

# Bicovariance and Bispectrum of ENSO index and its impact in nonlinear predictability

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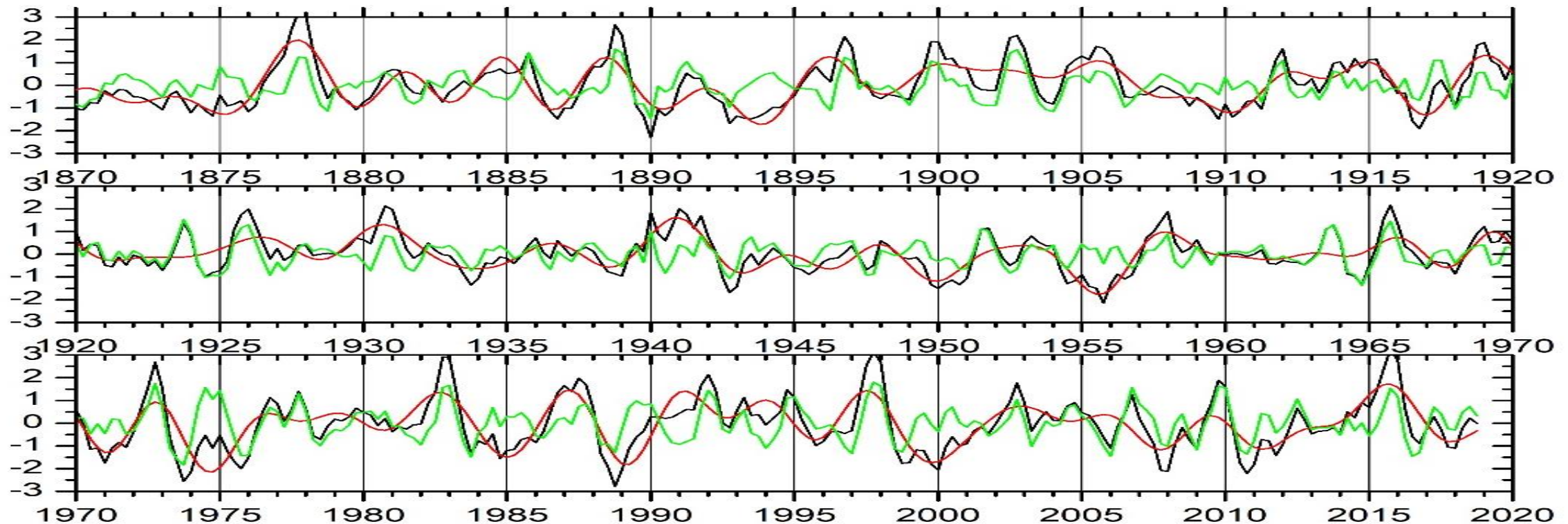
The bicovariance function  $B(\tau_1, \tau_2) = E[x(t)x(t + \tau_1)x(t + \tau_2)]$  is the 3<sup>rd</sup> order (two-lag) generalization of the (one-lag) auto-covariance function.

It works as a diagnostic of non-Gaussianity and nonlinearity of a time-series  $x(t)$

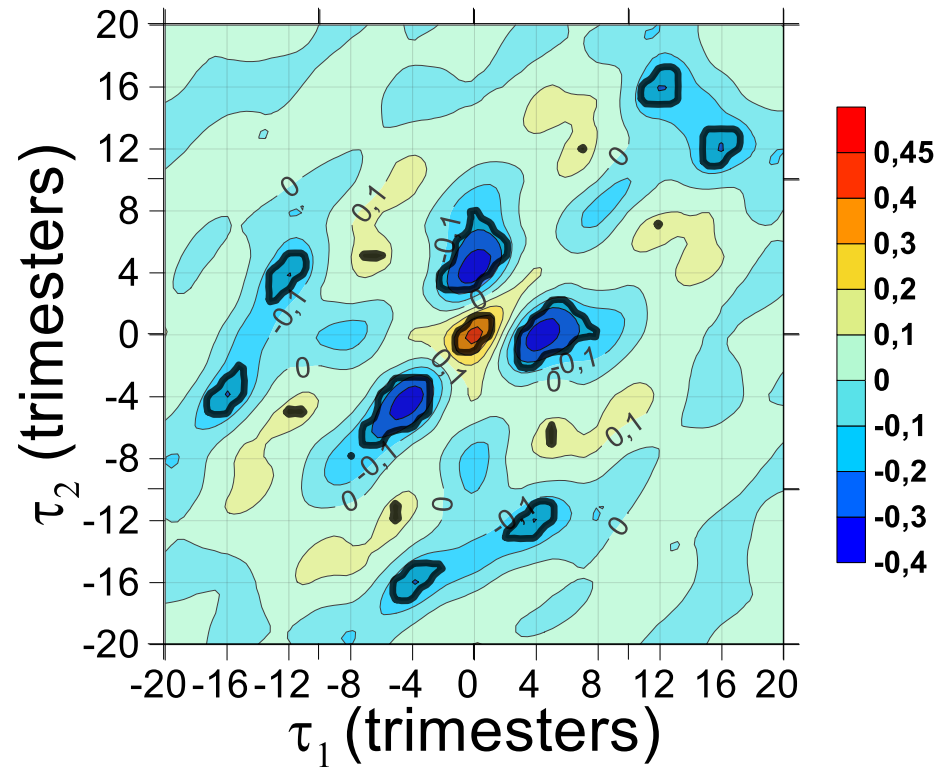


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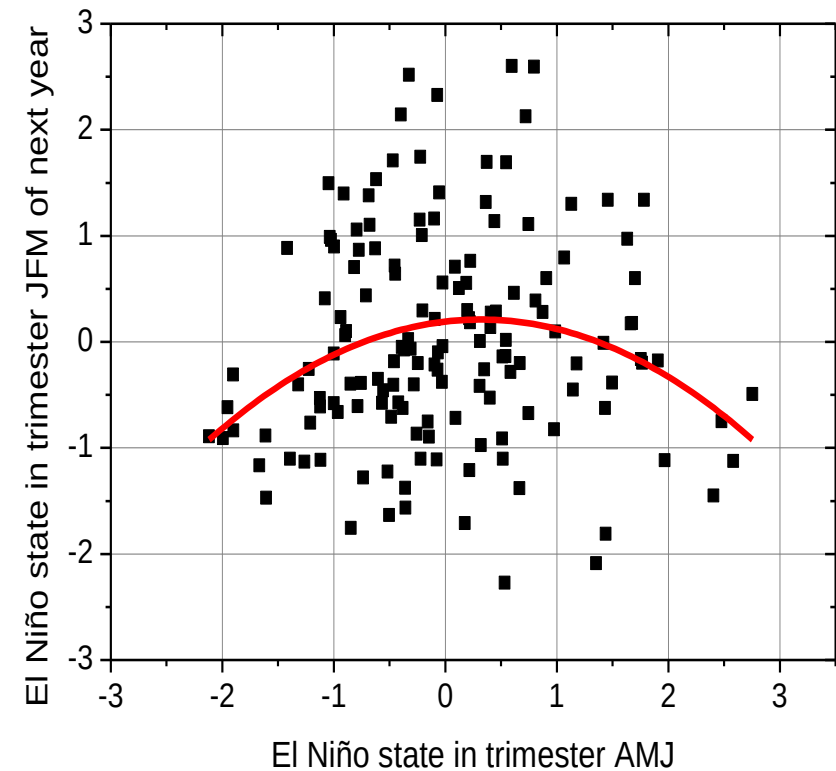
Data: Trimestral (trm) average anomalies of the standardized 3.4 El-Niño index (1870-2019) (black curve)  
Slow inter-triannual (red curve) and fast intra-triannual (green curve) components



El-Niño bicornelation function  $B(\tau_1, \tau_2)$



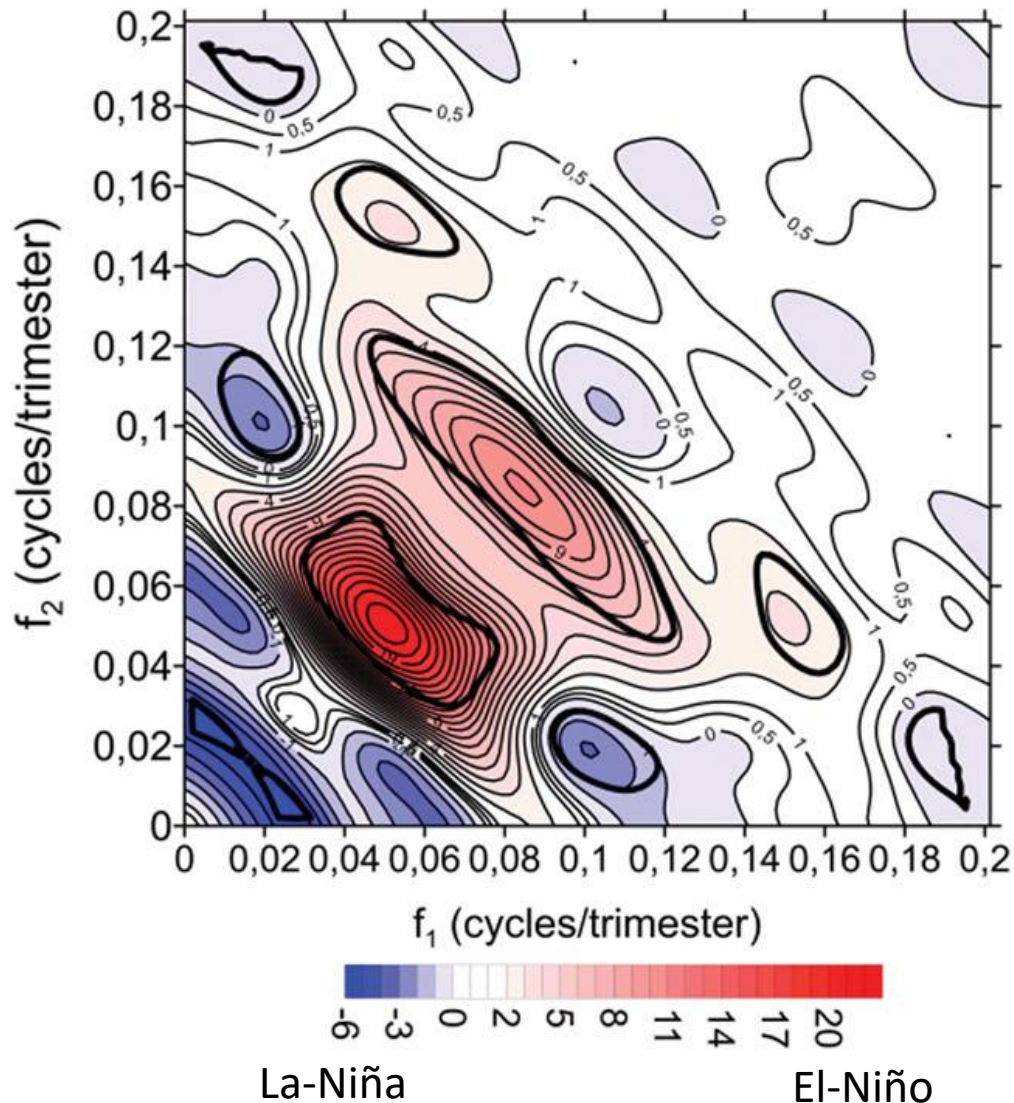
Winter El-Niño index versus previous Spring El-Niño index



The El-Niño skewness is positive:  $skew(x) = B(0,0) = 0.46$ , coming from more intense El-Niños than La-Niñas. Significant lagged bicornelation verifies at  $B(0, 3 \text{ trm}) = E[x(t)^2 x(t + 3 \text{ trm})] = -0.4$ , mostly because NH-Winter state  $x(t + 3 \text{ trm})$  correlates with the squared state  $x(t)^2$  of the previous NH-Spring (see Figure). The linear correlation is barely zero ( $\sim 0.05$ ). That represents a non-linear breakdown of the El-Niño Spring (linear) Predictability Barrier. Interpretation: Weak Spring states evolve mostly to Winter El-Niños whereas strong Spring La-Niñas persist longer (up to next winter), than Strong El-Niños, resulting from an asymmetric persistent SST response to anomalous trade-winds.

The two-dimensional Fourier transform of the bicoherence function is the complex bispectrum  $\Gamma_{3,x}(f_1, f_2)$ . It results from triadic phase synchronization of wave components at frequencies  $f_1$ ,  $f_2$  and  $f_3 = f_1 + f_2$

## El-Niño Bispectrum $Re[\Gamma_{3,x}(f_1, f_2)]$



Bicovariance is recovered by Bispectrum Inverse FT:

$$B(\tau_1, \tau_2) = \iint_{-1/2}^{1/2} Re[\Gamma_{3,x}(f_1, f_2) \exp[i(f_1\tau_1 + f_2\tau_2)]] df_1 df_2$$

In particular, the skewness  $B(0,0) = skew(x)$  is the (real-part) bispectrum decomposition over the bi-frequency domain, like the spectral decomposition of variance provided by the spectrum  $S(f)$

El-Niño (positive) extreme events are mostly occurring for

$$f_1 = f_2 \sim 0.05 \text{ cpt (5 yrs)}, f_3 = 0.1 \text{ cpt (2.5 yrs)}$$

$$f_1, f_2 \in [0.06, 0.12] \text{ cpt (2 - 4 yrs)}, f_3 = 0.165 \text{ cpt (1.5 yrs)}$$

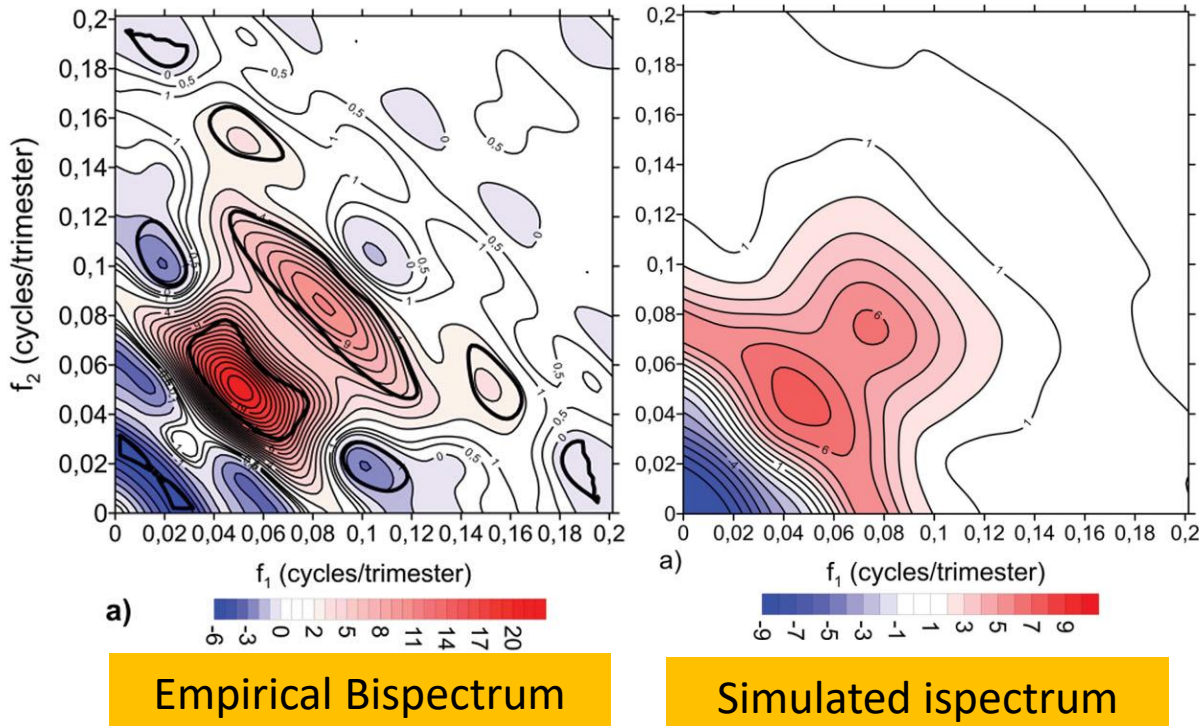
(Red regions)

La-Niña (negative) extreme events are mostly occurring for

$$f_1, f_2, f_3 < 0.07 \text{ cpt (3 yrs)}, \text{ i.e. resulting from triadic interaction between wave-periods larger than 3 yrs}$$

(Blue regions)

El-Niño reveals both **non-Gaussianity** due to non-vanishing skewness and **nonlinearity** due to the significant difference between the empirical bispectrum and that of a non-Gaussian linear model reproducing the empirical spectrum. Therefore, it is a challenge to fit a stochastic model reproducing the spectrum and bispectrum issued from observations.



The fitted model: **Bilinear model (BL)** i.e. an AR model driven by different delayed CAM (correlated-additive-multiplicative) noises generating non-linear and non-Gaussian behavior:

$$\tilde{x}(t) = \sum_{k=1}^{p_1} a_k \tilde{x}(t - l_k) + \sum_{k=1}^{p_2} b_k \tilde{x}(t - r_k) u(t - s_k) + \alpha + u(t) \quad (\tilde{x}: \text{signal}, u: \text{Gaussian white noise})$$

Maximum lags:  $p_1 = p_2 = 5$  trimesters

Hybrid Fitting: Find coefficients minimizing the residual variance of  $u$ , along the time-series and making the best fit between simulated and empirical spectrum/bispectrum

The bispectrum main features are reproduced by the fitted stochastic model, particularly the negative value in the bifrequency region  $f_1, f_2, f_3 < 0.07$  cpt (3 yrs). The cross-validation forecast correlation skill at 3 trimesters lag is 0.36 compared to the less performing AR5 linear model (0.26). The BL model is thus being able to reduce the El-Niño Spring Predictability Barrier coming from nonlinearity.