
Instantaneous multiscale properties of solar wind dynamical regimes and their attractors

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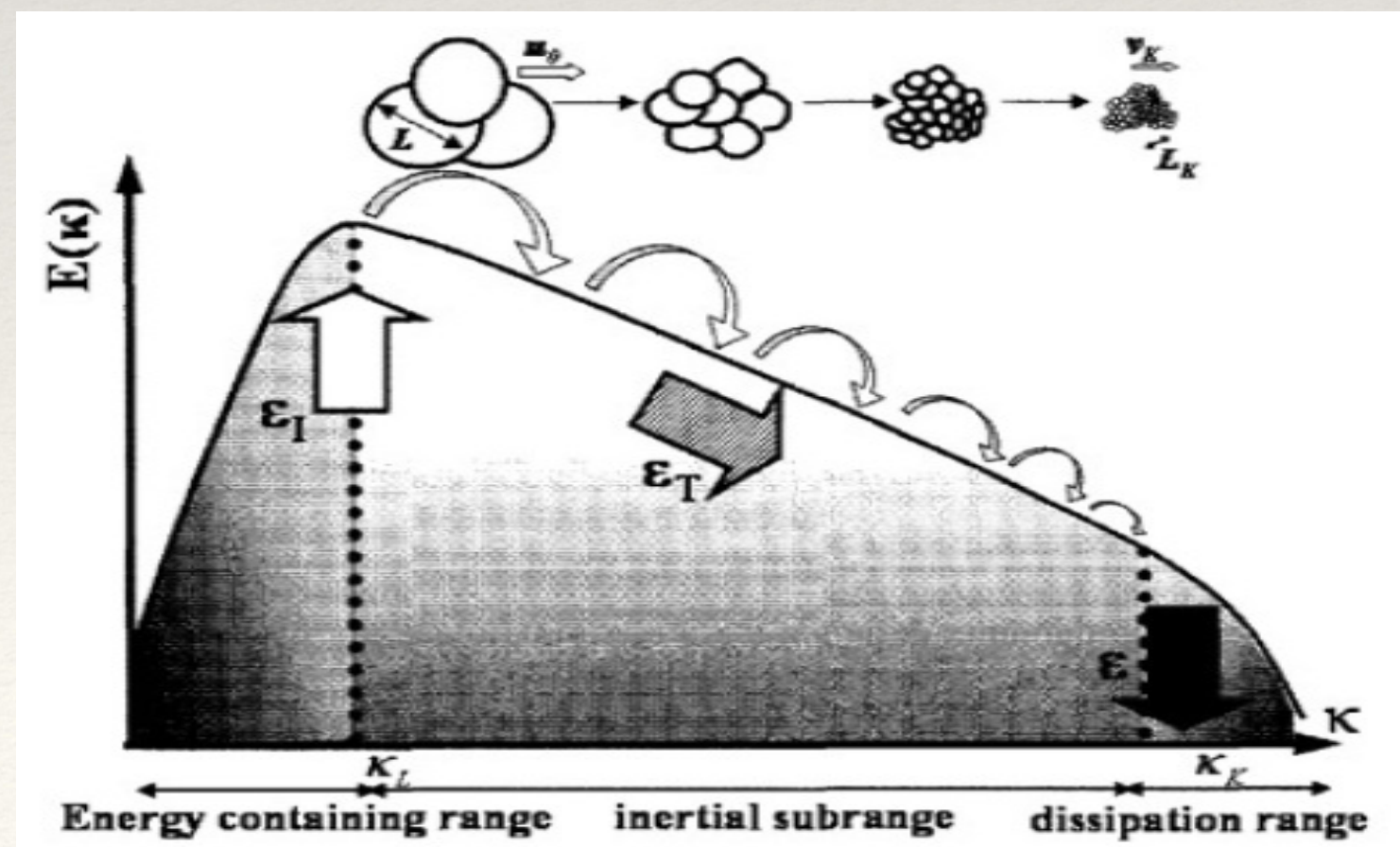
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A turbulent flow...

- ❖ can be seen as a “collection of scales”
- ❖ nonlinearly interacting each other
- ❖ transferring energy via an out-of-equilibrium cascade across scales
- ❖ whose mechanism is described more or less by Kolmogorov theory, with intermittency corrections



Being defined $\delta u_\ell \doteq u(x + \ell) - u(x)$, from NS equations we have

$$\frac{1}{2} \partial_t \langle (\delta u_\ell)^2 \rangle + \epsilon = -\frac{1}{4} \nabla_\ell \langle (\delta u_\ell)^3 \rangle + \nu \nabla_\ell^2 \langle (\delta u_\ell)^2 \rangle$$

assuming stationarity ($\partial_t \rightarrow 0$) and neglecting viscous effects ($\nu \rightarrow 0$)

$$\epsilon = -\frac{1}{4} \nabla_\ell \langle (\delta u_\ell)^3 \rangle$$

and homogeneity and isotropy

$$(\delta u_\ell)^3 = -\frac{4}{3} \epsilon \ell$$

and self-similarity

$$(\delta u_\ell)^p \propto (\epsilon \ell)^{p/3}$$

with intermittency

$$\epsilon \rightarrow \epsilon_\ell \propto \ell^{\mu(p)}$$

such that

$$(\delta u_\ell)^p \propto (\epsilon \ell)^{\zeta(p)} \text{ with being } \zeta(p) = p/3 + \mu(p)$$

Scale-dependent dimensions $D_{q,\tau}$

- ❖ A complex system (like a turbulent flow) is formed by many interacting components at different scales
- ❖ Assume for the sake of simplicity to have a time dependent signal

$$x(t) = \langle x(t) \rangle + \delta x_{\tau_1}(t) + \dots + \delta x_{\tau_k}(t) = x_0 + \sum_{\tau} \delta x_{\tau}(t)$$

- ❖ let be $\mathcal{M}_{\tau}$ the phase-space representative of the system for all scales $\tau' < \tau$ and let $\mathcal{S}_{\tau}$ a partition of $\mathcal{M}_{\tau}$ into $\mathcal{B}_{i,\tau}$ hypercubes of size r
- ❖ let now define a normalized measure

$$\mathcal{P}_{i,\tau} = \frac{\mu(\mathcal{B}_{i,\tau})}{\mu(\mathcal{M}_{\tau})}$$

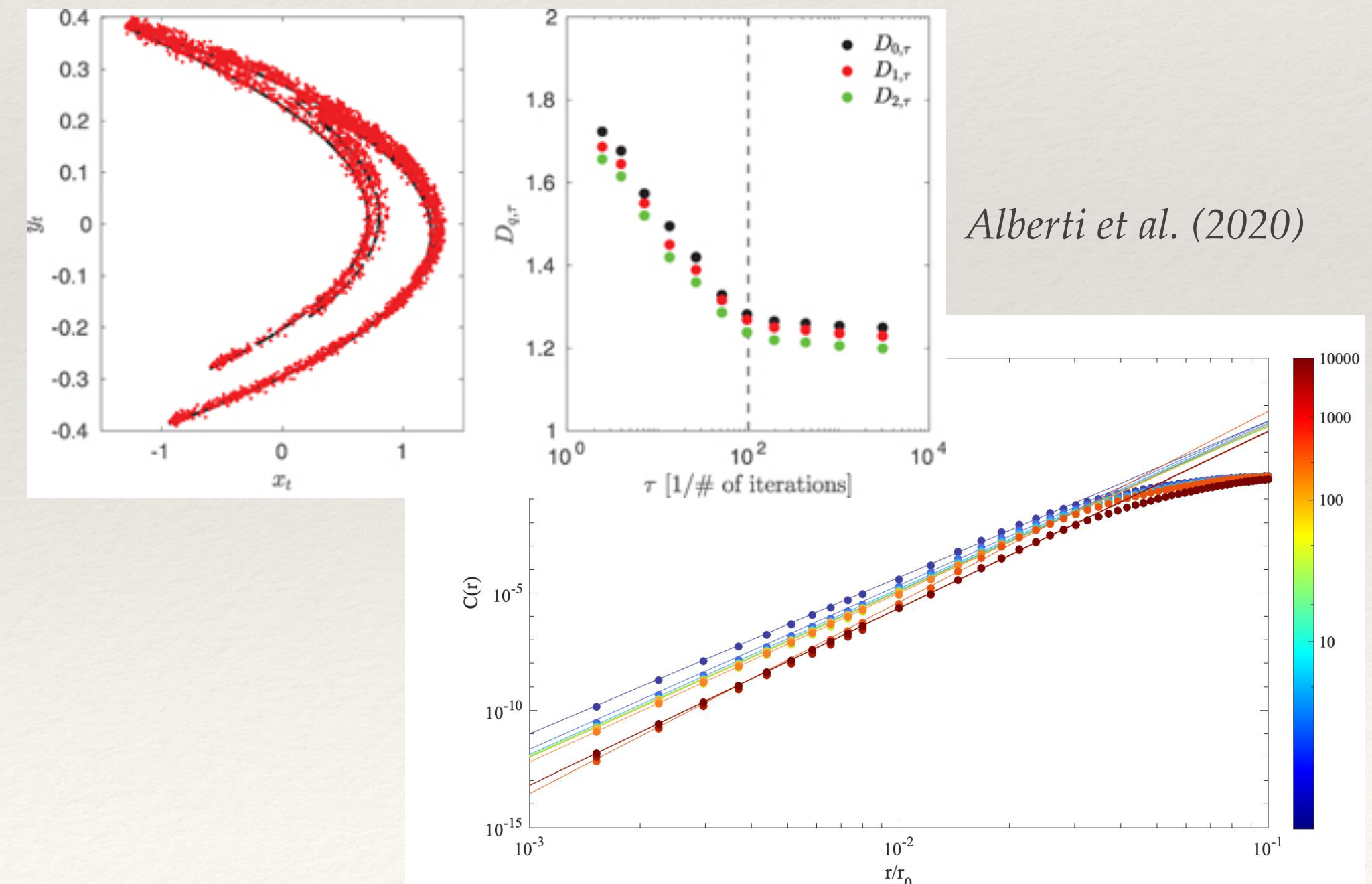
- ❖ if we define a partition function

$$\chi(r, q, \tau) = \sum_i \mathcal{P}_{i,\tau}^q$$

- ❖ it will scale as

$$\chi(r, q, \tau) \propto r^{(q-1)D_{q,\tau}}$$

- ❖ where $D_{q,\tau}$ are measures of the scale-dependent complexity of a system



Alberti et al. (2020)

Scale-dependent instantaneous dimensions $D_\tau(t)$

- ❖ Previous measures are based on the concept of “scaling” at different scales that is “local” but not “instantaneous”
- ❖ Is there any kind of measure to deal with instantaneous properties of chaotic attractors?

Yes, if you forgot “scalings” and use “recurrences”!

- ❖ How to do this? By looking what happens in the neighborhood of a point of the phase-space
- ❖ We need to introduce two concepts: i) instantaneous dimension, and ii) stability
- ❖ i) for a state ζ of the attractor the instantaneous dimension $d(\zeta)$ measures the density of neighboring points (i.e., recurrences) and is linked with the degrees of freedom
- ❖ ii) the stability of the state ζ is measured by the stickiness $\theta(\zeta)$ defined as the inverse of the average persistence time of trajectories around ζ and is linked to the fixed point nature

Scale-dependent instantaneous dimensions $D_\tau(t)$

- ❖ Given point ζ on a chaotic attractor, the probability P that a trajectory $x(t)$ returns within a sphere of radius r centered on the point ζ according to the Freitas-Freitas-Todd theorem modified by Lucarini is a generalized Pareto-like distribution
- ❖ let define $g(x) = -\log(\delta(x, \zeta))$ the logarithmic of the distance between $x(t)$ and ζ
- ❖ the probability of logarithmic returns is $\mathcal{P}(g(x) > \text{th}, \zeta) = \exp \left[-\frac{1}{\gamma(\zeta)} \frac{x - \mu(\zeta)}{\sigma(\zeta)} \right]$
- ❖ such that $\text{th} = \exp(-r)$, $d(\zeta) = \sigma(\zeta)^{-1}$, and $\theta(\zeta) = \gamma(\zeta)^{-1}$

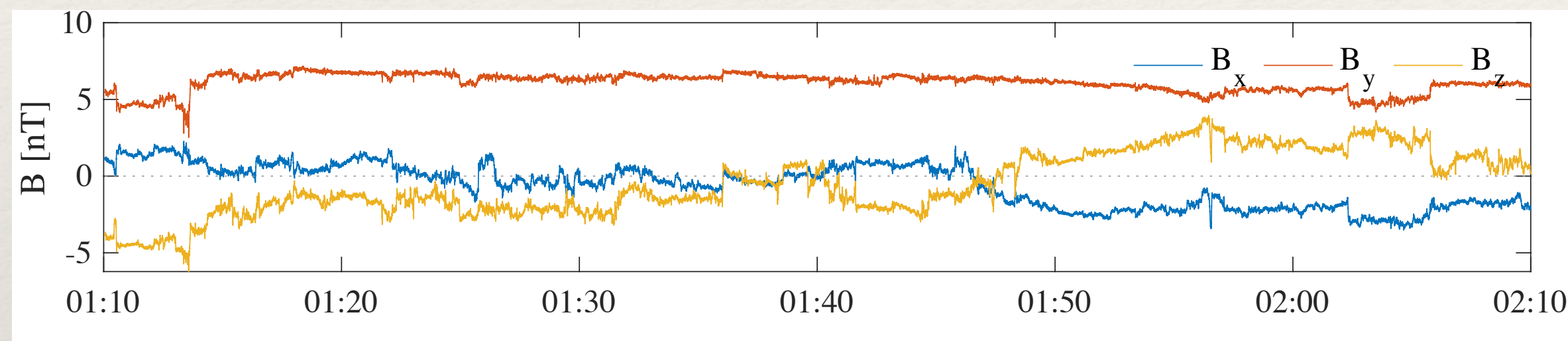
If we now write $x(t) = x_0 + \sum_{\tau} \delta x_\tau(t)$ for each τ we can define a **local instantaneous measure** $D_\tau(t)$ via

$$\mathcal{P}(g(\delta x_\tau(t)) > \text{th}, \zeta_\tau) = \exp \left[-\frac{1}{\gamma_\tau(\zeta_\tau)} \frac{\delta x_\tau - \mu_\tau(\zeta_\tau)}{\sigma_\tau(\zeta_\tau)} \right]$$

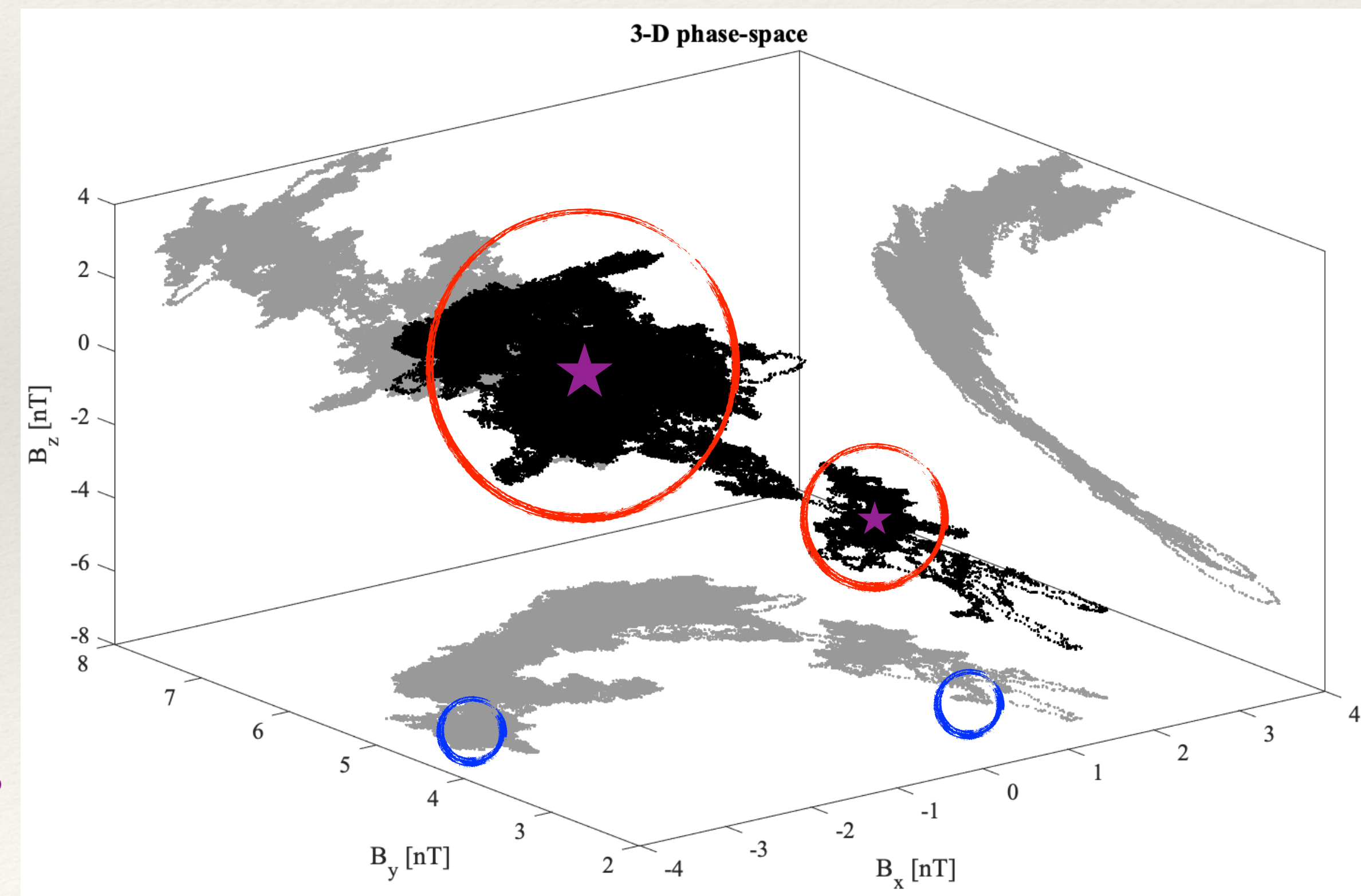
being $D_\tau(t) = \sigma_\tau(\zeta_\tau)^{-1}$ and $\theta_\tau(t) = \gamma_\tau(\zeta_\tau)^{-1}$

MMS measurements in the solar wind

- ❖ Magnetic field measurements in the solar wind in the GSE reference frame at 128 sample/s
- ❖ Clearly here we have a topological dimension $D=3$

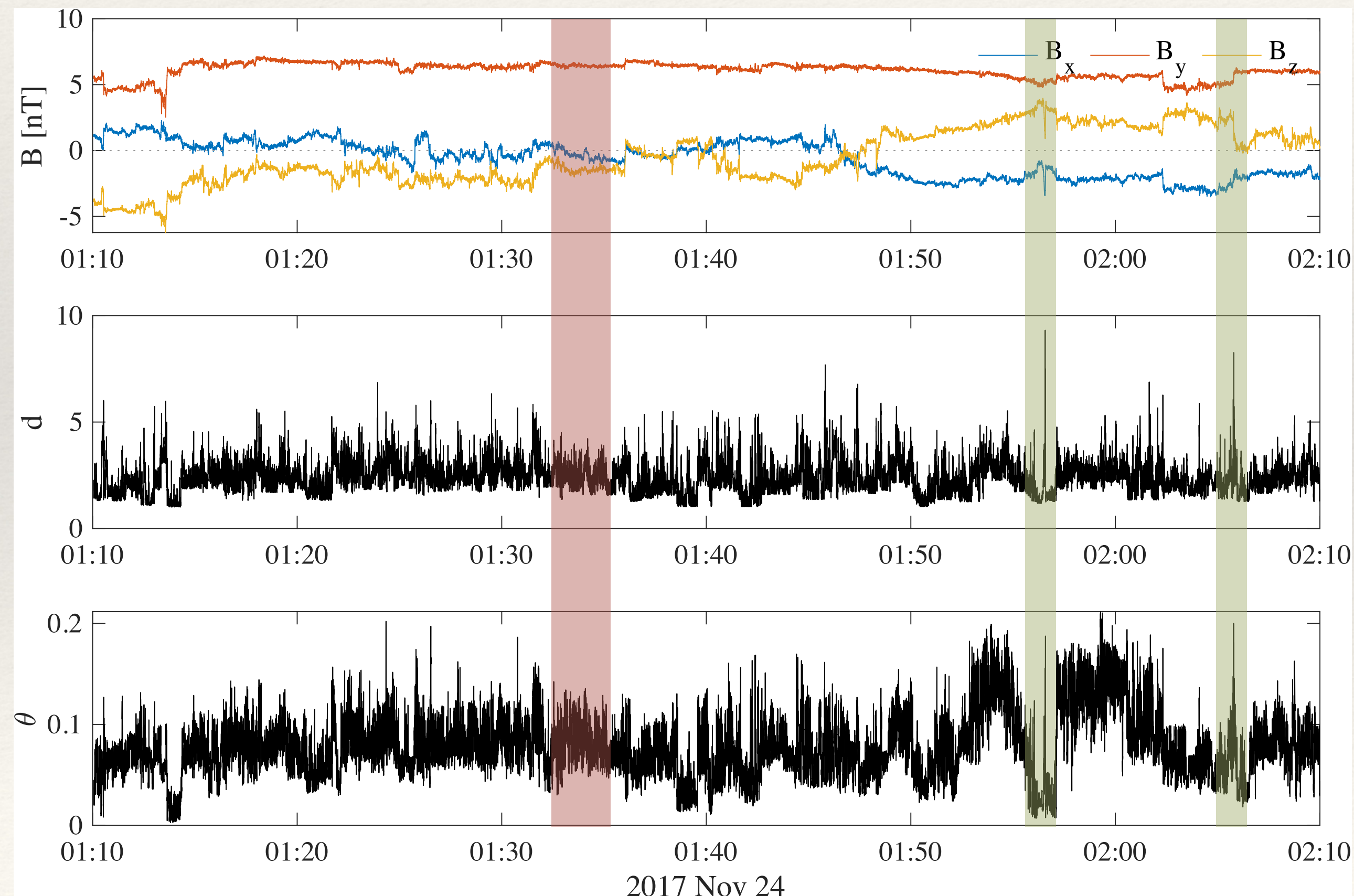


- ❖ An underlying strange attractor seems to be present with **small**- and **large**-scale recurrences
- ❖ It seems to be characterized by a 2D geometry
- ❖ There should be (some) **marginally stable states**



Instantaneous global properties

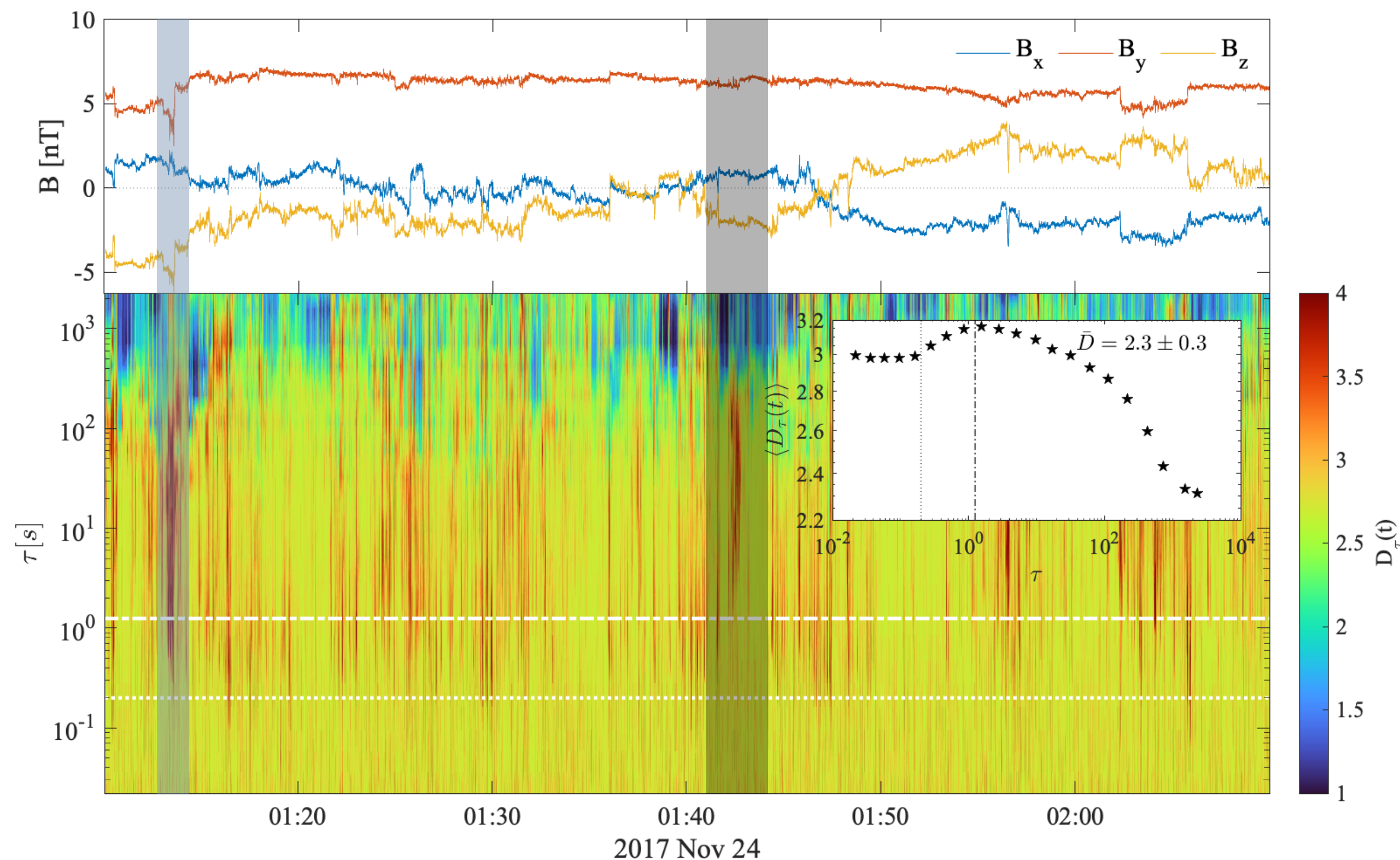
$$\mathbf{B}(t) \rightarrow (d(t), \theta(t))$$



- ❖ On average the dynamics is different from that of a stochastic process ($\bar{D} < 3$)
- ❖ larger $(d(t), \theta(t))$ are found in close correspondence with sudden changes in both B_x and B_z
- ❖ smaller $(d(t), \theta(t))$ are found in close correspondence with more regular dynamics

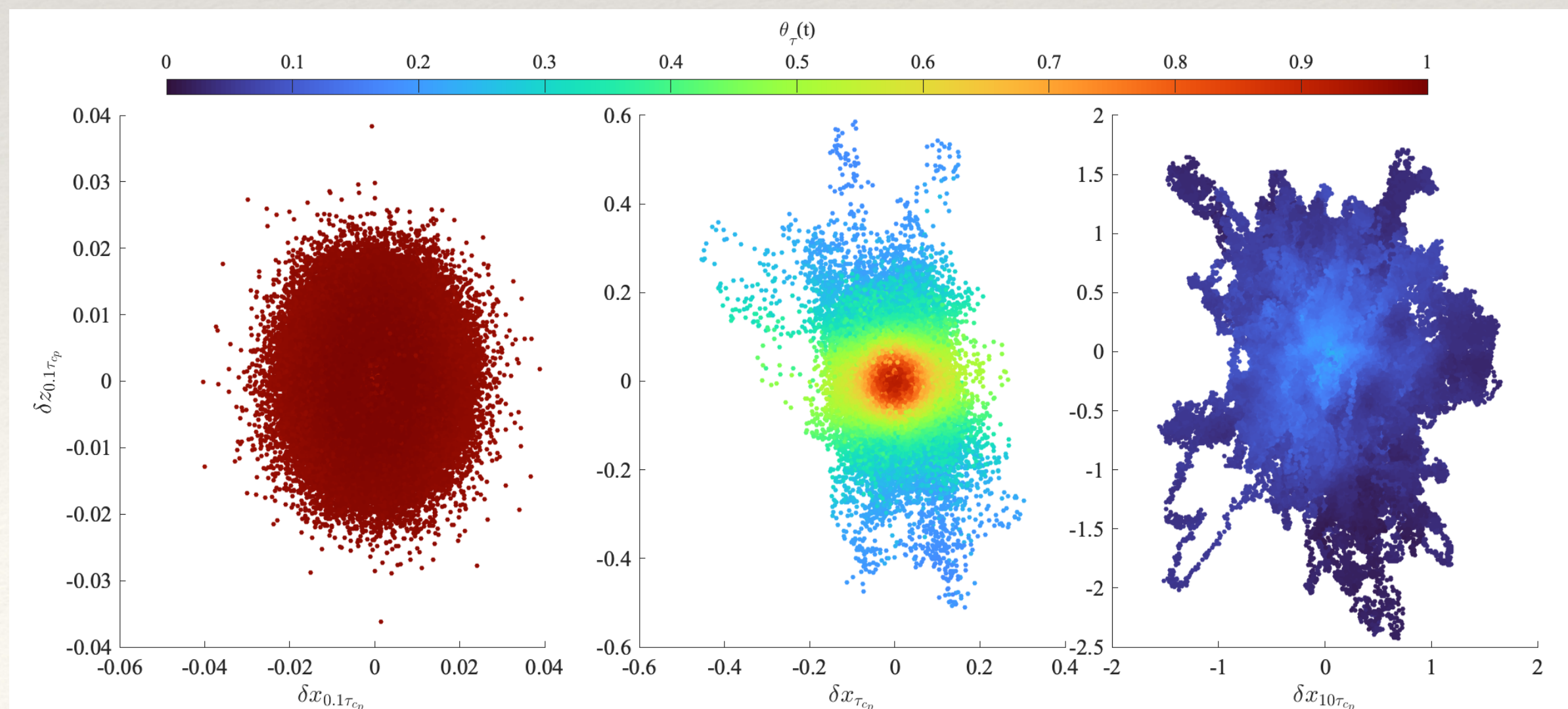
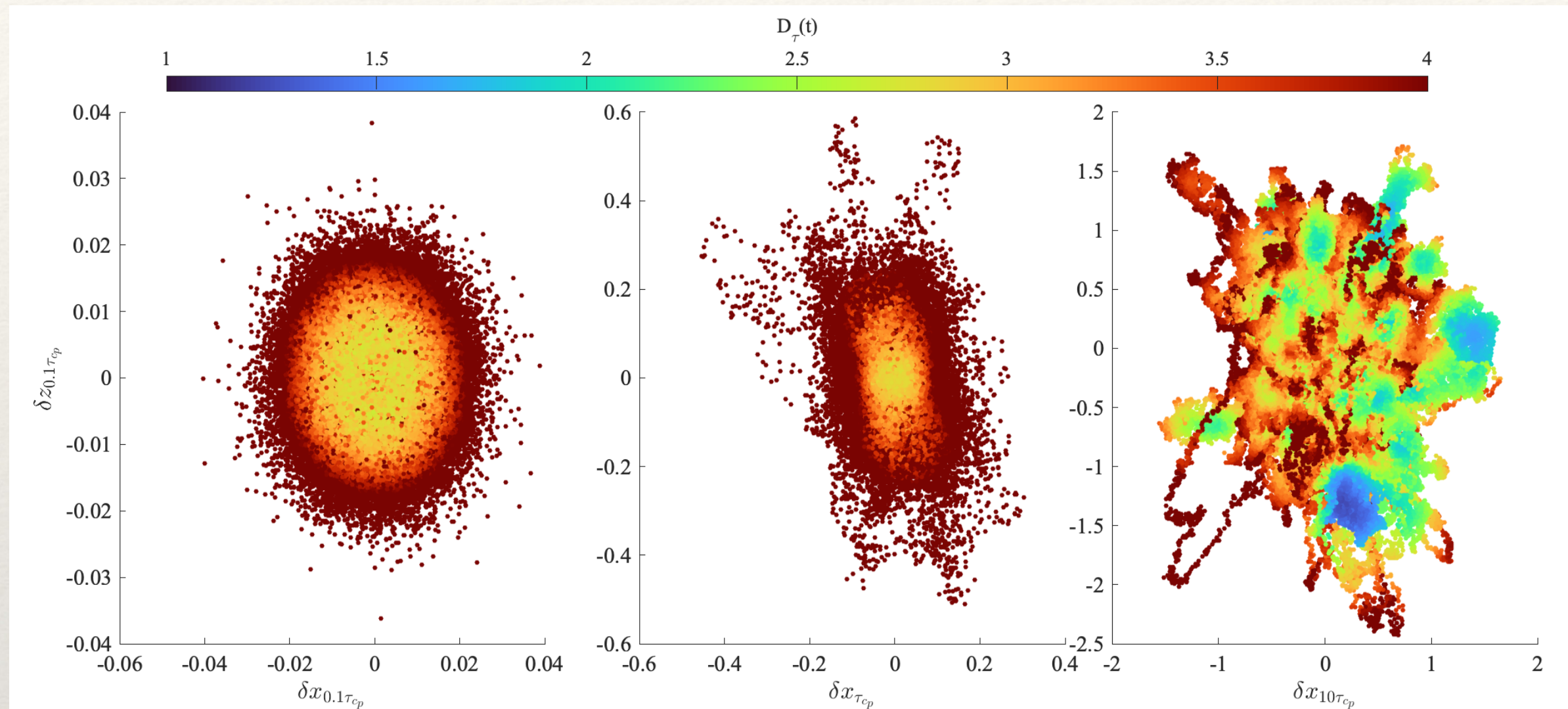
Instantaneous scale-dependent properties

$$\mathbf{B}(t) = \mathbf{B}_0 + \sum_{\tau} \delta \mathbf{B}_{\tau}(t) \rightarrow (D_{\tau}(t), \theta_{\tau}(t))$$



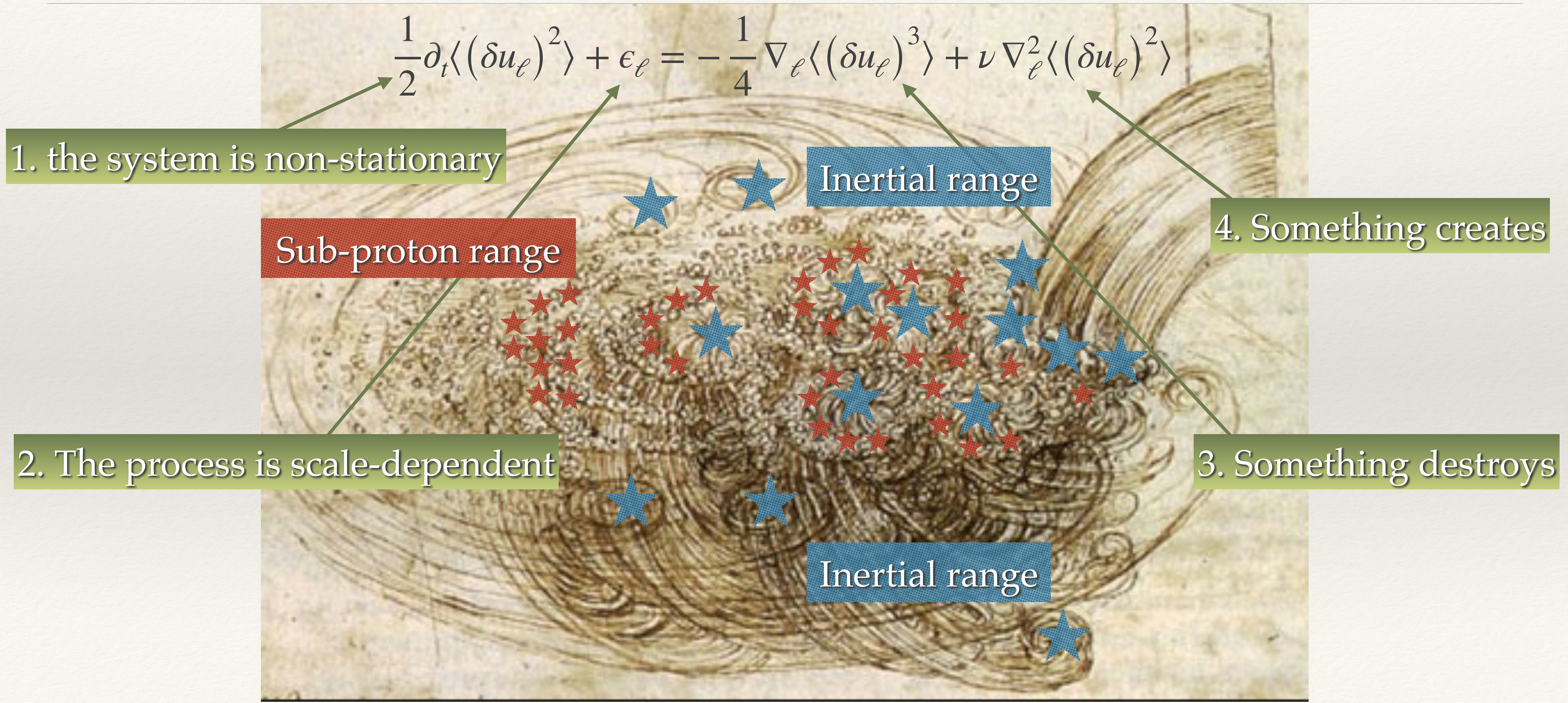
- ❖ A clear scale-dependent phase-space topology is observed
- ❖ $\bar{D} \approx 3$ for scales associated with the SCM noise, $\bar{D}_{\max}$ for $\tau \approx \tau_{c_p}$, decreasing $\bar{D}$ in the inertial range
- ❖ **global sudden variations are found in close correspondence with increases of $D_{\tau}(t)$ at $\tau \lesssim \tau_{c_p}$**
- ❖ conversely, no sudden changes are observed when $D_{\tau} > 3$ at $\tau > \tau_{c_p}$

MMS measurements in the solar wind



- ❖ a high number of degrees of freedom (> 3) suggests the existence of an unstable fixed point for $\tau \lesssim \tau_{cp}$
- ❖ the inertial range is less chaotic than the sub-proton one
- ❖ as for fluid turbulence we observe multi-stability pointing towards the existence of an underlying strange stochastic attractor
- ❖ the system would in principle prefer to stay in the inertial range but it needs to also explore a small-scale regime that really control transitions

Let's try to explain in other words...



Going towards the future...

- ❖ The instantaneous scale-dependent dimensions seem to be a promising tool for dealing with chaotic dynamical systems
- ❖ A novel concept of “scale-dependent instantaneous chaos” arises from $(D_\tau(t), \theta_\tau(t))$
- ❖ The novel formalism has been successfully applied on several dynamical systems
- ❖ It also allows to describe the overall dynamics of solar wind magnetic field fluctuations as consisting of multi-stable and multiscale fixed points, opening a novel way to describe the solar wind via stochastic low-dimensional models featuring a large number of degrees of freedom
- ❖ Deeper investigations are needed to provide a direct link with both direct energy cascade and inverse enstrophy cascade as well as to build up a corresponding theoretical framework
- ❖ Surprisingly results are also obtained with von Karman experiments providing a proof for the theoretical conjecture on the existence of bifurcations between different equilibria at large scales as well as restored symmetry at small scales