

Effects of thermal boundary conditions on rotating Rayleigh-Bénard convection in a cylinder with implications on geophysical and astrophysical systems

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Introduction

- Rayleigh-Bénard convection: from boiling water to molten planetary cores
- Experiments and numerical models seek to imitate these real conditions as well as each other.
- Compare numerical simulations with the most common, and most applicable boundary conditions
- Consider effects on heat transport and solution forms
- Determine appropriate parameters for comparing across boundary conditions

Motivation

- Improve understanding of relationships between large-scale physical flows, numerical models, and experiments
- Explore heat transport behavior
- Inform on stability for experimental design

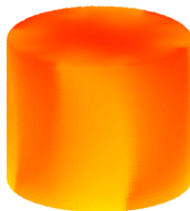
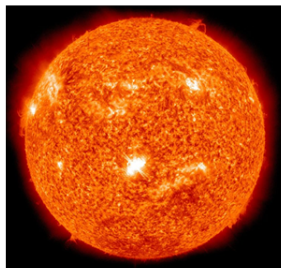


Figure: Image of the sun from National Geographic, a numerical simulation and a convection experiment at UCLA SpinLab.

Governing Equations

Boussinesq approximation applied to Navier Stokes equations:

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\frac{1}{\rho_0} \nabla p + \alpha T \mathbf{g} + \nu \nabla^2 \mathbf{u}, \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2)$$

$$\frac{DT}{Dt} = \kappa \nabla^2 T. \quad (3)$$

Obtain non-dimensional equations by apply scalings such that,

$$t = \hat{t} \frac{d^2}{\kappa} \quad \mathbf{x} = \hat{\mathbf{x}} d \quad \mathbf{u} = \hat{\mathbf{u}} \frac{\kappa}{d} \quad p = \hat{p} \frac{\kappa^2}{d^2} \quad T = \hat{T} \Delta T, \quad (4)$$

where d is the height of the system.

Governing Equations

Eqs (1)-(3) now become non-dimensional:

$$\frac{D\mathbf{u}}{Dt} = -\nabla p + Pr\nabla^2\mathbf{u} - RaPrT\tilde{\mathbf{z}} - \frac{Pr}{Ek}\tilde{\mathbf{z}} \times \mathbf{u}, \quad (5)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (6)$$

$$\frac{DT}{Dt} = \nabla^2 T. \quad (7)$$

With non-dimensional parameters Prandtl number, Rayleigh number, and Ekman number:

$$Pr = \frac{\nu}{\kappa} \quad Ra = \frac{\alpha\Delta Td^3g}{\nu\kappa} \quad Ek = \frac{\nu}{2\Omega d^2} \quad (8)$$

Boundary Conditions

Sidewalls:

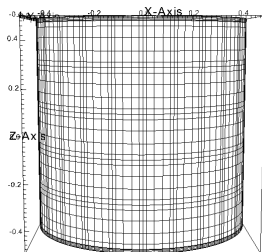
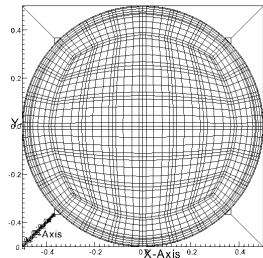
- Thermal insulation: $\nabla T \cdot \mathbf{n} = 0$
- No-slip velocity: $\mathbf{u} = 0$

Top and Bottom:

- One of the following temperature conditions:
 - FT** Fixed Temperature: $T = T_1, T_2$
 - FF** Fixed Flux: $\frac{dT}{dz} = a, a \text{ const.}$
 - M** Mixed: **FT** on the bottom, **FF** on the top
 - R** Robin Condition: $Bi \frac{dT}{dz} = T - \frac{T_m}{\Delta T}$
- No-slip velocity: $\mathbf{u} = 0$

Numerical Simulations

- Nek5000: Spectral Element Method solver
- Cylindrical domain with aspect ratio $\Gamma = r/d = 0.7$
- Parameter range:
 $Pr = 0.7$
 $Ra \leq 10^8$
 $Ek \geq 10^{-5}$
- Initial condition: $\mathbf{u} = 0$ and $T = -z + 0.5$ with perturbation

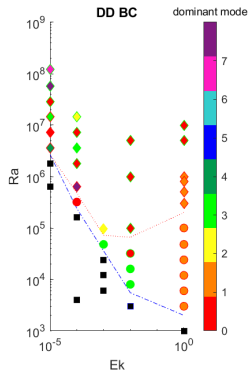
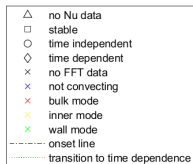
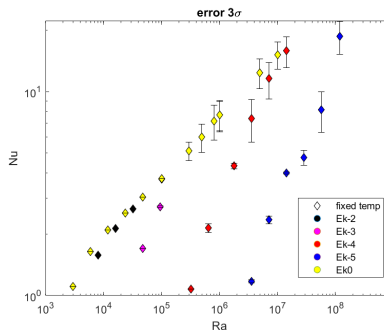


Fixed Temperature Results

$$Nu = \frac{\beta d}{\Delta T}$$

Relationship between Nu and Ra for fixed Ek

Summary of solution forms, main location of modes, and time dependence



Fixed Flux- Ra Consideration

Ra is the ratio of time scales for thermal transport due to diffusion and convection.

Used as a metric of how much/little a system is convecting

Dependant on thermal scaling, determined by boundary condition:

FT $\Delta T = \text{const.}$

So, scale temperature with ΔT

and define $Ra_{FT} = \frac{\alpha \Delta T d^3 g}{\nu \kappa}$

FF, M $\beta = \text{const.}$

So, scale temperature with βd

and define $Ra_{FF} = \frac{\alpha \beta d^4 g}{\nu \kappa}$

Recall, $Nu = \frac{\beta d}{\Delta T}$. Thus, for **FT**, $Nu \propto \beta$, and **FF** and **M**, $Nu \propto \frac{1}{\Delta T}$.

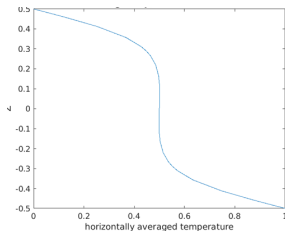
Find that the different Ra follow the relationship:

$$Ra_{FF} = Nu Ra_{FT}$$

Fixed Flux- Ra Consideration

Ex. applying $Ra_{FF} = Nu Ra_{FT}$:

FT

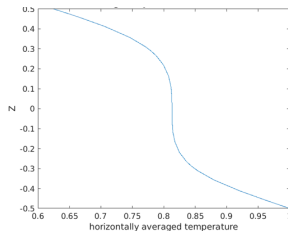


$$\Delta T = 1, \beta \approx -3$$

$$Ra_{FT} = 4.8 \times 10^5$$

$$Ra_{FF} = 1.4 \times 10^6$$

M

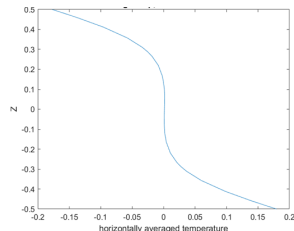


$$\Delta T \approx 0.33, \beta = -1$$

$$Ra_{FT} = 1.6 \times 10^5$$

$$Ra_{FF} = 4.8 \times 10^5$$

FF



$$\Delta T \approx 0.34, \beta = -1$$

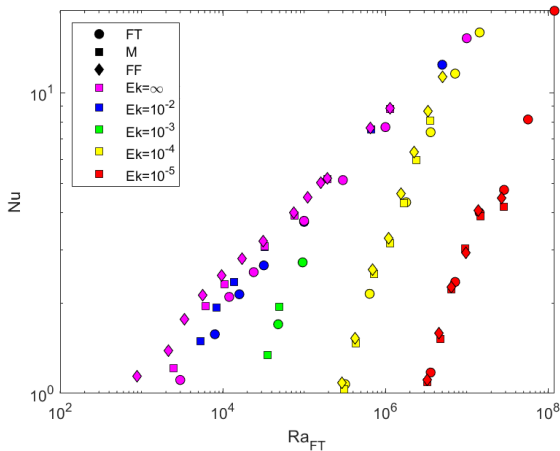
$$Ra_{FT} = 1.6 \times 10^5$$

$$Ra_{FF} = 4.8 \times 10^5$$

See that the effective Ra , Ra_{eff} , is dependant on BCs, i.e. $Ra_{FT} = Ra_{eff}$ for **FT** while $Ra_{FF} = Ra_{eff}$ for **FF** and **M**. Keep this concept in mind...

Results

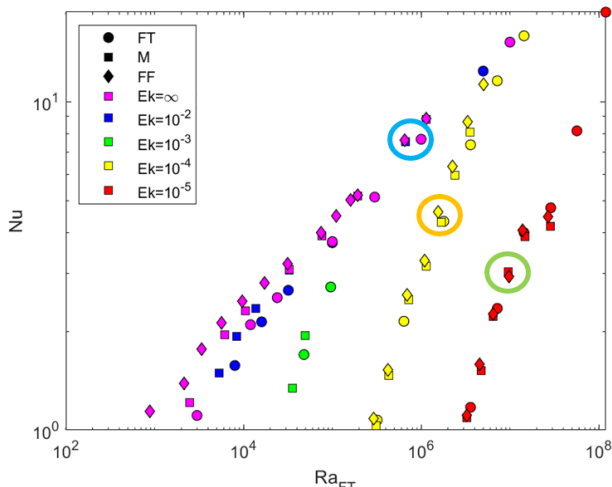
Apply $Ra_{FF} = Nu Ra_{FT}$ to simulation data:



Shows agreement with theory (Calkins et al, 2015) that all BC cases collapse onto a single Nu vs Ra_{FT} relationship for each Ek

Results

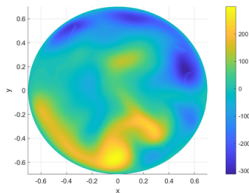
Now, investigate and compare solution forms and time-dependence of systems with similar Ra_{FT} for each BC using vertical velocity plots and Nu time series



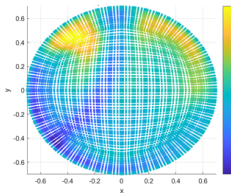
Results

$$Ek = \infty, Ra_{FT} \approx 8 \times 10^5, Ra_{FF} \approx 5 \times 10^6$$

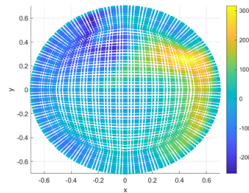
FT



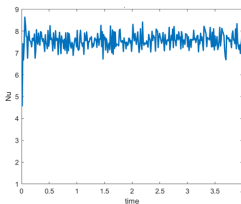
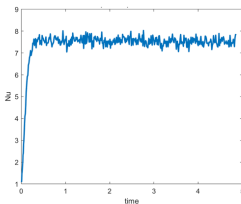
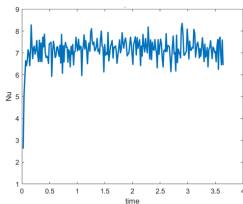
M



FF



All cases show a dominant $m = 1$ mode with similar structure close to the walls

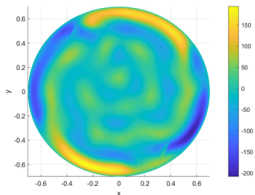


All cases have similar mean Nu , though **M** shows a slightly less chaotic time-dependence

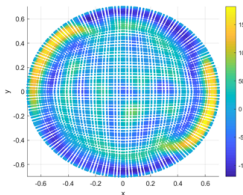
Results

$$Ek = 10^{-4}, Ra_{FT} \approx 2 \times 10^6, Ra_{FF} \approx 7 \times 10^6$$

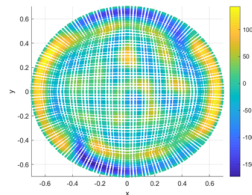
FT



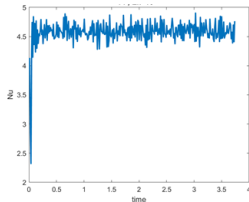
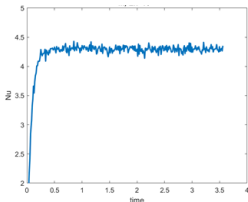
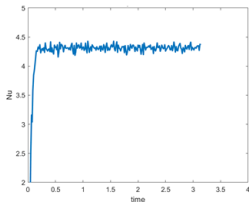
M



FF



All cases show dominant $m = 2$ mode at the sidewalls, as expected with rotation



The mean Nu is similar for all cases, largest for FF which also seems more chaotic

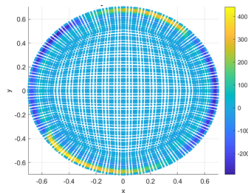
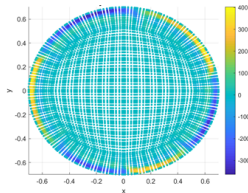
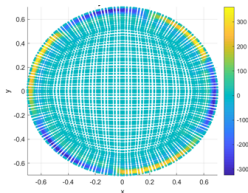
Results

$$Ek = 10^{-5}, Ra_{FT} \approx 1 \times 10^7, Ra_{FF} \approx 5 \times 10^7$$

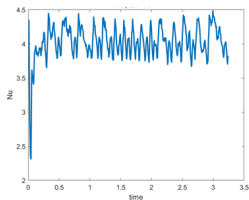
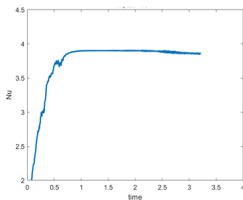
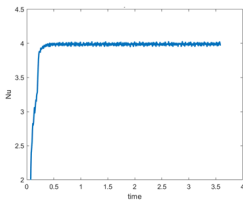
FT

M

FF

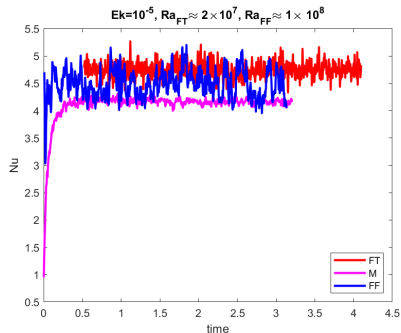
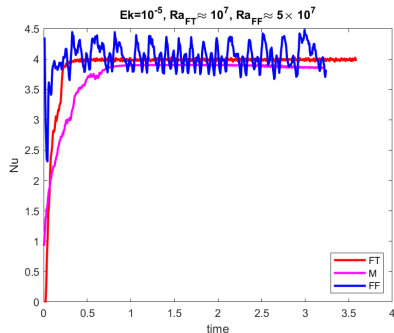


FT and M both show $m = 3$ wall modes, but FF is showing an $m = 2$ wall mode



Mean Nu is similar for all cases, but FF shows significantly more time-dependent

Intermediate Conclusions



- As $Ra_{FT,FF}$ increase, with rapid rotation, **FF** becomes more time-dependent and chaotic before **M** and **FT**
- Differences in dominant mode in **FF** appear at rapid rotation
- Without rotation, different BCs do not appear to affect solution form or time-dependence
- Agrees with results in Anders et al, 2020.

Robin Condition

- Intermediary between FT and FF
- Three ways to define it
 - ① weakly conductive boundaries (Busse and Riahi, 1980)
 - ② conductive-radiative Newton's law (O'Sullivan, 1990, Clarté et al, 2020)
 - ③ two layers of distinct length and thermal conductivity (Gringé et al, 2007)
- Considering 3rd definition here:

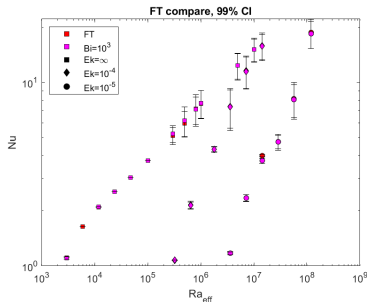
$$Bi \frac{dT}{dz} = T - \frac{T_m}{\Delta T} \quad \text{where Biot number, } Bi = \frac{d_m k}{dk_m},$$

where m indicates a property of the boundary material and k is thermal conductivity

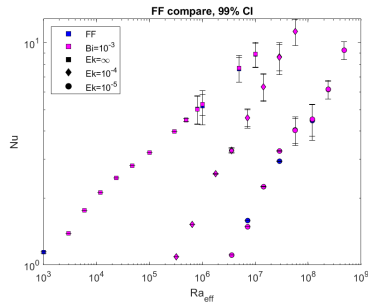
Robin Condition

From BC equation $Bi \frac{dT}{dz} = T - \frac{T_m}{\Delta T}$, it follows that,

as $Bi \rightarrow \infty$, the BC $\rightarrow$ **FT** and as $Bi \rightarrow 0$, the BC $\rightarrow$ **FF**



FT case plotted against high $Bi = 10^3$.



FF case plotted against low $Bi = 10^{-3}$

The data agrees with the theory, results for extreme Bi match **FT** and **FF**

Robin Condition- Ra consideration

No obvious condition is fixed in the thermal BC of the Robin condition to provide a thermal scaling constant. So an effective Rayleigh number, Ra_{eff} , must be determined to fit the limits that such that with large Bi , $Ra_{eff} = Ra_{FT}$, and with small Bi , $Ra_{eff} = Ra_{FF} = Nu Ra_{FT}$

Consider the relationship

$$Ra_{eff} = Nu^{\lambda(Bi)} Ra_{FT}, \quad (9)$$

where, $\lambda(Bi \rightarrow \infty) = 0$ and $\lambda(Bi \rightarrow 0) = 1$

Find $\lambda(Bi)$ by finding Nu scaling of the form $Nu \propto Ra_{eff}^{\gamma_{Bi}}$, where γ_{Bi} is measured from the data

Finding $\lambda(Bi)$

To find $\lambda(Bi)$, consider the limits on λ ,
Because $Bi \rightarrow \infty$ is equivalent to FT , and $\lambda(Bi \rightarrow \infty) = 0$,

$$Ra_{eff} = Ra_{FT}$$

$$\text{So, } Nu \propto Ra_{eff}^{\gamma_{\infty}} \rightarrow Nu \propto Ra_{FT}^{\gamma_{\infty}}$$

Thus for any Bi ,

$$Ra_{eff} = Nu^{\lambda(Bi)} Ra_{FT}, \text{ and,}$$

$$Nu \propto Ra_{eff} \rightarrow Nu \propto (Nu^{\lambda(Bi)} Ra_{FT})^{\gamma_{Bi}}$$

$$Nu \propto Ra_{FT}^{\frac{\gamma_{Bi}}{1 - \lambda(Bi)\gamma_{Bi}}}$$

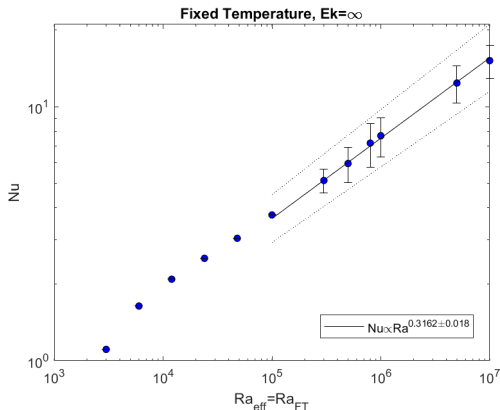
$$\text{So, } \gamma_{\infty} = \frac{\gamma_{Bi}}{1 - \lambda(Bi)\gamma_{Bi}} \text{ and,}$$

$$\lambda(Bi) = \frac{1}{\gamma_{Bi}} - \frac{1}{\gamma_{\infty}}$$

Finding $\lambda(Bi)$

As an example, consider only $Ek = \infty$

Use **FT** data to determine γ_∞ , by fitting a line to the Ra_{FT} vs Nu data with the form $Nu = aRa_{FT}^{\gamma_\infty}$.



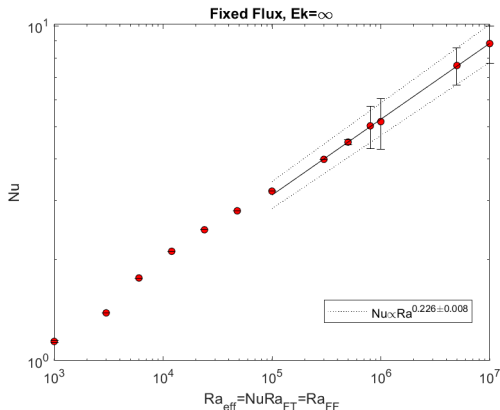
$$\gamma_\infty = 0.32 \pm 0.02$$

It is important to keep track of errors as more calculations are made.

Finding $\lambda(Bi)$

Test $\lambda(Bi) = \frac{1}{\gamma_{Bi}} - \frac{1}{\gamma_{\infty}}$ using FF data, where it is prescribed that $\lambda(0) = 1$.

Anticipate $\gamma_0 \approx 0.24$



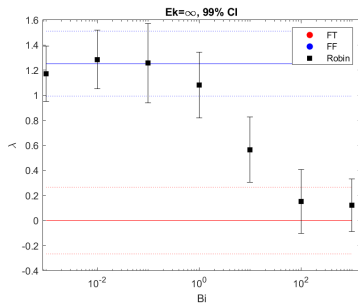
$$\gamma_0 = 0.23 \pm 0.01$$

This range agrees with the predicted scaling

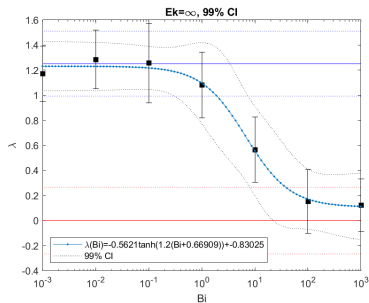
Finding $\lambda(Bi)$

The same fitting process is used to determine a λ value for each Bi with a fixed Ek .

Again using $Ek = \infty$ as an example,



the data points seem to have the form
of $-\tanh(\log(Bi))$,



so data points are fitted as
 $\lambda = A \tanh(B(\log(Bi) + C)) + D$

Finding $\lambda(Bi)$

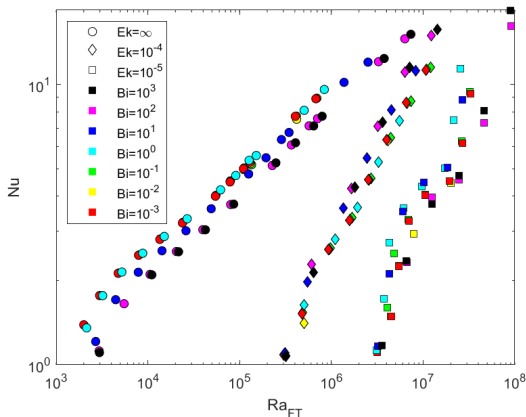
$$\lambda(Bi) = A \tanh(B(\log(Bi) + C)) + D$$

Ek	A	B	C	D
∞	-0.56 ± 0.08	1.2 ± 0.73	-0.83 ± 0.27	0.67 ± 0.07
10^{-4}	-0.36 ± 0.07	0.75 ± 0.47	-0.56 ± 0.35	0.34 ± 0.05
10^{-5}	-0.55 ± 0.09	8.8 ± 0.0	-1.0 ± 0.03	0.65 ± 0.09

Results for $\lambda(Bi)$ coefficients by Ek with 99% confidence interval. For each case, B and C are more uncertain, whereas A and D are more precise. Theory says the $A = -0.5$ and $D = 0.5$. The A criteria is met in the $Ek = \infty$ and 10^{-5} cases, but the D criteria is not met in any case.

Results

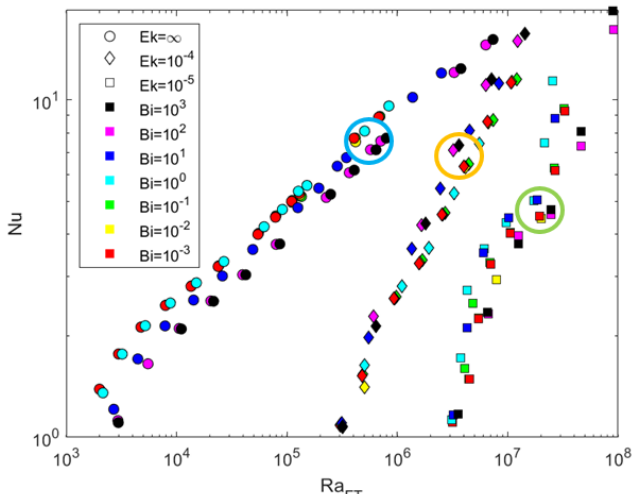
Apply $Ra_{eff} = Nu^{\lambda(Bi)} Ra_{FT}$ to simulation data using $\lambda(Bi)$ determined on previous slide:



Shows reasonable agreement with theory, all Bi cases seem to collapse onto a single $Nu \propto Ra_{FT}$ relationship for each Ek , though 10^{-5} spreads more than other cases

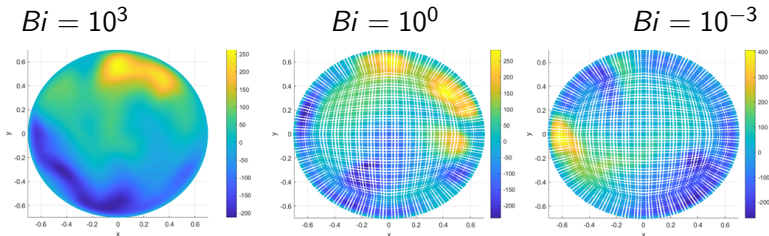
Results

Now, investigate and compare solution forms and time-dependence of system with similar Ra_{FT} for several Bi using vertical velocity plots and Nu time series

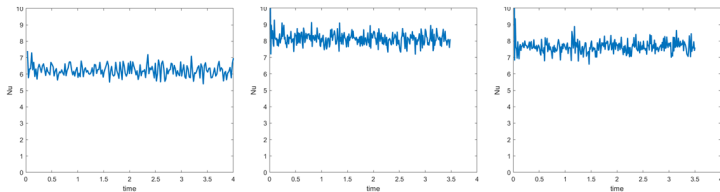


Results

$$Ek = \infty, Ra_{FT} \approx 5 \times 10^5$$



$Bi = 10^{-3}$ and 10^3 seem very similar in structure, $Bi = 10^0$ may be slightly different.

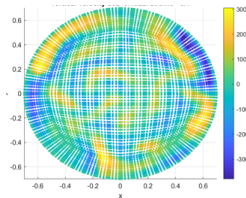


All three cases seem equally time-dependant.

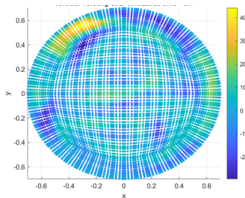
Results

$$Ek = 10^{-4}, Ra_{FT} \approx 3 \times 10^6$$

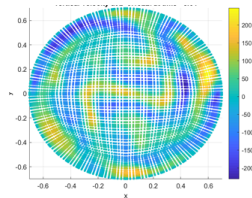
$$Bi = 10^3$$



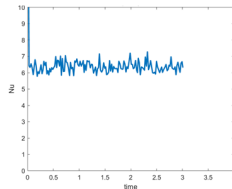
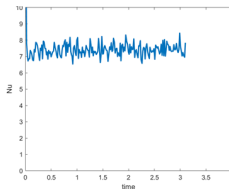
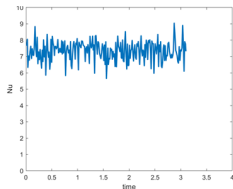
$$Bi = 10^0$$



$$Bi = 10^{-3}$$



$Bi = 10^3$ and 10^{-3} appear similarly chaotic, 10^0 seems more confined to wall modes

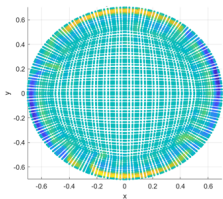


All cases seem similarly time dependant, $Bi = 10^{-3}$ is slightly more chaotic

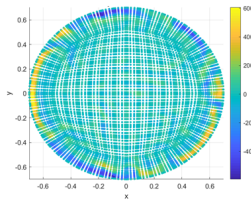
Results

$$Ek = 10^{-4}, Ra_{FT} \approx 3 \times 10^6$$

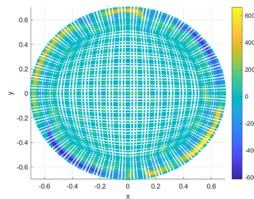
$$Bi = 10^3$$



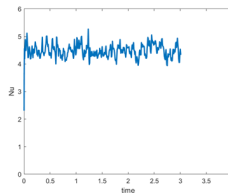
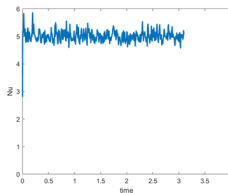
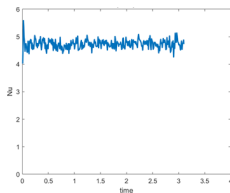
$$Bi = 10^0$$



$Bi = 10^{-3}$



All cases show $m = 2$ wall modes with slight deformation



All cases seem equally time-dependant, $Bi = 10^3$ may be more chaotic

Conclusions and Future Work

- **FT** and **FF** systems can be compared directly using the relationships $Ra_{FF} = Ra_{FT} Nu$
- Generally, **FT**, **M**, and **FF** systems behave similarly, though **M** remains time in-dependant to higher Ra_{FT} than **FF**
- **FF** can have different wall modes and be more chaotic at lower Ek
- Robin boundary conditions lie between **FF** and **FT** reaching them at the extremes of Bi
- Ra_{eff} can be defined using Nu scaling as $Ra_{eff} = Nu^{\lambda(Bi)} Ra_{FT}$
- Initial results suggest Bi away from 0 and ∞ have different solution forms than Bi closer to **FT** and **FF** (further analysis to be done)
- Overall, **FT** is a good approximation of experimental BCs, but becomes less effective as rotation become more rapid
- Future work: Curious case of inhomogeneous sidewall insulation...

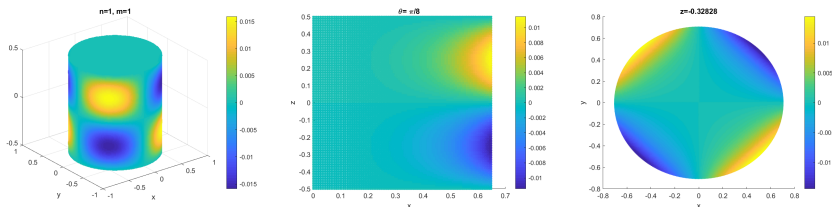
Inhomogeneous Sidewall Insulation

Molten planetary cores can have laterally varying heat flux (Mound and Davies, 2017).

To further understand the effects of inhomogeneous insulation, it is useful to conduct numerical simulations in a cylindrical geometry, which could inform better experimental studies.

Thus, the following thermal flux is applied at the sidewalls of a cylindrical domain,

$$\frac{\partial T}{\partial r} = \sum_{n=1}^{\infty} A_n \sin(k_n z) \sin(m\theta) \quad \text{at } r = 0.7, \quad (10)$$

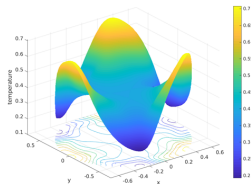


Temperature solution with $n = 1$, $m = 2$, and $A_n = 0.1$ where $\mathbf{u} = 0$ is prescribed

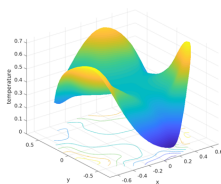
Inhomogeneous Sidewall Example

$$Ek = 10^{-4}, Ra_{FT} = 6.4 \times 10^5, n = 1 \text{ and } m = 1$$

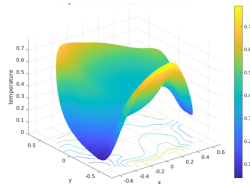
$A_n = 0$
(homogeneous)



$A_n = 1$



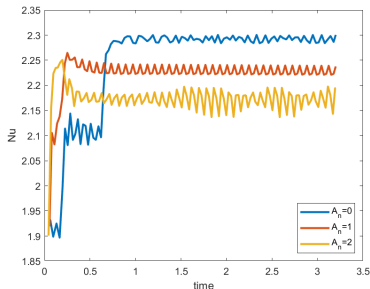
$A_n = 2$



Temperature at $z = 0.1$. As A_n increases, the sidewall modes have more effect on both number of modes and amplitude.

Inhomogeneous Sidewall Example

$$Ek = 10^{-4}, Ra_{FT} = 6.4 \times 10^5, n = 1 \text{ and } m = 1$$



Nu time series suggest that as A_n increases, mean Nu decreases.

The inhomogeneity also seems to allow the system to jump more quickly to the final solution form, unlike the homogeneous case which jumps twice before settling.

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