \xi' = -\sin x + \xi'$, we substitute such a representation in (6), (7), linearize the equations with respect to the primed quantities and, as a result, we obtain

$$\left(\frac{\partial}{\partial t} - \frac{1}{\text{Re}} \nabla^2 \right) \nabla^2 \psi' = -\frac{\partial \xi'}{\partial z}, \quad (8)$$

$$\left(\frac{\partial}{\partial t} - \frac{1}{\text{Re}} \nabla^2 \right) \xi' = (Q - \cos x) \frac{\partial \psi'}{\partial z}. \quad (9)$$

According to the Floquet theory, the general solution of equations (8), (9) has the form (cf., e.g., Yau et al 2004)

$$\begin{pmatrix} \psi' \\ \xi' \end{pmatrix} = \sum_{n=-\infty}^{\infty} \begin{pmatrix} \psi_n \\ \xi_n \end{pmatrix} \exp(\lambda t + i q z + i n x) \cdot \exp(i \alpha x), \quad (10)$$

where the real part is understood, and where q is a real number, n an integer, λ may be complex, and α is a real number such that $-1/2 < \alpha \leq 1/2$, the latter without loss of generality. Among the continuum of solutions (10), there are two linearly independent, spatially periodic solutions that correspond to $\alpha = 0$ and $\alpha = 1/2$. They have, respectively, the form

$$\begin{pmatrix} \psi'_a \\ \xi'_a \end{pmatrix} = \sum_{n=-\infty}^{\infty} \begin{pmatrix} \psi_n \\ \xi_n \end{pmatrix} \exp(\lambda t + i q z + i n x), \quad (11)$$

$$\begin{pmatrix} \psi'_b \\ \xi'_b \end{pmatrix} = \sum_{n=-\infty}^{\infty} \begin{pmatrix} \psi_{n+1/2} \\ \xi_{n+1/2} \end{pmatrix} \exp\{\lambda t + i q z + i(n+1/2)x\}. \quad (12)$$

Solution (11) has the same period in x as the external force; we will call it a 2π -periodic solution. Solution (12) has a doubled period compared to the external force; we call it a 4π -periodic solution.

Substituting (11), (12) into (8), (9), we arrive at the relations

$$\psi_{n-1} - c_n \psi_n + \psi_{n+1} = 0, \quad c_n = 2Q + \frac{2}{q^2} \left(\lambda + \frac{n^2 + q^2}{\text{Re}} \right)^2 (n^2 + q^2) \quad (13)$$

and correspondingly

$$\psi_{n-1/2} - c_{n+1/2}\psi_{n+1/2} + \psi_{n+3/2} = 0, \quad c_{n+1/2} = 2Q + \frac{2}{q^2} \left[\lambda + \frac{(n+1/2)^2 + q^2}{\text{Re}} \right]^2 \left[(n+1/2)^2 + q^2 \right]. \quad (14)$$

We use the method of continued fractions to obtain the solvability condition for (13)

$$\frac{1}{2}c_0 = \frac{1}{c_1 - \frac{1}{c_2 - \frac{1}{c_3 - \dots}}} \quad (15)$$

and the solvability condition for (14)

$$c_{1/2} - \frac{1}{c_{3/2} - \frac{1}{c_{5/2} - \frac{1}{c_{7/2} - \dots}}} = \pm 1. \quad (16)$$

4. Stability in a conservative problem

Let us first consider the conservative problem when $\text{Re} = \infty$ in equations (8), (9). Here, taking into account the periodic boundary conditions existing in the problem, there is a quadratic conservation law

$$E = \iint \left\{ \frac{(\nabla \psi')^2}{2} + \frac{(\xi')^2}{2(Q - \cos x)} \right\} dx dz, \quad (17)$$

ensuring the linear stability of the main flow $\bar{\psi} = 0$, $\bar{\xi} = -\sin x$ at $Q > 1$ ($\text{Ro} < 1$). This exactly corresponds to the well-known criterion for inertial stability, that is, the absolute vorticity of the main flow about a vertical axis must be everywhere positive if $f > 0$. This constitutes an approximation of the generalized Rayleigh's criterion for centrifugal instability (e.g., Park and Billant 2013) onto a case of parallel flows on an f plane. Let us compare the rigorous result (17) with what follows from equations (15), (16). We proceed from the assumption that the principle of exchange of stabilities is in effect (Drazin and Reid 2004), i.e. on the neutral stability curve $\lambda = 0$ ($\text{Re } \lambda = \text{Im } \lambda = 0$). For $\text{Re} = \infty$, $\lambda = 0$ and identically with respect to n we have that $c_n = c_{n+1/2} = c \equiv 2Q$, and the infinite continued fractions in (15), (16) are summed up and we find that $c = 2$, i.e. $Q = 1$.

5. Stability with account of viscosity

Let us now turn to the linear problem of stability with allowance for viscosity. We start from the validity also in this case of the principle of exchange of stabilities, when $\lambda = 0$ on the neutral curve of stability. Now, the coefficients c_n , $c_{n+1/2}$ in (14) increase very rapidly with the growth of n . This provides a fast convergence of convergents to the infinite continued fraction in (15), (16) and the ability to analyze stability based on the lowest-order convergents. Taking such convergents, we have $c_0 c_1 = 2$, $c_{1/2} = \pm 1$ from (15), (16), respectively. Therefore, taking into account the principle of exchange of stabilities, we obtain

$$\left(2Q + \frac{2}{q^2} \frac{q^6}{\text{Re}^2} \right) \left(2Q + \frac{2}{q^2} \frac{(1+q^2)^3}{\text{Re}^2} \right) = 2, \quad (18)$$

$$2Q + \frac{2}{q^2} \frac{(q^2 + 1/4)^3}{\text{Re}^2} = \pm 1. \quad (19)$$

The case of 4π -periodic perturbations corresponding to (19) is easier to analyze. Here, the neutral stability curve is given by the expression

$$\text{Re}^2 = \frac{2}{q^2} \frac{(q^2 + 1/4)^3}{1 - 2Q}. \quad (20)$$

Instability is possible only for $Q < 1/2$, which is consistent with the conservative problem analysis when using a convergent to the infinite continued fraction of the lowest order. Recalling the connection of Re^2 with the Reynolds number squared Re_*^2 for the classical two-dimensional Kolmogorov flow (Meshalkin and Sinai 1961), we will have $\text{Re}^2 = \text{Re}_*^2 Q$ and, therefore,

$$\text{Re}_*^2 = \frac{2}{q^2} \frac{(q^2 + 1/4)^3}{Q(1 - 2Q)}. \quad (21)$$

The right-hand side of (21) depends on Q and q^2 . The minimum of Re_*^2 as a function of Q is obtained when the denominator of the fraction on the right-hand side of (21) is maximum. This will be at $Q = 1/4$ and the maximum denominator of the fraction is $1/8$. Now minimizing the right side of (21) as a function of q^2 , we see that the minimum is reached at $q^2 = 1/8$. In this case, the multiplier based on the minimum value q^2 is $27/32$. Collecting the obtained values, we find that

$$\text{Re}_{*\min}^2 = 27/4. \quad (22)$$

The obtained $\text{Re}_{*\min}$ -value is slightly higher, but has the same order of magnitude as the critical value of the Reynolds number $\text{Re}_{*\text{crit}}$ for the long-wave instability of the Kolmogorov flow, which is determined by the well-known relationship $\text{Re}_{*\text{crit}}^2 = 2$ (Meshalkin and Sinai 1961). Returning back to the initial Reynolds number Re we see that the critical for instability Reynolds number squared, Re^2 , is minimal at $Q \rightarrow 0$ and its absolute minimum as a function of q^2 is achieved at $q^2 = 1/8$ and, therefore, $\text{Re}_{\min}^2 = 27/32$.

Let us now consider the case of 2π -periodic perturbations corresponding to (18). Here, as can be shown, the neutral stability curve is given by the equation

$$\text{Re}^2 = \frac{1}{q^2} \frac{\sqrt{2\varepsilon m m_1 + Q^2 (m + m_1)^2} + Q(m + m_1)}{\varepsilon}, \quad (23)$$

where the notations $\varepsilon = 1 - 2Q^2$, $m = (1 + q^2)^3$, $m_1 = q^6$ are introduced. It is seen that the neutral stability curve exists only for $Q < 1/\sqrt{2}$, which again exactly corresponds to the conservative problem using the convergent to the infinite continued fraction of the lowest order. The right-hand side of (23), as a function of q^2 , has asymptotes that go to infinity both for $q^2 \rightarrow 0$ and for $q^2 \rightarrow \infty$ and has a minimum at a certain value q_*^2 , which must be determined numerically. We note that q_*^2 decreases monotonically with decreasing of Q . Making a substitution $\text{Re}^2 = \text{Re}_*^2 Q$ in (23), we rewrite (23) in the form

$$\text{Re}_*^2 = \frac{1}{q^2} \frac{\sqrt{2mm_1 + Q^2(m - m_1)^2} + Q(m + m_1)}{Q(1 - 2Q^2)}, \quad (24)$$

which is more convenient for analysis and has principal physical advantages (see, section 2). The right side of (24) depends on Q and q^2 , and it is necessary to find its absolute minimum. We roughly assume that the Re_*^2 -minimum in Q is reached when the denominator of the fraction on the right-hand side of (24) is maximum. The maximum of the denominator of the fraction will be at $Q^2 = 1/6$ and the maximum itself is equal to $\sqrt{6}/9$. Substituting $Q = 1/\sqrt{6}$ into the right side of (24), we find numerically the minimum of this right side as a function of q^2 . It is achieved at $q_*^2 \approx 0.306049$. This q_*^2 -value corresponds to the minimum value of the Reynolds number, determined from the relation

$$\text{Re}_{*\min}^2 \approx 6.560454, \quad (25)$$

that is, the result close to (22) is obtained.

Thus, restricting ourselves to the first approximation in the characteristic equations (15), (16) by convergents of the lowest order, we obtain close, although not identical, results in terms of $\text{Re}_{*\min}^2$ for both 4π -periodic and 2π -periodic perturbations. In the search for corrections when higher-order convergents are considered, we first analyze the simpler case of instability with respect to 4π -periodic perturbations. We took two sequential convergents to the infinite continued fraction in (16) of sufficiently high order, the first truncated at $c_{11/2}$ and the other at $c_{13/2}$ and numerically solved the solvability equation (16) by varying q^2 in the neighborhood of $q^2 = 1/8$ that corresponds to the value $\text{Re}^2 = 27/32$ for the first approximation. We did it for $Q = 0, 1/4, 3/8, 7/16$. The validity of the principle of exchange of stabilities was assumed. The numerical results showed a remarkable convergence of results: for $c_{11/2}$ - and $c_{13/2}$ -truncations the results obtained coincide within the nine digits after the comma. In a more approximate way, they are summarized in table 1.

Table 1. Numerical values of the wavenumber squared of the fastest growing disturbance (the second row) and of the minimum critical Reynolds number squared (the third row) for 4π -periodic perturbations and four selected values of Q . The calculations are carried out using the higher order convergents to the infinite continued fraction in (16). The fourth and fifth rows show the corresponding values for the lowest order approximation, see (20).

Q	0	1/4	3/8	7/16
q_*^2	0.1245	0.1226	0.1147	0.0835
Re_*^2	0.8471	1.7146	3.6063	9.3414
q^2	$\frac{1}{8} = 0.125$	$\frac{1}{8} = 0.125$	$\frac{1}{8} = 0.125$	$\frac{1}{8} = 0.125$
Re^2	$\frac{27}{32} = 0.84375$	$\frac{27}{16} = 1.6875$	$\frac{27}{8} = 3.375$	$\frac{27}{4} = 6.75$

We conclude that for a qualitative analysis, and in many aspects quantitative, it is enough to limit ourselves to the results of the first approximation. To get more information on the dependence

$\text{Re}^2 = \text{Re}^2(Q, q^2)$, two cross-sections have been made. In the first cross-section the value $Q = 1/4$ was set and the value of q^2 was increased in the geometric progression with the common ratio 2 starting from $q^2 = 1/4096$. Truncation at $c_{15/2}$ was applied. Both the lower and upper neutral stability curves, which limit the instability domain, have been calculated and are shown in figure 1 together with the neutral stability curve (20).

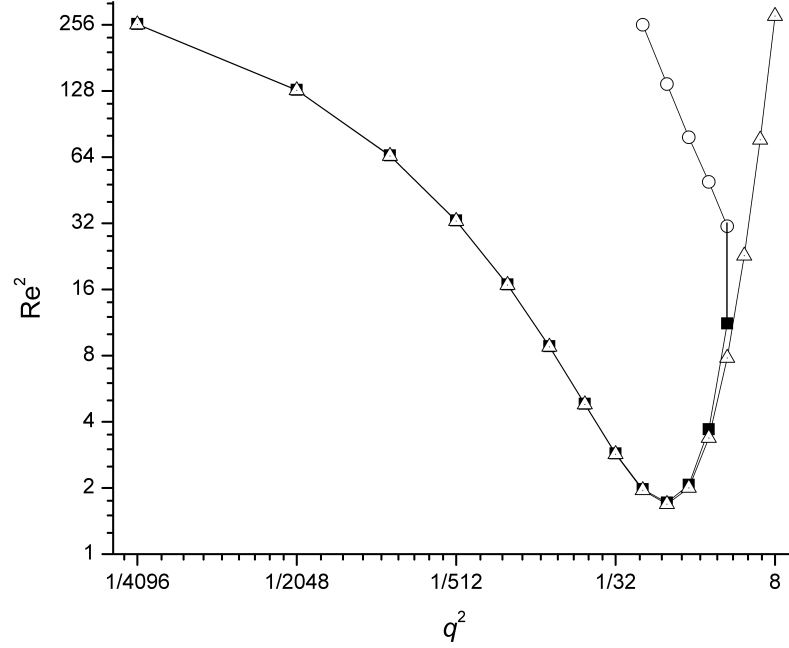


Figure 1. Dependence of the square of the Reynolds number Re^2 on q^2 for the lower (black squares) and upper (white circles) neutral stability curve for the higher-order convergents to the infinite continued fraction in (16). The curve labeled with white triangles corresponds to (20). The value $Q = 1/4$ is used. Notice how the curves labeled with black squares and white circles are very close at $q^2 \ll 1$, for the chosen scale on the vertical axis.

The results show a good agreement for small q^2 , but reveal the existence of a “tongue” of instability that extends from small q^2 to $q_{\text{lim}}^2 \cong 1.165$, where the lower and upper neutral stability curves merge. The Reynolds number squared corresponding to this merger is $\text{Re}^2 \cong 20.76$. For $q^2 > q_{\text{lim}}^2$ another branch of the neutral stability curve appears with $\text{Re}^2 = O(10^3)$ (not shown); the minimum of $\text{Re}^2 \sim 3 \times 10^2$ is reached at $q^2 \sim 8$. For other Q -values the dependence $\text{Re}^2(q^2)$ is qualitatively the same. In the second cross section, $q^2 = 1/8$ was fixed while the Q -values were varied with some discretization from zero to the upper limit value $Q = 1$ corresponding to the inviscid theory. The results are shown in figure 2.

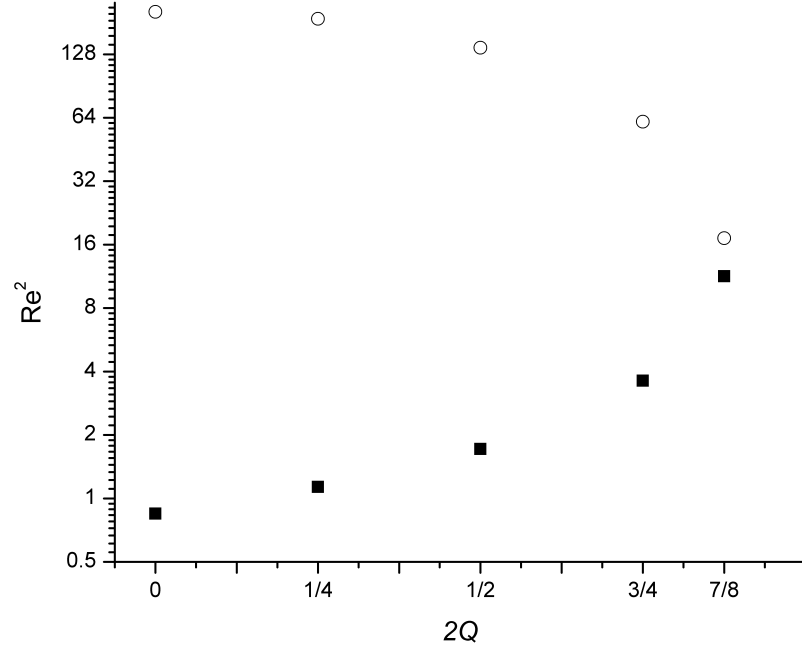


Figure 2. Dependence of the square of the Reynolds number Re^2 on $2Q$ for the lower (black squares) and upper (white circles) neutral stability curves for the higher-order convergents to the infinite continued fraction in (16). The value $q^2 = 1/8$ is used.

A “tongue” of instability extends from $Q = 0$ to $Q_{\text{lim}} \approx 0.439$ where the lower and upper neutral stability curves merge, and $Re^2 \cong 14.28$ at this merging point. For $Q > Q_{\text{lim}}$ another branch of the lower stability curve appears and the minimum critical Re^2 value increases monotonically from $Re^2 \sim 70570$ to infinity at $Q \rightarrow 0.912$ – (not shown).

It is instructive to do the same procedure for 2π -periodic perturbations. The infinite continued fraction with c_7 -truncation was applied in the solvability condition (15). The results obtained are identical, up to nine digits after the comma, to those for c_8 -truncation, which shows a good convergence to the “true” solution corresponding to the infinite continued fraction in (15). Again, two cross sections were made. The first cross section was for $Q = 1/4$ and q^2 increasing in the geometric progression with the common ratio 2 from $q^2 = 1/4096$. The results for the neutral stability curve are shown in figure 3 in comparison with the neutral stability curve of the first approximation, see (23).

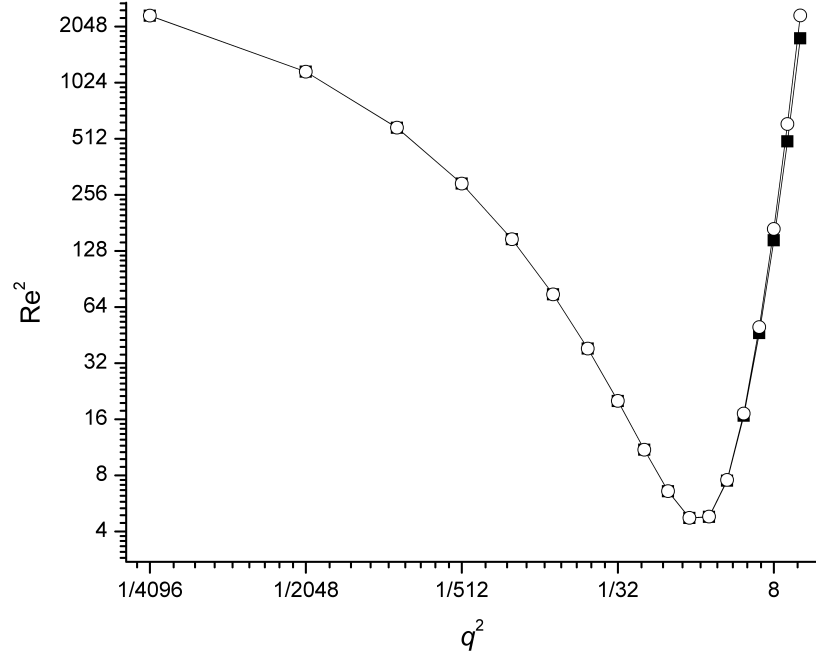


Figure 3. Dependence of the square of the Reynolds number Re^2 on q^2 for the lower neutral stability curve (black squares) for the higher order convergents to the infinite continued fraction in (15). The curve labeled with white circles corresponds to (23). The value $Q = 1/4$ is used. Notice how the curves are indistinguishable except for the largest q^2 -values for the chosen scale on the vertical axis.

The results appeared to be strikingly close, except for the largest q^2 -values. The second cross section was made for $q^2 = 1/4$, that better than $q^2 = 1/8$ fits the absolute minimum of Re^2 in figure 3, and for $Q \in (0,1)$. The results are shown in figure 4. They show a monotonic increase of Re^2 with Q throughout the interval $(0,1)$. However, despite the simpler pattern of instability with respect to 2π -periodic perturbations compared to 4π -periodic perturbations, the latter may be more unstable for small Q -values (i.e. large Rossby numbers). That is, the minimum critical Re^2 value for them is lower than for 2π -periodic perturbations (cf. figures 1 and 3). Taking this into account, we focus in the following more on an analytically simpler case of 4π -periodic disturbances.

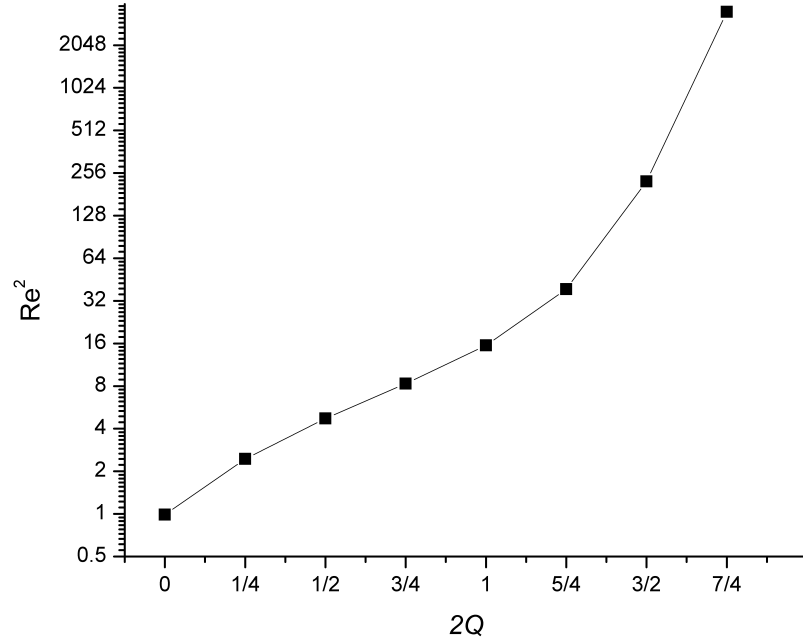


Figure 4. Dependence of the square of the Reynolds number Re^2 on $2Q$ for the neutral stability curves corresponding to the higher order convergents to the infinite continued fraction in (15). The value $q^2 = 1/4$ is used.

6. Nonlinear problem

Let us move on to the nonlinear system of equations (6), (7), concentrating on 4π -periodic disturbances. We represent solution (6), (7) in the form

$$\zeta = \bar{\zeta}(x, t) + \zeta'(x, z, t),$$

$$\zeta'(x, z, t) = \hat{\zeta}(x, t) \exp(iqz) + \hat{\zeta}^*(x, t) \exp(-iqz),$$

where $\zeta = (\psi, \xi)$ and the asterisk above dependent variables denotes a complex conjugation. We arrive at a closed system of equations

$$\frac{\partial \bar{\xi}}{\partial t} + iq \frac{\partial}{\partial x} (-\hat{\psi} \hat{\xi}^* + \hat{\psi}^* \hat{\xi}) = \frac{1}{\text{Re}} \frac{\partial^2 \bar{\xi}}{\partial x^2} - \frac{1}{\text{Re}} \sin x, \quad (27a)$$

$$\left(\frac{\partial}{\partial t} - \frac{1}{\text{Re}} \nabla^2 \right) \nabla^2 \hat{\psi} = -iq \hat{\xi}, \quad (27b)$$

$$\left(\frac{\partial}{\partial t} - \frac{1}{\text{Re}} \nabla^2 \right) \hat{\xi} = iq \left(Q + \frac{\partial \bar{\xi}}{\partial x} \right) \hat{\psi}, \quad (27c)$$

plus the corresponding equations for $\hat{\psi}^*$ and $\hat{\xi}^*$. We restrict ourselves in (12) to the minimum number of components $\psi_{\pm 1/2}$ and $\xi_{\pm 1/2}$. Without loss of generality, the stream function components can be considered real, and the Coriolis acceleration components purely imaginary. Introducing the notation $a_1 = \psi_{1/2}$, $a_2 = \psi_{-1/2}$, $ib_1 = \xi_{1/2}$, $ib_2 = \xi_{-1/2}$ and assuming $\bar{\xi} = -b_0(t) \sin x$, we arrive on the basis of equations (27) to a system of five real equations. The class of solutions can be further reduced by choosing one of the options: (i) $a_1 = a_2$, $b_1 = b_2$ and

(ii) $a_1 = -a_2$, $b_1 = -b_2$. Considering the first possibility without loss of generality, we arrive at the system ($q_1^2 = q^2 + 1/4$)

$$\left(\frac{d}{dt} + \frac{q_1^2}{\text{Re}}\right) q_1^2 a_1 = -q b_1, \quad \left(\frac{d}{dt} + \frac{q_1^2}{\text{Re}}\right) b_1 = q \left(Q - \frac{1}{2} b_0\right) a_1, \quad (28a)$$

$$\left(\frac{d}{dt} + \frac{1}{\text{Re}}\right) b_0 - 4q a_1 b_1 = \frac{1}{\text{Re}}, \quad (28b)$$

which transforms to the famous Lorenz system (Lorenz 1963) if to make a change of variables $(t = \kappa^{-1} \tau, a_1 = Ax, -b_1 = By, 1 - b_0 = Cz)$, where $\kappa = q_1^2 \text{Re}^{-1}$, $A = \kappa 2^{-1/2} q^{-1}$, $B = \kappa^2 2^{-1/2} q^{-2}$, $C = 2\kappa^2 q^{-2}$, but is simpler in the sense that dissipation acts in the same way on two components of motion in (28a), which corresponds to the case when the Prandtl number is equal to unity in the Lorenz model. In these new variables $1/2 - Q$ is proportional to the re-scaled Rayleigh number r in the Lorenz system. The system of equations (28) has a primary stationary solution $b_0 = 1$, $a_1 = b_1 = 0$, whose stability we already investigated and found the value of the critical Reynolds number for the instability Re_{crit} , determined by (20), and it is necessary that the inequality $1 - 2Q > 0$ be satisfied. There are two secondary stationary solutions

$$\bar{b}_0 = 2Q + (1 - 2Q) \frac{\text{Re}_{\text{crit}}^2}{\text{Re}^2}, \quad (29a)$$

$$\bar{a}_1^2 = \frac{1}{4q_1^4} (1 - 2Q) \frac{\text{Re}^2 - \text{Re}_{\text{crit}}^2}{\text{Re}^2}, \quad \bar{b}_1 = -\frac{q_1^4}{\text{Re} q} a_1. \quad (29b)$$

These secondary solutions exist only for $\text{Re}^2 > \text{Re}_{\text{crit}}^2$ when the primary solution loses its stability and, in turn, a natural question arises about their stability. If we consider the equations linearized with respect to the perturbations imposed on the secondary flow, and look for the time dependence in the form $\exp(\lambda t)$, then the question of stability is reduced to the analysis of the complete cubic equation

$$\lambda^3 + p_1 \lambda^2 + p_2 \lambda + p_3 = 0, \quad (30)$$

$$p_1 = \frac{1}{\text{Re}} (1 + 2q_1^2), \quad p_2 = \frac{q^2}{2q_1^4} (1 - 2Q) \frac{\text{Re}^2 + \text{Re}_{\text{crit}}^2}{\text{Re}^2}, \quad p_3 = \frac{q^2}{q_1^2} (1 - 2Q) \frac{\text{Re}^2 - \text{Re}_{\text{crit}}^2}{\text{Re}^3}.$$

Given that $p_1 > 0$, $p_2 > 0$, $p_3 > 0$ for $1 - 2Q > 0$ and, as can be shown $p_1 p_2 - p_3 > 0$, then according to the Routh–Hurwitz criterion all roots of equation (30) have negative real parts, that is, the necessary and sufficient conditions are met for the stability of secondary flows. Against the background of the primary flow losing stability, stable stationary vertical circulations develop with the vortex axis parallel to the primary flow and resembling cloud streets that are frequently observed in the atmosphere when cold Arctic air flows over open warm sea.

Preliminary analysis shows that the same conclusion about the stability of secondary stationary solutions is also valid for 2π -periodic disturbances.

7. Account for stable stratification

Let us complicate the problem by considering the case of a stable density stratification of a fluid with a constant Brunt–Väisälä frequency N . The direction of action of gravity is considered to

coincide with the axis of fluid rotation. In this case, instead of (3), (4), we have a more complex system of equations

$$\frac{\partial}{\partial t} \nabla^2 \psi + J(\psi, \nabla^2 \psi) = -\frac{\partial \xi}{\partial z} + \frac{\partial \eta}{\partial x} + \nu \nabla^4 \psi, \quad (31)$$

$$\frac{\partial}{\partial t} \xi + J(\psi, \xi) = f^2 \frac{\partial \psi}{\partial z} + \nu \nabla^2 \xi - G_0 \sin k_0 x, \quad (32)$$

$$\frac{\partial}{\partial t} \eta + J(\psi, \eta) = -N^2 \frac{\partial \psi}{\partial x} + \kappa \nabla^2 \eta, \quad (33)$$

where η is the buoyancy and κ the thermal diffusivity. The system of equations (31)-(33) again has an exact solution

$$\bar{\psi} = \bar{\eta} = 0, \quad \bar{\xi} = -G_0 k_0^{-2} \nu^{-1} \sin k_0 x, \quad (34)$$

which describes a spatially periodic stationary flow corresponding to the Kolmogorov flow. Since ξ and η are quantities of the same physical dimension (acceleration), then reducing equations (31)-(33) to dimensionless form as indicated above, we arrive at the system of dimensionless equations

$$\frac{\partial}{\partial t} \nabla^2 \psi + J(\psi, \nabla^2 \psi) = -\frac{\partial \xi}{\partial z} + \frac{\partial \eta}{\partial x} + \frac{1}{\text{Re}} \nabla^4 \psi, \quad (35)$$

$$\frac{\partial}{\partial t} \xi + J(\psi, \xi) = Q \frac{\partial \psi}{\partial z} + \frac{1}{\text{Re}} \nabla^2 \xi - \frac{1}{\text{Re}} \sin x, \quad (36)$$

$$\frac{\partial}{\partial t} \eta + J(\psi, \eta) = -B \frac{\partial \psi}{\partial x} + \frac{1}{\text{Re Pr}} \nabla^2 \eta. \quad (37)$$

Here two new dimensionless parameters appeared: $B = N^2 G_0^{-1} k_0 \nu > 0$ is a dimensionless stratification parameter equal to the ratio of the square of the Brunt–Väisälä frequency to the square of the frequency determined by the external force; $\text{Pr} = \nu / \kappa$ is the Prandtl number. If to introduce the velocity scale $V_0 = G_0 k_0^{-2} \nu^{-1} f^{-1}$ (see, section 2), then $B = (N/f) N V_0^{-1} k_0^{-1} \equiv (N/f) \text{Fr}^{-1}$, where Fr is the internal Froude number. Alternatively, $B = (N^2/f^2) Q \equiv (N^2/f^2) \text{Ro}^{-1}$, where Ro is the Rossby number. Sometimes, it is $S = N^2/f^2$ that is named the stratification parameter (e.g., Weber 1980); now, $B = S^{1/2} \text{Fr}^{-1} = S \text{Ro}^{-1}$, such that $\text{Ro} = S^{1/2} \text{Fr}$.

Let us investigate the linear stability of a stationary solution $\bar{\psi} = \bar{\eta} = 0$, $\bar{\xi} = -\sin x$, similar to (5). Assuming $\psi = \bar{\psi} + \psi' = \psi'$, $\eta = \bar{\eta} + \eta' = \eta'$, $\xi = \bar{\xi} + \xi' = -\sin x + \xi'$, we substitute such a representation in (35) - (37), linearize the resulting equations with respect to the primed quantities and, as a result, we obtain

$$\left(\frac{\partial}{\partial t} - \frac{1}{\text{Re}} \nabla^2 \right) \nabla^2 \psi' = -\frac{\partial \xi'}{\partial z} + \frac{\partial \eta'}{\partial x}, \quad (38)$$

$$\left(\frac{\partial}{\partial t} - \frac{1}{\text{Re}} \nabla^2 \right) \xi' = (Q - \cos x) \frac{\partial \psi'}{\partial z}, \quad (39)$$

$$\left(\frac{\partial}{\partial t} - \frac{1}{\text{Re Pr}} \nabla^2 \right) \eta' = -B \frac{\partial \psi'}{\partial x}. \quad (40)$$

In the conservative limit case when $\text{Re} = \infty$ and $\text{Re Pr} = \infty$ by taking into account the periodic boundary conditions existing in the problem, we obtain a quadratic conservation law (cf. (17))

$$E = \iint \left\{ \frac{(\nabla \psi')^2}{2} + \frac{(\xi')^2}{2(Q - \cos x)} + \frac{(\eta')^2}{2B} \right\} dx dz,$$

which ensures the linear stability of the main flow $\bar{\psi} = \bar{\eta} = 0$, $\bar{\xi} = -\sin x$ at $Q > 1$ ($\text{Ro} < 1$) irrespective of the stratification parameter $B > 0$ value.

We first limit ourselves to studying the simpler case of 4π -periodic solutions of (38) - (40), which have a period in x twice that of the external force (cf. (12))

$$\begin{pmatrix} \psi' \\ \xi' \\ \eta' \end{pmatrix} = \sum_{n=-\infty}^{\infty} \begin{pmatrix} \psi_{n+1/2} \\ \xi_{n+1/2} \\ \eta_{n+1/2} \end{pmatrix} \exp \{ \lambda t + iqz + i(n+1/2)x \}. \quad (41)$$

Substituting (41) in (38) - (40), we arrive at the relations

$$\begin{aligned} \psi_{n-1/2} - c_{n+1/2} \psi_{n+1/2} + \psi_{n+3/2} &= 0, \\ c_{n+1/2} &= 2Q + \frac{2}{q^2} \frac{B(n+1/2)^2 \left[\lambda + \frac{(n+1/2)^2 + q^2}{\text{Re}} \right]}{\lambda + \frac{(n+1/2)^2 + q^2}{\text{Re Pr}}} + \frac{2}{q^2} \left[\lambda + \frac{(n+1/2)^2 + q^2}{\text{Re}} \right]^2 \left[(n+1/2)^2 + q^2 \right], \end{aligned}$$

with the solvability condition (16). The linear stability problem reduces to numerically solving the characteristic equations of successive order corresponding to the convergents to an infinite continued fraction in (16). The coefficients $c_{n+1/2}$ increase very rapidly with the growth of n if the complex values of λ are limited in modulus. We analyze stability based on the lowest-order convergent (cf. section 5). Taking such a convergent, we have $c_{1/2} = \pm 1$ from (16). We will perform the analysis of the resulting complete cubic characteristic equation, see (42) below, in the framework of more general analysis of the nonlinear system of equations (35) - (37), limiting ourselves to considering 4π -periodic perturbations. We represent solution (35) - (37) in the form

$$\begin{aligned} \zeta &= \bar{\zeta}(x, t) + \zeta'(x, z, t), \\ \zeta'(x, z, t) &= \hat{\zeta}(x, t) \exp(iqz) + \hat{\zeta}^*(x, t) \exp(-iqz), \end{aligned}$$

where $\zeta = (\psi, \xi, \eta)$ and an asterisk above dependent variables denotes a complex conjugation.

We arrive at a closed system of equations

$$\begin{aligned} \frac{\partial \bar{\xi}}{\partial t} + iq \frac{\partial}{\partial x} (-\hat{\psi} \hat{\xi}^* + \hat{\psi}^* \hat{\xi}) &= \frac{1}{\text{Re}} \frac{\partial^2 \bar{\xi}}{\partial x^2} - \frac{1}{\text{Re}} \sin x, \\ \left(\frac{\partial}{\partial t} - \frac{1}{\text{Re}} \nabla^2 \right) \nabla^2 \hat{\psi} &= -iq \hat{\xi} + \frac{\partial}{\partial x} \hat{\eta}, \\ \left(\frac{\partial}{\partial t} - \frac{1}{\text{Re}} \nabla^2 \right) \hat{\xi} &= iq \left(Q + \frac{\partial \bar{\xi}}{\partial x} \right) \hat{\psi}, \\ \left(\frac{\partial}{\partial t} - \frac{1}{\text{Re Pr}} \nabla^2 \right) \hat{\eta} &= -B \frac{\partial}{\partial x} \hat{\psi}, \end{aligned}$$

plus the corresponding equations for, $\hat{\psi}^*$, $\hat{\xi}^*$ and $\hat{\eta}^*$. Again, as in the case of a rotating fluid of uniform density, we restrict ourselves in (41) to the minimum number of components, $\psi_{\pm 1/2}$, $\xi_{\pm 1/2}$ and $\eta_{\pm 1/2}$. Without loss of generality, $\psi_{\pm 1/2}$ can be considered real and $\xi_{\pm 1/2}$, $\eta_{\pm 1/2}$ purely imaginary quantities. Introducing the notation $a_1 = \psi_{1/2}$, $a_2 = \psi_{-1/2}$, $ib_1 = \xi_{1/2}$, $ib_2 = \xi_{-1/2}$, $ic_1 = \eta_{1/2}$, $ic_2 = \eta_{-1/2}$ and assuming $\bar{\xi} = -b_0(t) \sin x$, we arrive at a system of seven real equations. One can further narrow down the class of solutions by choosing one of the possibilities: (i) $a_1 = a_2$, $b_1 = b_2$, $c_1 = -c_2$ and (ii) $a_1 = -a_2$, $b_1 = -b_2$, $c_1 = c_2$. Considering the first possibility without loss of generality, we arrive at the system ($q_1^2 = q^2 + 1/4$)

$$\begin{aligned} \left(\frac{d}{dt} + \frac{q_1^2}{\text{Re}} \right) q_1^2 a_1 &= -q b_1 + \frac{1}{2} c_1, & \left(\frac{d}{dt} + \frac{q_1^2}{\text{Re}} \right) b_1 &= q \left(Q - \frac{1}{2} b_0 \right) a_1, \\ \left(\frac{d}{dt} + \frac{q_1^2}{\text{Re Pr}} \right) c_1 &= -\frac{B}{2} a_1, & \left(\frac{d}{dt} + \frac{1}{\text{Re}} \right) b_0 - 4q a_1 b_1 &= \frac{1}{\text{Re}}. \end{aligned}$$

This system of equations has a primary stationary solution $a_1 = b_1 = c_1 = 0$, $b_0 = 1$. Let us investigate its linear stability, assuming that the deviations from the stationary solution depend on time as $\exp(\lambda t)$. We arrive at the characteristic cubic equation (corresponding to $c_{1/2} = 1$, see above)

$$\lambda^3 + p_1 \lambda^2 + p_2 \lambda + p_3 = 0, \quad (42)$$

$$\begin{aligned} p_1 &= \frac{q_1^2}{\text{Re}} (2 + \text{Pr}^{-1}), & p_2 &= \frac{q_1^4}{\text{Re}^2} (2 \text{Pr}^{-1} + 1) + \frac{B}{4q_1^2} + \frac{q^2}{2q_1^2} (2Q - 1), \\ p_3 &= \frac{q_1^6}{\text{Re}^3 \text{Pr}} + \frac{B}{4\text{Re}} + \frac{q^2}{2\text{Re Pr}} (2Q - 1). \end{aligned}$$

Necessary and sufficient conditions for stability (when all three roots of the equation (42) have negative real parts) are in the fulfillment of the inequalities $p_1 > 0$, $p_2 > 0$, $p_3 > 0$ and $p_1 p_2 - p_3 > 0$. It can be shown that the condition $2Q - 1 > 0$ guarantees the fulfillment of all the indicated inequalities, i.e. is a sufficient condition for stability. Thus, failure to meet this condition, when on the contrary $1 - 2Q > 0$, is a necessary condition for instability. Let us now turn to the sufficient conditions for instability. First, the inversion of the condition $p_3 > 0$ corresponds to the neutral stability curve with the minimum critical Reynold number squared

$$\text{Re}_{\min 1}^2 = \frac{27/32}{(1 - 2Q)} (1 + 4q_{\min 1}^2)^2, \quad q_{\min 1}^2 = \frac{B \text{Pr}}{2(1 - 2Q)} \quad (43)$$

Second, the inversion of the condition $p_2 > 0$ corresponds to

$$\text{Re}_{\min 2}^2 = \frac{27/32}{(1 - 2Q)} (1 + 2 \text{Pr}^{-1}) (1 + 4q_{\min 2}^2)^2, \quad q_{\min 2}^2 = \frac{B}{2(1 - 2Q)} \quad (44)$$

Third, the inversion of the condition $p_1 p_2 - p_3 > 0$, corresponds to

$$\text{Re}_{\min 3}^2 = \frac{27/32}{(1 - 2Q)} (1 + \text{Pr}^{-1})^2 (1 + 4q_{\min 3}^2)^2, \quad q_{\min 3}^2 = \frac{B(1 + \text{Pr}^{-1})}{4(1 - 2Q)} \quad (45)$$

Comparison of (43), (44) and (45) leads to the following conclusions. When $Pr < 1$, then $q_{\min 1}^2 < q_{\min 2}^2 < q_{\min 3}^2$, $Re_{\min 1}^2 < Re_{\min 2}^2 < Re_{\min 3}^2$ and therefore instability is primarily determined by violation of the condition $p_3 > 0$, i.e. the principle of exchange of stabilities works. In a particular case $Pr = 1$, we have that $q_{\min 1}^2 = q_{\min 2}^2 = q_{\min 3}^2$ and $Re_{\min 1}^2 = (1/3)Re_{\min 2}^2 = (1/4)Re_{\min 3}^2$, i.e. the principle of exchange of stabilities continues to operate. The situation becomes more complicated when $Pr > 1$. Here, for $Pr \gg 1$ and B -values greater than a certain critical value depending on Pr , the inequalities $Re_{\min 1}^2 > Re_{\min 2}^2, Re_{\min 3}^2$ can be satisfied, i.e. instability occurs when either a condition $p_2 > 0$ or a weaker condition $p_1 p_2 - p_3 > 0$ is violated, and the principle of exchange of stabilities no longer works (figure 5).

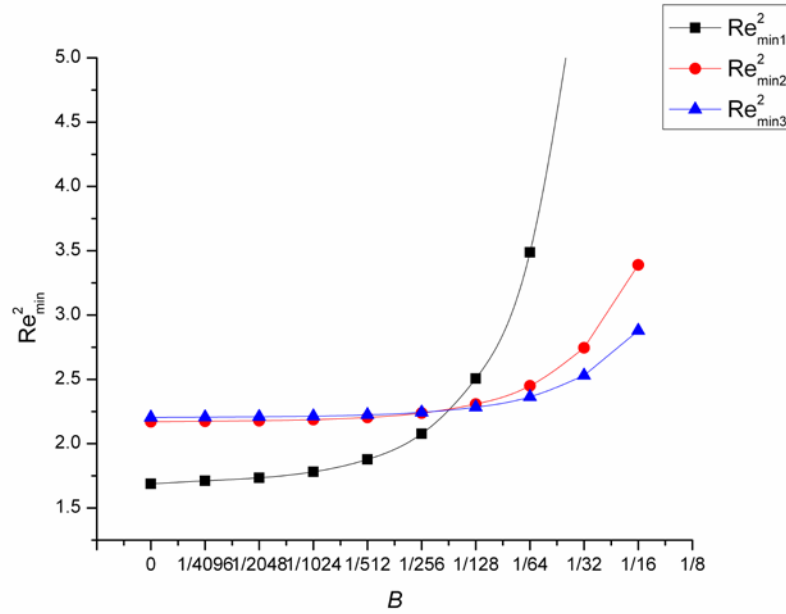


Figure 5. Dependence of the square of the minimum critical Reynolds number $Re_{\min}^2$, determined, respectively, by expressions (43) (black line), (44) (red line) and (45) (blue line), on the stratification parameter B in the case when $Q = 1/4$ and the Prandtl number $Pr = 7$ (for water).

The problem also has two secondary stationary solutions defined by the relations

$$\bar{b}_0 = 2Q + \frac{2q_1^6}{Re^2 q^2} + \frac{BPr}{2q^2} \equiv 2Q + A, \quad (46a)$$

$$\bar{a}_1^2 = \frac{1}{4q_1^4} \frac{1 - 2Q - \frac{2q_1^6}{Re^2 q^2} - \frac{BPr}{2q^2}}{1 + BRe^2 Pr/q_1^6} \equiv \frac{q_1^2}{2q^2 Re^2 A} (1 - 2Q - A), \quad (46b)$$

$$\bar{b}_1 = -\left(\frac{q_1^4}{Re q} + \frac{BRePr}{4q_1^2} \right) \bar{a}_1 \equiv -\frac{q Re A}{2q_1^2} \bar{a}_1, \quad \bar{c}_1 = -\frac{BRePr}{2q_1^2} \bar{a}_1. \quad (46c)$$

A short notation is introduced here ($q_1^2 = q^2 + 1/4$)

$$A = \frac{2q_1^6}{Re^2 q^2} + \frac{BPr}{2q^2}. \quad (47)$$

Using (47), the coefficient p_3 of equation (42) can be written in the form $p_3 = \frac{q^2}{2\text{RePr}}(2Q + A - 1)$ and the instability condition $p_3 < 0$ (within the framework of the validity of the principle of exchange of stabilities) is equivalently written in the form $1 - 2Q - A > 0$, which ensures the existence of stationary secondary solutions, namely $\bar{a}_1^2 > 0$. Note that in this case, by virtue of (43)

$$1 - 2Q - A \equiv \frac{2q_1^6}{q^2} \left(\frac{1}{\text{Re}_{\min}^2} - \frac{1}{\text{Re}^2} \right) > 0.$$

Let us turn to the question of the linear stability of secondary stationary solutions (46) at $\text{Pr} \leq 1$. It can be shown that under the assumption that the deviations from the stationary solution depend on time as $\exp(\lambda t)$, we obtain the characteristic equation of the fourth degree

$$\lambda^4 + p_1\lambda^3 + p_2\lambda^2 + p_3\lambda + p_4 = 0, \quad (48)$$

$$p_1 = \frac{q_1^2}{\text{Re}} \left(2 + \frac{1}{\text{Pr}} \right) + \frac{1}{\text{Re}},$$

$$p_2 = \frac{q_1^4}{\text{Re}^2} \left(1 + \frac{1}{\text{Pr}} \right) + \frac{q_1^2}{\text{Re}^2} \left(2 + \frac{1}{\text{Pr}} \right) + \frac{q^2 A}{2q_1^2} \left(\frac{1}{\text{Pr}} - 1 \right) + \frac{q_1^2}{\text{Re}^2 A} (1 - 2Q - A),$$

$$p_3 = \frac{q_1^4}{\text{Re}^3} \left(1 + \frac{1}{\text{Pr}} \right) + \frac{q^2 A}{2q_1^2 \text{Re}} \left(\frac{1}{\text{Pr}} - 1 \right) + \frac{q_1^4}{\text{Re}^3 A} \left(1 + \frac{1}{\text{Pr}} \right) (1 - 2Q - A) + \frac{q^2}{2q_1^2 \text{Re}} (1 - 2Q - A),$$

$$p_4 = \frac{q^2}{\text{Re}^2 \text{Pr}} (1 - 2Q - A).$$

For $\text{Pr} \leq 1$ all the coefficients of the equation are obviously positive. According to the Liénard–Shipard criterion, instability in this case is only possible for $p_3(p_1 p_2 - p_3) - p_1^2 p_4 < 0$. Due to the complexity of the resulting calculations and in order to focus on the qualitative effect of stable density stratification, we first consider the simplest case $\text{Pr} = 1$. This satisfactorily describes the situation in the atmosphere and ocean, when small-scale turbulent viscosity and heat conduction are considered as dissipative factors.

We proceed as follows. We use $q^2 = (1/8) + (3B/4)(1 - 2Q)^{-1}$, which at $\text{Pr} = 1$ corresponds to the minimum value of the Reynolds number that is critical for the instability of the primary flow, given by (43). We estimate for this q^2 -value the quantity A (at $\text{Pr} = 1$), see (47), and substitute it into (46). After that, we calculate the criterion $p_3(p_1 p_2 - p_3) - p_1^2 p_4 < 0$ that will be given by the functional dependence of Re^2 on Q and B . Let us simplify the problem by limiting ourselves to two values $Q = 0, 1/4$. The calculation results are shown in figure 6. For $\text{Pr} = 1$, the curve $\text{Re}^2 = \text{Re}^2(B)$ has two infinite asymptotes, $\text{Re}^2 \rightarrow \infty$, for $B \rightarrow 0$ and $B \rightarrow \infty$. At $Q = 0$, the minimum $\text{Re}_{\min}^2 \approx 12.33$ is achieved at $B = B_0 \approx 0.184$; at $Q = 1/4$, the minimum $\text{Re}_{\min}^2 \approx 24.67$ is attained at $B_0 \approx 0.092$. For these minimum values, due to the presence of a factor $(1 - 2Q)$ in the formulas, approximately a simple scaling is valid. For

$Q = 1/4$, the value $Re_{\min}^2$ is two times greater than for $Q = 0$, and the value B_0 at which this minimum is reached, respectively, is two times less.

Now let us move on to a more complicated case when $Pr < 1$. We assume that $Q = 1/4$ and will consider three possibilities: (a) $Pr = 3/4$, (b) $Pr = 1/2$ and (c) $Pr = 1/4$. Case (a) corresponds closely to the relationship between molecular viscosity and thermal diffusivity of atmospheric air. Figure 7 shows a typical diagram of the stability of secondary regimes in this case, and table 1 shows the corresponding numerical values of the parameters.

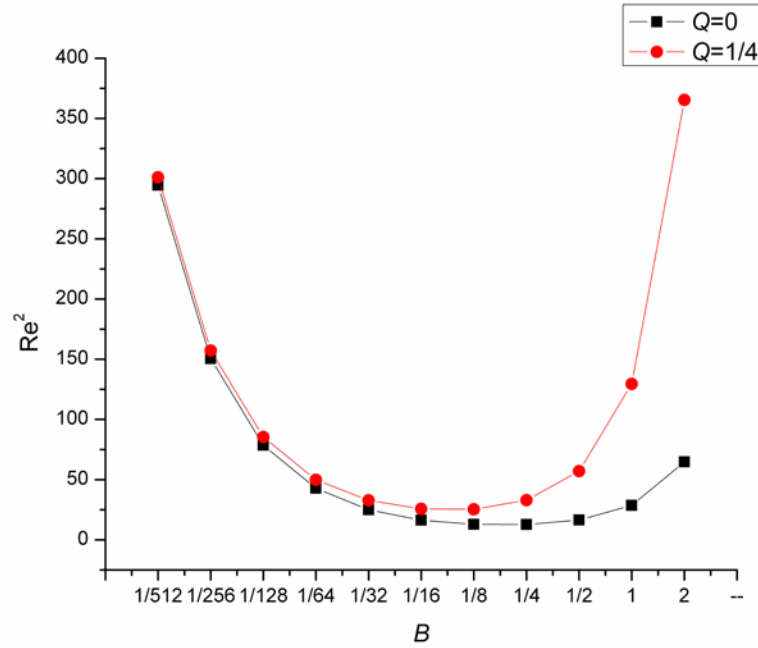


Figure 6. Dependence of the square of the minimum critical Reynolds number Re^2 on the stratification parameter B at $Q = 0$ (black line) and $Q = 1/4$ (red line) when the Prandtl number $Pr = 1$.

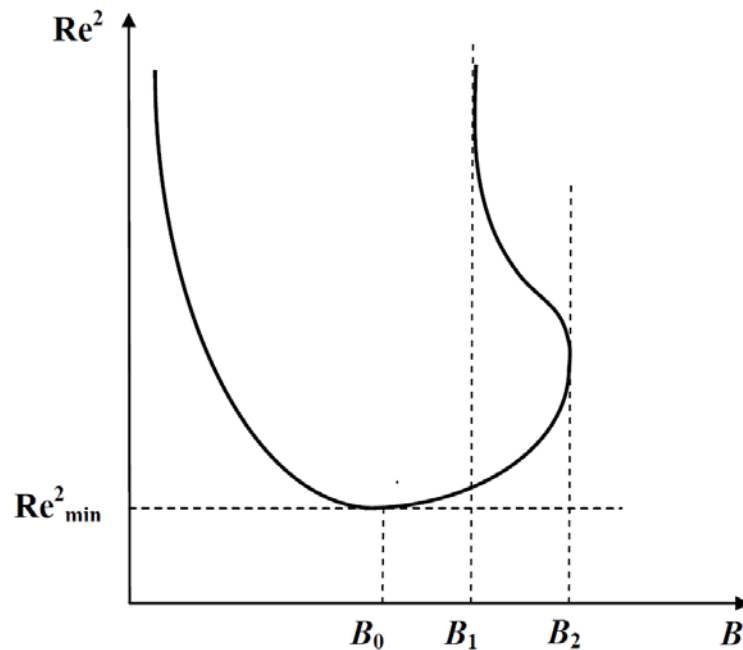


Figure 7. Schematic view of the dependence of the square of the minimum critical Reynolds number Re^2 on the stratification parameter B in the case when the Prandtl number is less than one. Parameters B_0 , B_1 and B_2 are explained in the text and their numerical values are given in table. 2.

In the case of $Pr < 1$, the asymptote $Re^2 \rightarrow \infty$ at $B \rightarrow 0$ continues to exist, but the second asymptote $Re^2 \rightarrow \infty$ is realized at a finite value $B = B_1$, and the region of existence of the neutral stability curve extends to the value $B = B_2 > B_1$. The minimum value of the squared Reynolds number $Re_{\min}^2$ is attained at $B = B_0 < B_1$.

Table 2. Numerical values of the characteristic parameters of the neutral stability curve in figure 7.

Pr	3/4	1/2	1/4
$Re_{\min}^2$	29.489	38.845	65.589
B_0	0.0815	0.0715	0.0636
B_1	0.3996	0.1667	0.0739
B_2	0.4667	0.1767	0.1100

The most difficult case to analyze is when $Pr > 1$. Here are two complications. First, the coefficients p_2 and p_3 of equations (48) can be negative, and the instability conditions do not reduce to inequality $p_3(p_1p_2 - p_3) - p_1^2p_4 < 0$. Second, which is more important, and as noted above, for $Pr \gg 1$ and the stratification parameter B values greater than the critical value B_{crit} , which depends inversely on the Prandtl number, the instability of the primary solution is set when the condition $p_1p_2 - p_3 > 0$ is violated (see figure 5) and the principle of exchange of stabilities does not work. Similarly, both monotonic instability ($Re\lambda = Im\lambda = 0$ on the neutral stability curve) and growing oscillations (“overstability”; $Re\lambda = 0$ but $Im\lambda \neq 0$ on the neutral stability curve) are involved in the problem of the stability of a baroclinic circular vortex for Prandtl numbers that differ from unity (McIntyre 1970).

Therefore, the secondary stationary solution (46), which arises after the loss of stability of the primary regime under the validity of the principle of exchange of stabilities, for $Pr = 7$ exists only for $B < B_{\text{crit}} = 7/1244 \approx 1/178$ (cf. figure 5) and we restrict ourselves to this range when we study the stability of the secondary solution (46). In this case, the condition $p_4 > 0$ is obviously fulfilled and, in accordance with the Liénard–Shipard criterion, for the instability of solution (46), it is necessary and sufficient to satisfy one of the inequalities, $p_2 < 0$, $p_3 < 0$ and/or $p_3(p_1p_2 - p_3) - p_1^2p_4 < 0$. Numerical calculations show that the only essential condition for instability at $0 < B < B_{\text{crit}}$ is the condition $p_2 < 0$ (figure 8). In the limit $B \ll 1$, but $Re^2 \gg B^{-1}$, when in this case $q^2 \cong 1/8$ and $q_1^2 \cong 3/8$, the simple asymptotic is valid for the coefficient p_2 in (48),

$$p_2 \cong -\frac{q^2 B Pr}{4q_1^2} \left(1 - \frac{1}{Pr}\right) + \frac{2q_1^2 q^2 (1 - 2Q)}{Re^2 B Pr}.$$

Therefore, the instability condition for solution (46) is written in the asymptotic form

$$\text{Re}^2 > \frac{8q_l^4 q^2 (1-2Q)}{\text{Pr}(\text{Pr}-1)} \frac{1}{B^2} \propto \frac{1}{B^2},$$

generally consistent with the calculations shown in figure 8 at very small B -values.

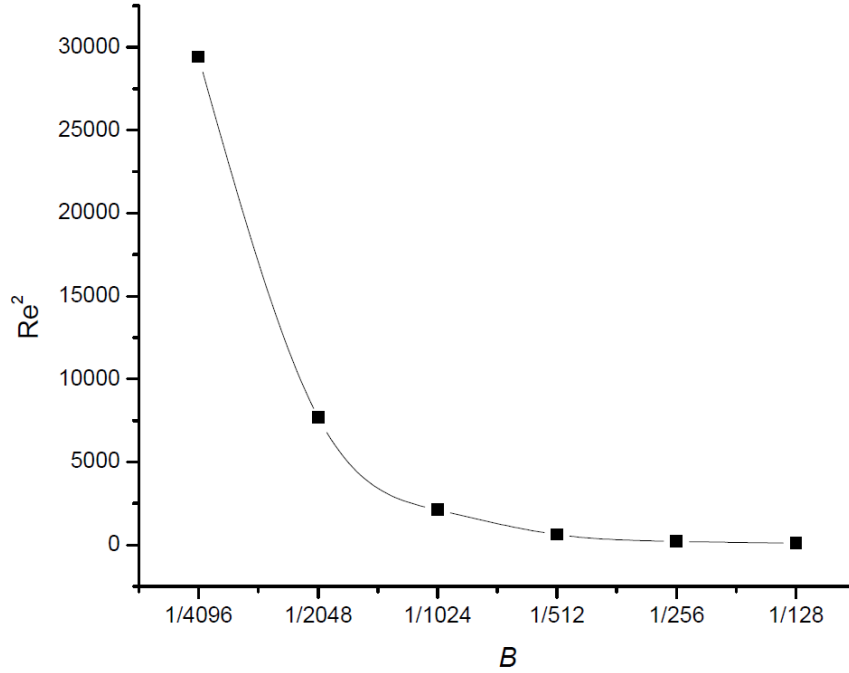


Figure 8. Dependence of the square of the minimum critical Reynolds number Re^2 on the stratification parameter B at $Q = 1/4$ in the case when the Prandtl number $\text{Pr} = 7$.

At the end of this section we touch upon the case of 2π -periodic disturbances when the solution of (38) - (40) reads

$$\begin{pmatrix} \psi' \\ \xi' \\ \eta' \end{pmatrix} = \sum_{n=-\infty}^{\infty} \begin{pmatrix} \psi_n \\ \xi_n \\ \eta_n \end{pmatrix} \exp\{\lambda t + i q z + i n x\}. \quad (49)$$

Substituting (49) in (38) - (40), we arrive at the relations

$$\begin{aligned} \psi_{n-1} - c_n \psi_n + \psi_{n+1} &= 0, \\ c_{n+1/2} &= 2Q + \frac{2}{q^2} \frac{Bn^2 \left(\lambda + \frac{n^2 + q^2}{\text{Re}} \right)}{\lambda + \frac{n^2 + q^2}{\text{Re Pr}}} + \frac{2}{q^2} \left(\lambda + \frac{n^2 + q^2}{\text{Re}} \right)^2 (n^2 + q^2), \end{aligned}$$

with the solvability condition (15). Again, the coefficients c_n increase very rapidly with the growth of n if the complex values of λ are limited in modulus. This provides a fast convergence of convergents to the infinite continued fraction in (15) and the ability to analyze stability based on the lowest-order convergent. Taking such a convergent, we have $c_0 c_1 = 2$ from (15). We will perform the analysis of the resulting characteristic equation (see below) in the framework of more general analysis of the nonlinear system of equations (35) - (37), now in the case of 2π -periodic perturbations. We restrict ourselves to the minimum number of components

$\psi_0, \psi_{\pm 1}, \xi_0, \xi_{\pm 1}, \eta_0$ and $\eta_{\pm 1}$. Again, the stream function components are considered real, while the Coriolis acceleration components and the buoyancy components are considered purely imaginary. Introducing the notation $a_1 = \psi_0$, $a_2 = \psi_1 + \psi_{-1}$, $ib_1 = \xi_0$, $ib_2 = \xi_1 + \xi_{-1}$, $ic_1 = \eta_0$, $ic_2 = \xi_1 - \xi_{-1}$, assuming $\bar{\xi} = -b_0(t)\sin x$ and proceeding as in the case of 4π -periodic perturbations, we arrive to a system of seven real equations

$$\begin{aligned} \left(\frac{d}{dt} + \frac{q^2}{\text{Re}}\right)qa_1 &= -b_1, & \left(\frac{d}{dt} + \frac{q^2+1}{\text{Re}}\right)(q^2+1)a_2 &= -qb_2 + c_2, \\ \left(\frac{d}{dt} + \frac{q^2}{\text{Re}}\right)b_1 &= qQa_1 - \frac{1}{2}qb_0a_2, & \left(\frac{d}{dt} + \frac{q^2+1}{\text{Re}}\right)b_2 &= qQa_2 - qb_0a_1, \\ \left(\frac{d}{dt} + \frac{q^2}{\text{Re Pr}}\right)c_1 &= 0, & \left(\frac{d}{dt} + \frac{q^2+1}{\text{Re Pr}}\right)c_2 &= -Ba_2, \\ \left(\frac{d}{dt} + \frac{1}{\text{Re}}\right)b_0 - 2q(a_1b_2 + a_2b_1) &= \frac{1}{\text{Re}}. \end{aligned}$$

The equation for c_1 splits off and we have a closed system of six equations whose analysis is more complicated than for the case of 4π -periodic disturbances. Below, we limit ourselves to studying the asymptotic behavior of the solutions of these equations in the limit of very large but finite Reynolds numbers Re (cf. Libin et al. 1987). We start with a primary stationary solution $b_0 = 1$, $a_1 = a_2 = b_1 = b_2 = c_2 = 0$, the linear stability of which is described by a fifth-order characteristic equation that has a zero root and the other four roots satisfy a biquadratic equation

$$\lambda^4(q^2+1) + \lambda^2[B + Q(2q^2+1)] + BQ + q^2(Q^2 - 1/2) = 0.$$

The question of stability reduces to the question of the sign of the constant (absolute) term in this equation. Stable density stratification imposes a stabilizing action and provides a lower cut-off for growing unstable perturbations (cf. Kloosterziel and Carnevale (2008) and references therein), consistent with an inequality $q^2 > 2BQ/(1-2Q^2)$ in our case. Very similar conclusions are obtained in the limit $\text{Re} \rightarrow \infty$ for 4π -periodic disturbances where an inequality $q^2 > B/2$ for unstable growing disturbances holds, as can be inferred from equation (42).

There are two secondary stationary solutions $a_1 = O(\text{Re}^{-1})$, $a_2 = O(\text{Re}^{-1})$, $b_1 = O(\text{Re}^{-2})$, $b_2 = O(1)$, $c_2 = O(1)$ determined from the set of equations taken in the leading order of approximation in Re^{-1}

$$\begin{aligned} -qb_2 + c_2 &= 0, & qQa_1 - \frac{1}{2}qb_0a_2 &= 0, \\ \frac{q^2+1}{\text{Re}}b_2 &= qQa_2 - qb_0a_1, & \frac{q^2+1}{\text{Re Pr}}c_2 &= -Ba_2. \end{aligned}$$

From these equations it follows

$$\frac{2q^2(q^2+1)}{\text{Pr } B}b_2^2 = 1 - b_0, \quad b_0^2 = 2Q^2 + \frac{2Q \text{Pr } B}{q^2}.$$

An analysis of the linear instability of these solutions keeping in the equations only terms without explicit dependence on Re^{-1} results in the biquadratic equation (a duplicate root $\lambda = 0$ is disregarded)

$$(q^2 + 1)\lambda^4 + [B + (2q^2 + 1)Q]\lambda^2 + BQ(1 - \text{Pr}) = 0, \quad (50)$$

and a robust result is obtained that the stable density stratification destabilizes the secondary flow and this occurs predominantly for $\text{Pr} > 1$. Note that for 4π -periodic disturbances, a similar analysis leads to the characteristic equation $\lambda^2 + \frac{1}{4}B(1 - \text{Pr}) = 0$. The main conclusion about the destabilization of the secondary solutions in the presence of stable density stratification remains qualitatively the same for the 2π - and 4π -periodic disturbances, which justifies restricting ourselves in most of section 7 to the simpler case of 4π -periodic disturbances. For $\text{Pr} \leq 1$ the destabilization mechanism is more subtle and was discussed above in detail for the last case.

8. Concluding remarks

The linear and nonlinear inertial stability of the Kolmogorov flow in a rotating viscous fluid of uniform density is investigated with respect to zonal symmetric disturbances using the method of continued fractions and low-order Galerkin approximations. A necessary condition for instability is that the Rossby number exceeds the critical value of the order of unity, and the sufficient condition for instability is formulated in the form of the Reynolds criterion. The existence of stable secondary stationary flow structures in the problem is shown, developing against the background of loss of stability of the main flow and in the form of rolls oriented along it (like cloud streets in the atmosphere).

Stable density stratification is taken into account in the problem, when the direction of the gravity force coincides with the direction of rotation of the fluid, and the instability of the Kolmogorov flow with respect to zonal symmetric disturbances is considered. In this more complicated problem, the necessary condition for the instability of the main flow remains the same, that is, the general rotation of the fluid is a stabilizing factor, but the critical Reynolds number for the instability depends on two additional dimensionless parameters that appear in the problem: the stratification parameter B and the Prandtl number Pr .

The most studied case in the paper is that of $\text{Pr} \leq 1$, when the existence of a secondary stationary regime is possible in our low-order model setup, which, however, can now be unstable – in contrast to the case of a fluid uniform in density – and, paradoxically, stable density stratification is a destabilizing factor. It is an example of a case where stratification makes the flow more unstable, similar to strato-rotational instability (Wang and Balmforth 2018, and references therein) and other but related problems (see, the newest paper by Le Gal et al 2021). In this paper, the reason for the instability might be highlighted a bit more. For rotation only it is inertial instability. Adding stable stratification augments the number of degrees of freedom involved in the instability problem and increases the probability for merging of the complex roots of the resulting characteristic equation which leads to enhancement of instability.

We note that the secondary stationary solutions in question, apparently, reliably exist only for $\text{Pr} \leq 1$, but for $\text{Pr} > 1$ one should generally speak about more complex secondary regimes, possibly oscillatory, albeit a clue to the existence of secondary stationary solutions for large but finite Reynolds number was provided at the end of section 7.

The results obtained qualitatively agree with those obtained for the inertial instability of a uniform barotropic shear flow of a diffusive stratified rotating Boussinesq fluid confined between two parallel vertical walls with prescribed free-slip conditions on them, although a case of unbounded flows was also discussed (Kloosterziel and Carnevale 2008). From the perspective of fluid dynamics per se, the merit of investigating the inertial instability of the Kolmogorov flow, as was done in this paper, is that one is free from the need to consider realistic no-slip boundary conditions for viscous fluids, which significantly complicates the analysis, but instead much simpler conditions of the periodicity of the flow in space can be posed (cf. Arnol'd and Meshalkin 1960; Meshalkin and Sinai, 1961; Obukhov 1983). From the perspective of the applications of the results obtained to real atmospheres and oceans, we first mention the potential applicability of our results to cloud streets frequently observed in the atmosphere, at a height of the order of one kilometer, when cold Arctic air flows over open warm sea; cloud streets have also been observed at much higher levels (cirrus streets), and often associated with the jet stream (cf. Weber 1980 and references therein). Second, we mention that such spatially periodic flows can be considered a geophysical prototype of a recently discovered multi-jet system in the Antarctic Ocean (Maximenko et al 2005; Sokolov and Rintoul (2007)). Last but not least, zonal flows with periodic or quasi-periodic structure are observed in the atmospheres of giant planets, particularly Jupiter (Vasavada and Showman 2005) and an important role can be expected from inertial (symmetric) instability in the atmospheres of such rapidly rotating planets as Jupiter, Saturn, Uranus and Neptune (O'Neill and Kaspi 2016).

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