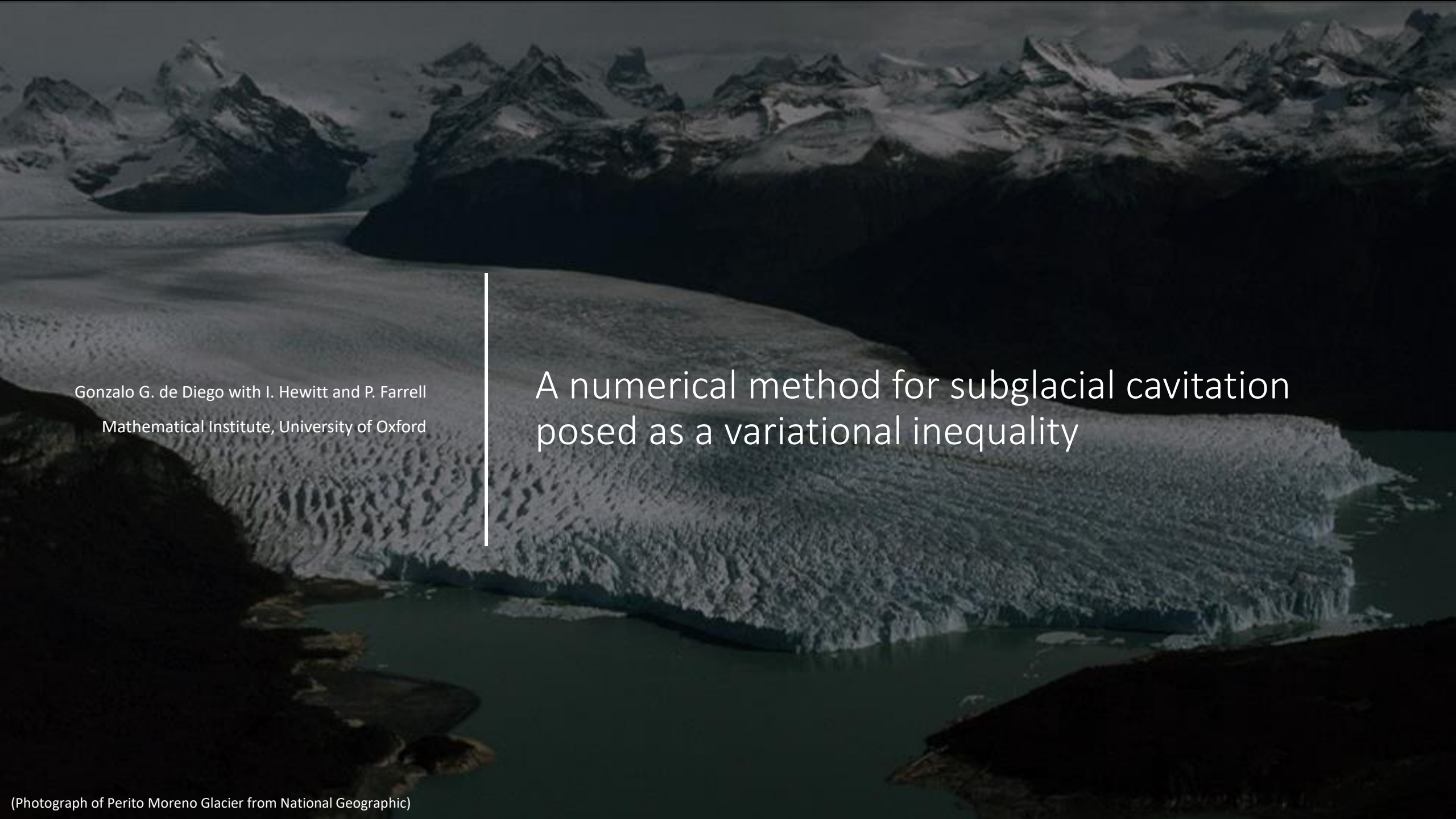




Gonzalo G. de Diego with I. Hewitt and P. Farrell
Mathematical Institute, University of Oxford

A numerical method for subglacial cavitation posed as a variational inequality

Subglacial cavitation is a phenomenon that occurs at the base of a glacier and consists in the detachment of the ice from the bed due to high water pressures. It is a fundamental mechanism in glacial sliding over hard beds.

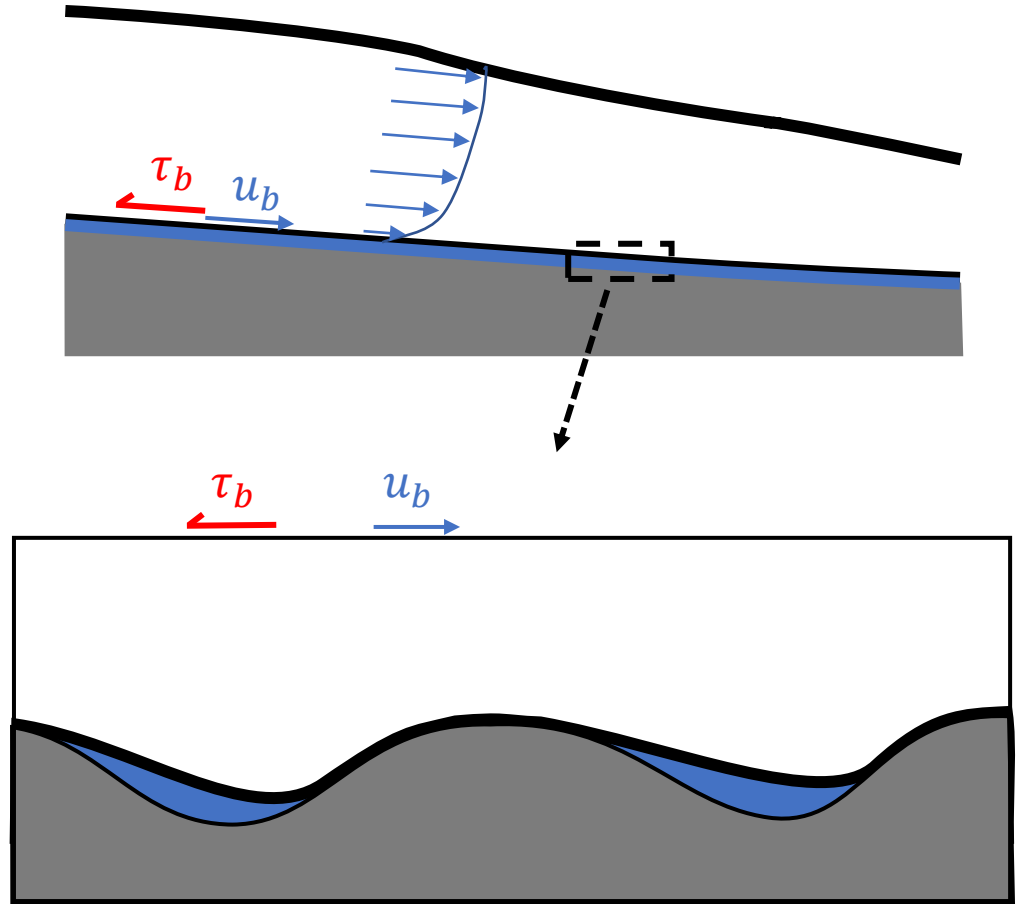
By modelling subglacial cavitation, we can calculate the **sliding law**:

$$\frac{\tau_b}{N} = f\left(\frac{u_b}{N}\right)$$

Here, N is the **effective pressure**

$$N = p_i - p_w,$$

the difference between the overburden ice pressure p_i and water pressure p_w .

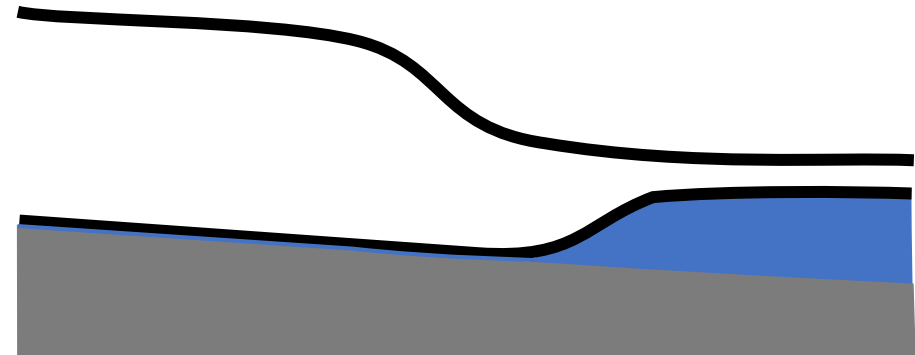
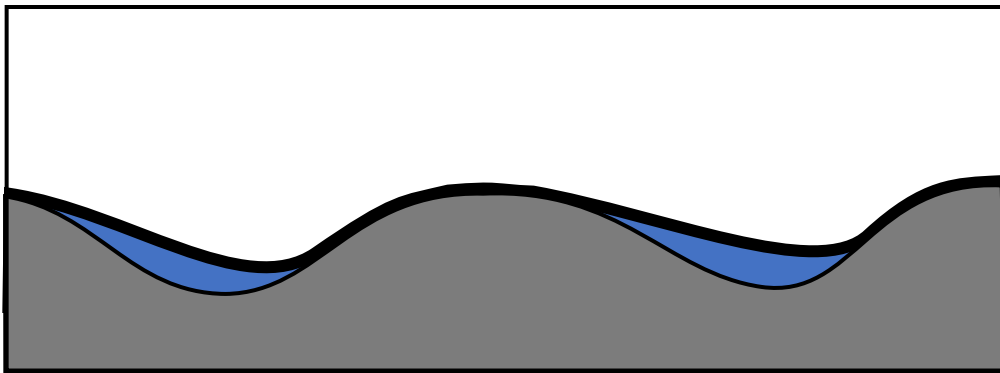


In the following slides,

1. We first show how a variational inequality arises naturally when modelling subglacial cavitation,
2. We then present some numerical results,
3. Finally, we briefly explain how the structure of this variational inequality can be exploited to build accurate and robust finite element solvers for computing the evolution of a subglacial cavity.

Robust solvers are important because

1. Subglacial cavitation is a fundamental mechanism in glacial sliding,
2. The evolution of the grounding line of a marine ice sheet presents the same mathematical structure: they are both dynamic contact problems.



A model for subglacial cavitation:

Stokes equation:

$$-\nabla \cdot (\mu \varepsilon(\vec{u})) + \nabla p = 0,$$

$$\nabla \cdot \vec{u} = 0$$

Contact boundary conditions
on **attached region** Γ_a :

$$\vec{u} \cdot \vec{n} \leq 0$$

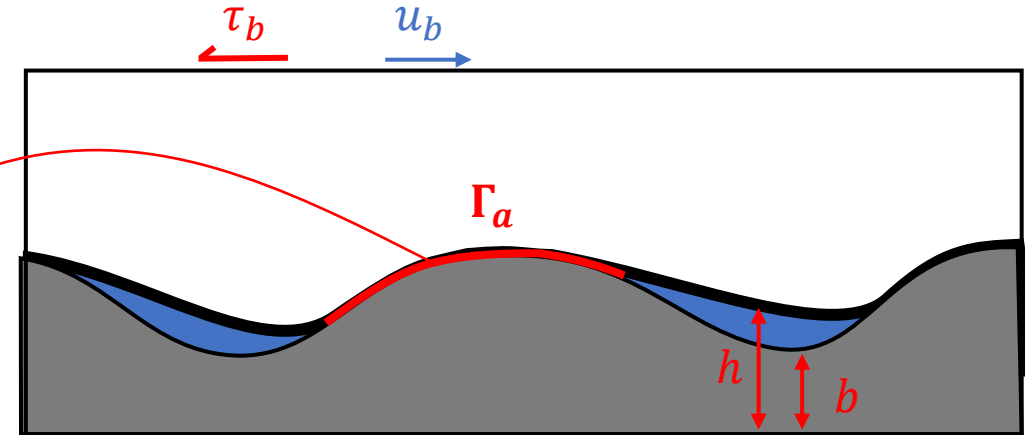
$$\sigma_{nn} + p_w \leq 0$$

$$(\sigma_{nn} + p_w)(\vec{u} \cdot \vec{n}) = 0$$

Free boundary equation:

$$\frac{\partial h}{\partial t} + \vec{u} \cdot \left(\frac{\partial h}{\partial x}, -1 \right) = 0,$$

$$h \geq b$$



Sketch of an Algorithm

At each timestep

1. Find attached and detached regions
2. Solve the Stokes equations with contact boundary conditions
3. Given the newly computed velocity field, compute the cavity height h with the free boundary equation

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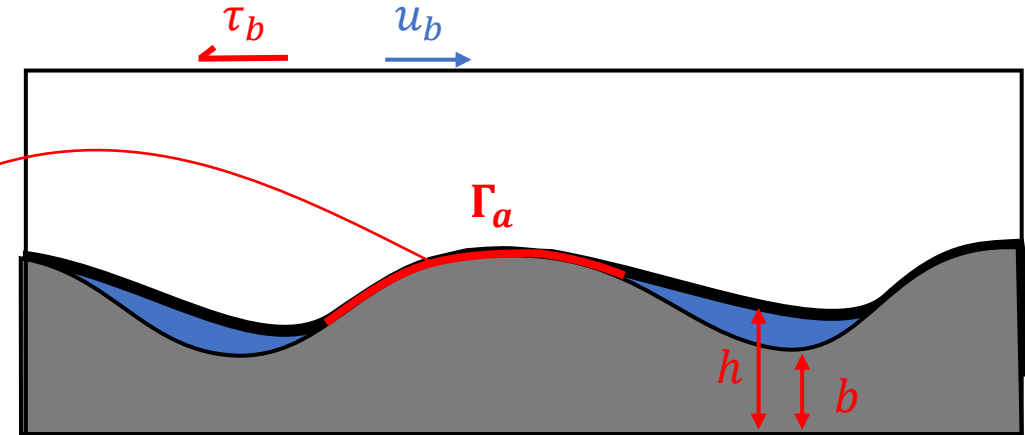
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$$\vec{u} \cdot \vec{n} \leq 0 \quad \leftarrow \text{Kinematic condition}$$

$$\sigma_{nn} + p_w \leq 0 \quad \leftarrow \text{Dynamic condition}$$

$$(\sigma_{nn} + p_w)(\vec{u} \cdot \vec{n}) = 0 \quad \leftarrow \text{Complementarity condition}$$



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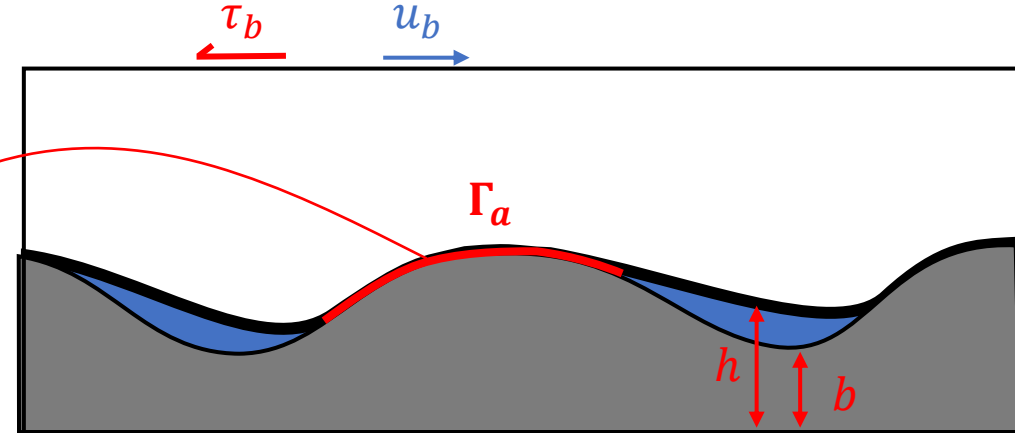
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This system of equations can be written as a variational inequality

There exist many different approaches to solving variational inequalities.

- Different approaches exploit different properties of this mathematical structure.
- Of these, we found that enforcing the contact boundary conditions with a Lagrange multiplier works very well in conjunction with the free boundary equation.

In the final slides, we write the variational inequality with and without Lagrange multipliers.

We next show two numerical results obtained with a finite element solver built with Firedrake.

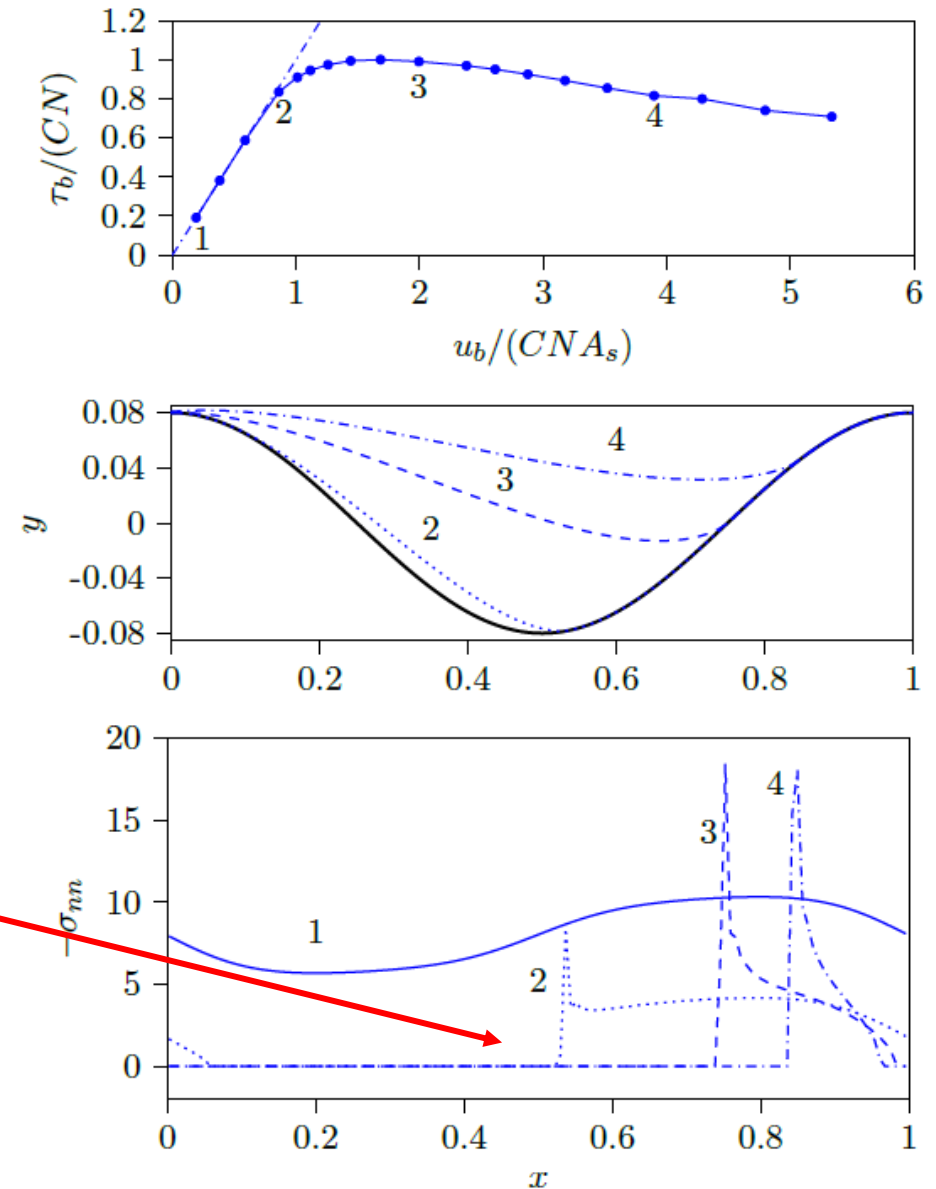
The steady sliding law over a sinusoidal bed

We compute the steady cavities for different sliding speeds u_b and effective pressures N . Then we compute the resulting shear stress with

$$\tau_b = \int_0^1 \lambda \vec{n} \cdot (1,0) ds$$

We reproduce the results from Gagliardini et al 2007

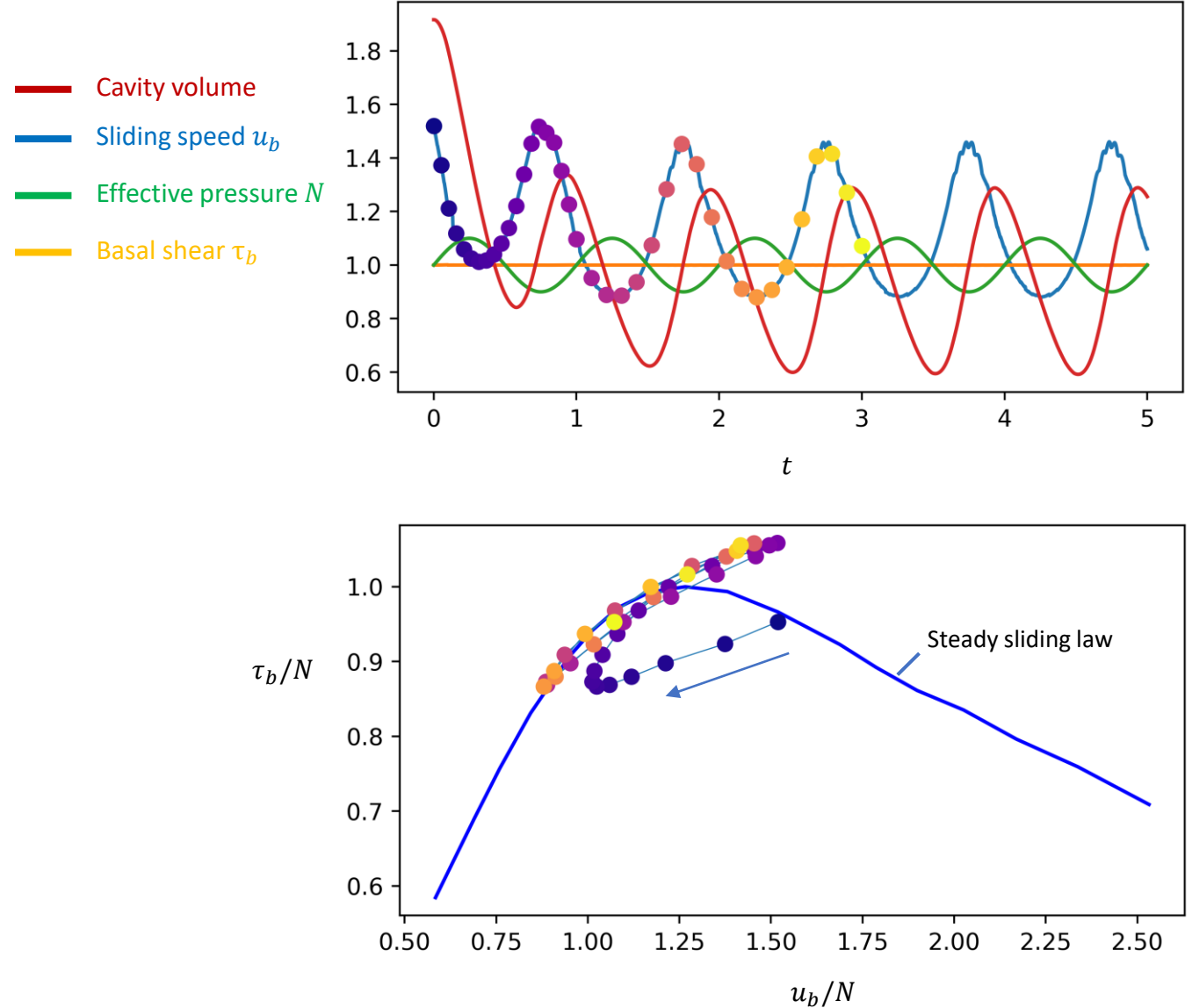
For any mesh size, we exactly enforce the discrete contact boundary conditions. This means that we will always have $\sigma_{nn} \leq 0$ on the attached region and $\sigma_{nn} = 0$ on the detached regions



Unsteady sliding

Field observations suggest that subglacial water pressures oscillate in many cases. We are currently studying the effects of oscillating effective pressures on the sliding speed u_b .

We present the results from a computation in which a steady cavity shape is evolved by varying the effective pressure with a frequency of 1 time unit (≈ 3 days) and a 10% amplitude. We fix the basal shear stress as a boundary condition and allow the sliding speed to vary.



A weak formulation for the Stokes equations

We aim to build a finite element solver for the Stokes equations with contact boundary conditions. A finite element scheme for a PDE is based on a weak formulation of that PDE. In general, these weak formulations are variational *equalities*; however, in the presence of contact boundary conditions, they take the form of a *variational inequality*.

Let V and Q be adequate spaces for the velocity and pressure fields. We introduce the space of admissible velocity fields

$$K = \{\vec{v} \in V: \vec{v} \cdot \vec{n} \leq 0 \text{ on attached region } \Gamma_a\}.$$

We explicitly enforce the **kinematic** contact condition

Then, the weak formulation of the Stokes equations with contact boundary conditions is:

Find $(\vec{u}, p) \in K \times Q$ such that

$$\int_{\Omega} \mu \varepsilon(\vec{u}) : \varepsilon(\vec{v} - \vec{u}) dx - \int_{\Omega} p \nabla \cdot (\vec{v} - \vec{u}) dx \geq - \int_{top} p_i (\vec{v} - \vec{u}) \cdot \vec{n} ds + \int_{\Gamma_a} p_w (\vec{v} - \vec{u}) \cdot \vec{n} ds \quad \forall \vec{v} \in K$$

The **dynamic** and **complementarity** conditions are implicitly enforced in this formulation

$$\int_{\Omega} q \nabla \cdot \vec{u} dx \quad \forall q \in Q$$

A weak formulation for the Stokes equations: Use of Lagrange Multipliers

There exist many different methods for solving a variational inequality with finite element methods. In general, we need to transform the *inequality* into an *equality*. One way to do this is by adding a nonlinear term that penalises the kinematic condition $\vec{u} \cdot \vec{n} \leq 0$.

An alternative which we found to work best in conjunction with the free boundary equation is to insert a **Lagrange multiplier** λ which represents the normal stresses σ_{nn} along the attached region Γ_a . If Σ is an adequate space of scalar functions along Γ_b , we may define

$$\Lambda = \{\mu \in \Sigma: \mu + p_w \leq 0 \text{ on } \Gamma_a\}$$

Then, the variational inequality may be rewritten as: find $(\vec{u}, p, \lambda) \in V \times Q \times \Sigma$

$$\int_{\Omega} \mu \varepsilon(\vec{u}) : \varepsilon(\vec{v}) \, dx - \int_{\Omega} p(\nabla \cdot \vec{v}) \, dx - \int_{\Gamma_a} (\lambda + p_w)(\vec{v} \cdot \vec{n}) \, ds = - \int_{top} p_i \vec{v} \cdot \vec{n} \, ds + \int_{\Gamma_a} p_w \vec{v} \cdot \vec{n} \, ds \quad \forall \vec{v} \in V$$

$$\left\{ \begin{array}{l} \int_{\Omega} q(\nabla \cdot \vec{u}) \, dx \quad \forall q \in Q \\ \underbrace{\int_{\Gamma_a} (\mu + p_w)(\vec{u} \cdot \vec{n}) \, ds = 0}_{\vec{u} \cdot \vec{n} \leq 0} \quad \forall \mu \in \Lambda, \quad \underbrace{\lambda \in \Lambda}_{\sigma_{nn} + p_w \leq 0}, \quad \underbrace{\int_{\Gamma_a} (\lambda + p_w)(\vec{u} \cdot \vec{n}) \, ds = 0}_{(\sigma_{nn} + p_w)(\vec{u} \cdot \vec{n}) \leq 0} \end{array} \right.$$

These are a weak equivalent of the contact boundary conditions

Solving the Stokes equation with Lagrange multipliers

Rewriting the variational inequality with Lagrange multipliers offers several advantages:

- We can rewrite it as a nonlinear equation by using the equivalence:

$$\int_{\Gamma_a} (\mu + p_w)(\vec{u} \cdot \vec{n}) ds = 0 \quad \forall \mu \in \Lambda, \quad \lambda \in \Lambda, \quad \int_{\Gamma_a} (\lambda + p_w)(\vec{u} \cdot \vec{n}) ds = 0$$
$$\longleftrightarrow \quad \lambda + p_w + \max\{0, -(\lambda + p_w) + \vec{u} \cdot \vec{n}\} = 0 \quad \text{on } \Gamma_a$$

- The resulting nonlinear equation can be solved with a (semismooth) Newton method directly. We implemented the Stokes variational inequality coupled to the free boundary equation in Firedrake.
- The *weak* contact boundary conditions hold exactly at the discrete level. This means that we can clearly distinguish the regions which wish to detach from those that remain attached. This makes the solver for subglacial cavitation much more robust.

The **numerical results** were obtained with Firedrake using p.w. quadratic elements for the velocity and p.w. constant elements for the pressure and the Lagrange multiplier.