

Bound Coherent Structures Propagating on the Free Surface of Deep Water

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Initial problem statement

We consider one-dimensional gravitational waves propagating on the free surface of an ideal incompressible deep fluid. The flow is considered to be potential:

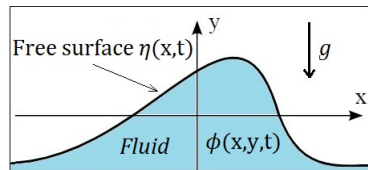
$$\vec{V} = \vec{\nabla} \cdot \phi, \vec{\nabla} \cdot \vec{V} = 0 \Rightarrow \boxed{\Delta \phi = 0}$$

$$-\infty < x < \infty, \quad -\infty < y < \eta(x, t)$$

Boundary conditions:

$$\left[\begin{array}{l} \frac{\partial \psi}{\partial t} + \frac{1}{2} |\nabla \psi|^2 + g\eta = 0 \\ \frac{\partial \eta}{\partial t} + \eta_x \psi_x = \psi_y \end{array} \right] \text{ at } y = \eta(x, t)$$

The system is Hamiltonian, and η и ψ – Hamiltonian canonical variables
[V. E. Zakharov, 1968].



$y = \eta(x, t)$ – the surface shape

$\psi(x, t)$ – velocity potential at the surface

$\phi(x, y, t)$ – velocity potential inside the fluid

Hamiltonian system of equations

$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}, \quad \frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta}$$

$$H = \frac{1}{2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\eta} (\nabla \phi)^2 dy + \frac{g}{2} \int \eta^2 dx$$

Zakharov V. E. Stability of periodic waves of finite amplitude on the surface of a deep fluid // Journal of Applied Mechanics and Technical Physics, 1968

Canonical transformations

In the assumption of small value of wave steepness $\mu = \eta'_x \ll 1$ the Hamiltonian can be expanded in a power series of η and ψ :

Expansion of the Hamiltonian

$$H(\eta, \psi) = \frac{1}{2} \int \{g\eta^2 + \psi \hat{k} \psi\} dx - \frac{1}{2} \int \{(\hat{k} \psi)^2 - (\psi_x)^2\} \eta dx + \frac{1}{2} \int \{\psi_{xx} \eta^2 \hat{k} \psi + \psi \hat{k} (\eta \hat{k} (\eta \hat{k} \psi))\} dx$$

The canonical transformation $\eta, \psi \rightarrow b, b^*$ is aimed at simplifying the original Hamiltonian by removing all, so-called, non-resonant terms.

The simplified Hamiltonian

$$H(b, b^*) = \int w_k b_k b_k^* dk + \frac{1}{2} \int T_{k, k_1}^{k_2, k_3} b_k^* b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk dk_1 dk_2 dk_3$$

The equation of motion is the traditional Zakharov equation:

$$\frac{\partial b_k}{\partial t} + i \frac{\delta H}{\delta b_k^*} = 0$$

Canonical transformations

The unique feature of four-wave interaction was discovered in work [A. I. Dyachenko, V. E. Zakharov, 1994].

$T_{k,k_1}^{k_2,k_3} \equiv 0$ on the resonance manifold with non-trivial solutions.

$T_{k,k_1}^{k_2,k_3} \equiv 0$ if the product of $kk_1k_2k_3 \leq 0$

Physically, it means that a system initially consisting of unidirectional waves will retain this property over time.

From now and on we consider the case of unidirectional waves

In this case it is convenient to use new variables $b, b^* \rightarrow c, c^*$ to rewrite the expression for the Hamiltonian in a simpler form:

The resulting Hamiltonian for unidirectional waves

$$H(c, c^*) = \int c^* \hat{V} c dx + \frac{1}{2} \int \left[\frac{i}{4} \left(c^2 \frac{\partial}{\partial x} c^{*2} - c^{*2} \frac{\partial}{\partial x} c^2 \right) - |c^2| \hat{k} |c^2| \right] dx$$

The equation of motion in such a case can be written in the following form:

$$\frac{\partial c}{\partial t} + \partial_x^+ \frac{\delta H}{\delta c^*} = 0$$

Dyachenko A. I., Zakharov V. E. Is free-surface hydrodynamics an integrable system? // Physics Letters A, 1994

The supercompact Dyachenko-Zakharov equation

The supercompact equation for unidirectional deep water waves

$$\frac{\partial c}{\partial t} + i\hat{\omega}_k c - i\partial_x^+ \left(|c|^2 \frac{\partial c}{\partial x} \right) = \partial_x^+ \left(\hat{k} (|c|^2) c \right)$$

Operators $\hat{k}$ and $\hat{\omega}_k$ are Fourier multipliers by the $|k|$ and $\sqrt{gk}$ respectively.

Operator ∂_x^+ acts as multiplication by $ik\Theta(k)$ in k -space where $\Theta(k)$ — Heaviside's step function

Physical variables of surface $\eta(x)$ and potential $\psi(x)$ can be recovered by using the next expressions:

$$\eta(x) = \eta^{(1)}(x) + \eta^{(2)}(x) + \dots \quad \psi(x) = \psi^{(1)}(x) + \psi^{(2)}(x) + \dots$$

$$\eta^{(1)}(x) = \frac{\hat{k}^{-\frac{1}{4}}}{\sqrt{2g}^{\frac{1}{4}}} [c(x) + c^*(x)] \quad \psi^{(1)}(x) = -i \frac{\hat{k}^{-\frac{3}{4}}}{\sqrt{2g}^{\frac{1}{4}}} [c(x) + c^*(x)]$$

$$\eta^{(2)}(x) = \frac{\hat{k}}{4\sqrt{g}} \left[\hat{k}^{-\frac{1}{4}} c(x) + \hat{k}^{-\frac{1}{4}} c^*(x) \right]^2$$

$$\psi^{(2)}(x) = \frac{i}{2} \left[\hat{k}^{-\frac{1}{4}} c^*(x) \hat{k}^{\frac{1}{4}} c^*(x) - \hat{k}^{-\frac{1}{4}} c(x) \hat{k}^{\frac{1}{4}} c(x) \right] + \frac{1}{2} \hat{H} \left[\hat{k}^{-\frac{1}{4}} c(x) \hat{k}^{\frac{1}{4}} c^*(x) + \hat{k}^{-\frac{1}{4}} c^*(x) \hat{k}^{\frac{1}{4}} c(x) \right]$$

One can also recover steepness $\mu(x)$ knowing the surface $\eta(x)$:

$$\mu(x) = \frac{\partial \eta(x)}{\partial x}$$

A breather solution

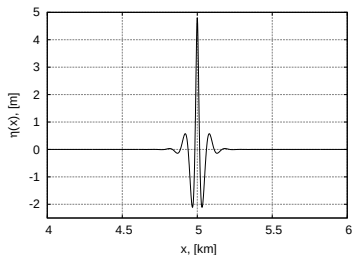
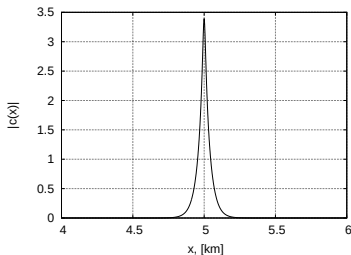
The supercompact equation has a solution in the form of a single breather:

Single breather solution

$$c(x, t) = c_{br}(x - V_0 t; \delta) e^{ik_0 x - i\omega_{k_0} t - i\delta^2 t} \quad \text{where } V_0 = \frac{\omega_{k_0}}{2k_0}$$

The solution can be obtained numerically with the use of the Petviashvili method [V. I. Petviashvili, 1976]. The value of V defines the carrier wave number k , and the value of δ implicitly defines shape and amplitude of the breather.

An example of a breather with $\mu \approx 0.4$ is shown below:



The NLS approximation

The supercompact equation can be written in terms of envelope $C(x, t)$ of wave packet such as $c(x, t) = C(x, t)e^{ik_0 x - i\omega_{k_0} t}$ [A. I. Dyachenko et al., 2017]:

Dyachenko-Zakharov envelope equation

$$\frac{\partial C}{\partial t} + i[\hat{\omega}_{k_0+k} - \omega_{k_0} - \frac{\partial \omega_{k_0}}{\partial k_0} \hat{k}]C - i \frac{\partial^{k_0}}{\partial x} (|C|^2 \frac{\partial^{k_0}}{\partial x} C) = \frac{\partial^{k_0}}{\partial x} (\hat{k} |C|^2 C)$$

$$\text{where } \frac{\partial^{k_0}}{\partial x} = (ik_0 + \frac{\partial}{\partial x}) \hat{\theta}_{k_0+k} = i(k_0 + k) \theta(k_0 + k) \text{ in k-space}$$

Under the additional assumption that the spectral bandwidth is narrow the equation turns to the well-known integrable model for deep water waves – nonlinear Schrödinger equation (NLS):

Nonlinear Schrödinger equation

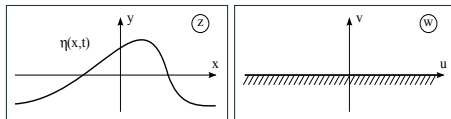
$$\frac{\partial C}{\partial t} + \frac{i\omega_{k_0}}{8k_0^2} \frac{\partial^2 C}{\partial x^2} + ik_0^2 [|C|^2 C] = 0. \quad (1)$$

Equation (1) is written in the reference frame moving with velocity $V_0 = \frac{\omega_{k_0}}{2k_0}$.

Dyachenko A. I., Kachulin D. I., Zakharov V. E. Envelope equation for water waves //Journal of Ocean Engineering and Marine Energy, 2017

Equations in conformal variables

Equations in conformal variables are based on the conformal mapping of fluid-filled domain Z to the lower half-plane of a new complex variable $w = u + iv$.



The transformation is realized by the analytic function $Z = Z(W)$, $Z = x + iy$

In terms of functions $R = \frac{1}{Z_w}$ and $V = i\Phi_z = i\frac{\Phi_w}{Z_w}$ the equations can be written as follows:

RV-equations

$$R_t = i(UR_w - RU_w)$$

$$V_t = i(UV_w - RB_w) + g(R - 1)$$

here $U = \hat{P}(V\bar{R} + \bar{V}R)$, $B = \hat{P}(V\bar{V})$, where $\hat{P} = \frac{1}{2}(1 + i\hat{H})$ – projection operator.

Details can be found in the articles below:

- Dyachenko A. I. et al. Analytical description of the free surface dynamics of an ideal fluid (canonical formalism and conformal mapping) //Physics Letters A, 1996
- Dyachenko A. I. On the dynamics of an ideal fluid with a free surface //Doklady Mathematics. – Pleiades Publishing, Ltd., 2001

Thus, three different models for deep water waves are used to obtain and compare numerical results: the integrable NLS model, the non-integrable supercompact Dyachenko - Zakharov equation — more accurate model compared to the NLS, and the exact model of RV equations written in conformal variables.

The Nonlinear Schrödinger equation

$$\frac{\partial C}{\partial t} + \frac{i\omega_{k_0}}{8k_0^2} \frac{\partial^2 C}{\partial x^2} + ik_0^2 [|C|^2 C] = 0.$$

The supercompact equation

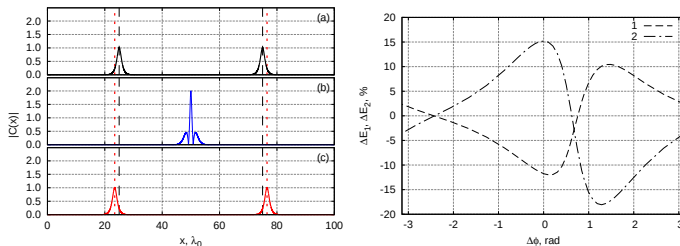
$$\frac{\partial c}{\partial t} + i\hat{\omega}_k c - i\partial_x^+ \left(|c|^2 \frac{\partial c}{\partial x} \right) = \partial_x^+ \left(\hat{k} (|c|^2) c \right)$$

RV-equations

$$\begin{aligned} R_t &= i(UR_w - RU_w) \\ V_t &= i(UV_w - RB_w) + g(R - 1) \end{aligned}$$

Pair collisions of solitons and breathers

- In work [D. I. Kachulin, A. A. Gelash, 2018] pair collisions of the NLS solitons were studied, as well as breathers collision in the supercompact equation.



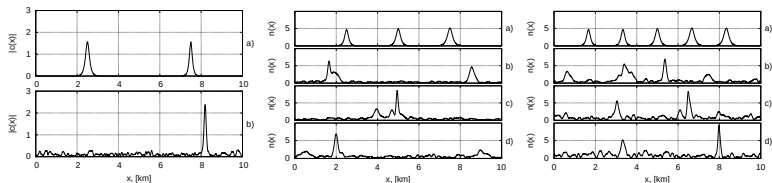
- It was shown that there is an energy exchange between the breathers during their interaction, they also can change their amplitudes and velocities which is not observed for the NLS solitons.

Kachulin D., Gelash A. On the phase dependence of the soliton collisions in the Dyachenko–Zakharov envelope equation //Nonlinear Processes in Geophysics, 2018

What happens to the system when the breathers collide multiple times?

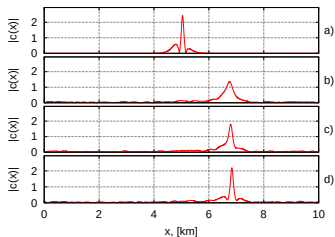
Multiple breathers collisions

- Long-time dynamics of multiple breather collisions in the supercompact equation was studied in work [D. I. Kachulin, A. I. Dyachenko, S. V. Dremov, 2020].



- It was shown that only one breather will remain after large number of collisions regardless of initial number of breathers.

It was revealed that in some cases breathers can bind into one periodically oscillating structure stably existing for a large number of wave periods without losing its energy and resembling the NLS bi-soliton.



Perhaps there are also stable solutions similar to the NLS bi-solitons in the supercompact equation and in the exact model of RV-equations?

The NLS bi-soliton solution

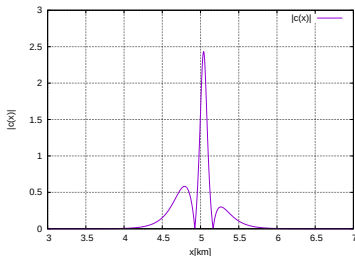
The expression for an exact bound bi-soliton solution of the NLS [V. E. Zakharov, A. B. Shabat, 1972], [D. H. Peregrine, 1983] is as follows :

$$C_{bs}(x, t) = \frac{2 \frac{C_1 - C_2}{C_1 + C_2} \left(C_1 \cosh \left[\frac{2C_2 k_0^2}{\sqrt{\omega_{k_0}}} (x - x_2) \right] e^{-\frac{1}{2} i k_0^2 C_1^2 t} - C_2 \cosh \left[\frac{2C_1 k_0^2}{\sqrt{\omega_{k_0}}} (x - x_1) \right] e^{-\frac{1}{2} i k_0^2 C_2^2 t} \right)}{\left(\frac{C_1 - C_2}{C_1 + C_2} \right)^2 \cosh \left[\frac{2k_0^2}{\sqrt{\omega_{k_0}}} (C_1(x - x_1) + C_2(x - x_2)) \right] + \cosh \left[\frac{2k_0^2}{\sqrt{\omega_{k_0}}} (C_1(x - x_1) - C_2(x - x_2)) \right] - \frac{4C_1 C_2}{(C_1 + C_2)^2} \cos \left[(C_1^2 - C_2^2) \frac{k_0^2}{2} t \right]}$$

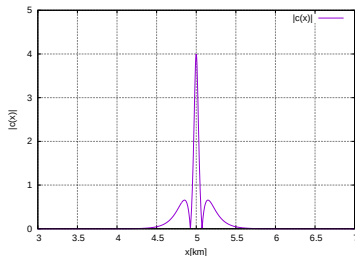
The solution is periodic in time: $T = \frac{4\pi}{k_0^2(C_1^2 + C_2^2)}$; C_1, C_2 – solitons amplitudes, and $\kappa = \frac{C_1}{C_2}$ – amplitude ratio.

The examples of bi-soliton

$x_1 \neq x_2$



$x_1 = x_2$



- Zakharov V., Shabat A. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media //Soviet physics JETP, 1972
- Peregrine D. H. Water waves, nonlinear Schrödinger equations and their solutions //The ANZIAM Journal, 1983

Finding a giant breather

A strong nonlinear solution in the form of a giant breather in the exact equations was discovered in work [A. I. Dyachenko, V. E. Zakharov, 2008]. This discovery was the starting point for a number of scientific works devoted to the study of such objects.

A large number of experiments devoted to the study of such breathers [A. V. Slunyaev, 2009], [A. V. Slunyaev et al., 2013], [A. V. Slunyaev et al., 2017]. The interaction of breathers in exact model of RV-equations was also studied in [D. I. Kachulin, A. I. Dyachenko, A. A. Gelash, 2019].

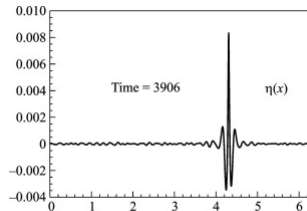


Fig. 5. Typical profile of a breather.

- Dyachenko A. I., Zakharov V. E. On the formation of freak waves on the surface of deep water //JETP letters, 2008
- Slunyaev, A. Numerical simulation of “limiting” envelope solitons of gravity waves on deep water. J. Exp. Theor. Phys. 2009
- Slunyaev, A.; Clauss, G.F.; Klein, M.; Onorato, M. Simulations and experiments of short intense envelope solitons of surface water waves. Phys. Fluids 2013
- Slunyaev, A.; Klein, M.; Clauss, G. Laboratory and numerical study of intense envelope solitons of water waves: Generation, reflection from a wall, and collisions. Phys. Fluids 2017
- Kachulin D., Dyachenko A., Gelash A. Interactions of coherent structures on the surface of deep water //Fluids, 2019

Bound structure obtaining

The algorithm used for obtaining bound structures is similar to the one used in work [A. I. Dyachenko, V. E. Zakharov, 2008]. Roughly speaking, it consists of taking the known NLS soliton solution as initial conditions for more accurate models and see what happens to it. In the process of time evolution such a soliton begins to radiate incoherent waves which are suppressed by a damping procedure.

The damping procedure was carried out by adding the appropriate term to the right-hand side of the corresponding equation:

Damping procedure

$$\frac{\partial c(x, t)}{\partial t} = \dots - \gamma f(x) c(x, t)$$

$$\text{where } f(x) = \cos^6\left(\frac{\pi x}{L}\right)$$

$L = 10 \text{ km}$ — length of the periodic domain

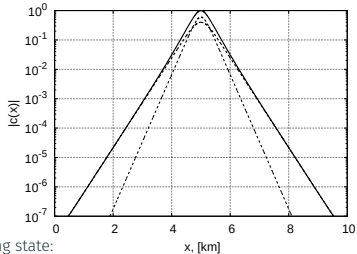
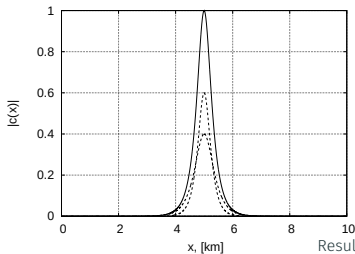
Damping coefficient γ and function $f(x)$ suppress radiation at the edges of the computational domain and do not affect the structure itself which is always located in the center of the domain. To keep the bound structure in the center, the velocity of the reference frame V_0 is constantly adjusted during the calculation.

Dyachenko A. I., Zakharov V. E. On the formation of freak waves on the surface of deep water //JETP letters, 2008

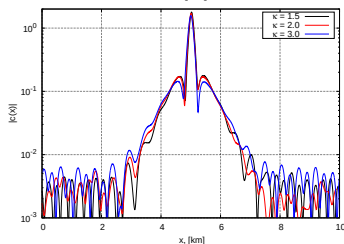
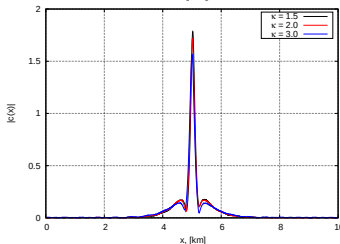
Numerical experiments with two single breathers

The initial state is set by the sum (solid line) of two separate breathers (dashed lines) placed in one point of the domain. It will look the same for all κ values.

Initial state:



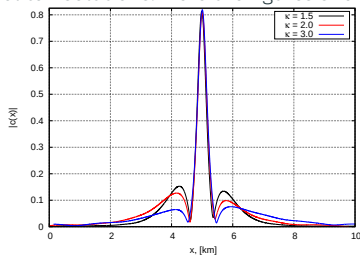
Resulting state:



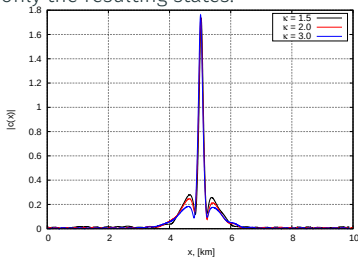
Remained after damping procedure, the structures stably propagate on the free surface for at least 10^6 s. $\approx 100000 T_0$ where $T_0 = \frac{2\pi}{\sqrt{gk_0}} \approx 10$ s. – wave period, $\mu \approx 0.2$

Numerical experiments with the NLS bi-solitons:

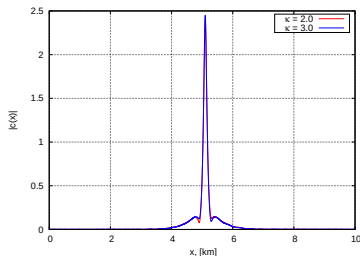
Another way to obtain the bound structure is to set initial conditions using the NLS bi-soliton solutions. Here the figures show only the resulting states.



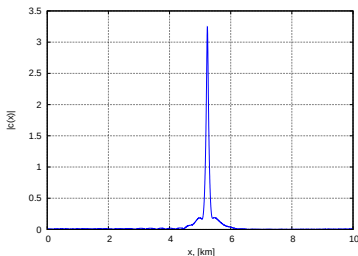
$\mu \approx 0.09$



$\mu \approx 0.18$



$\mu \approx 0.26$

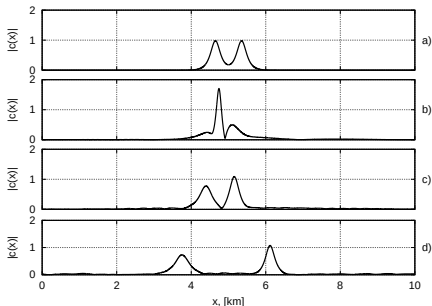


$\mu \approx 0.33$

It can be seen that as μ increases, the dependence on κ weakens.

Splitting of bound state

- There are cases when obtaining the bound structure becomes impossible using neither two single breathers nor NLS-solitons. This occurs when steepness μ exceeds the value of 0.5 (RV-equations).

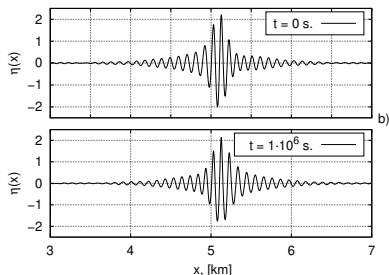
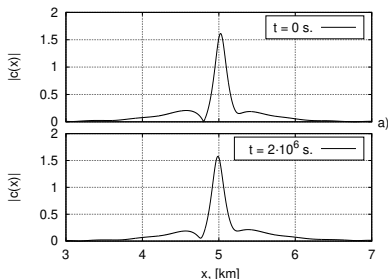


Splitting of the NLS bi-soliton

- In the case of using the NLS bi-solitons the bound structure cannot be obtained at amplitude ratios $\kappa \leq 1.1$, not with any steepness. Thus, solitons or breathers do not bind into the structure at close or the same amplitudes.

The bound structure in RV-equations

Bound structures were also obtained in RV-equations. Firstly, the structure was obtained in the supercompact equation and it exists there for a $2 \cdot 10^6$ s.



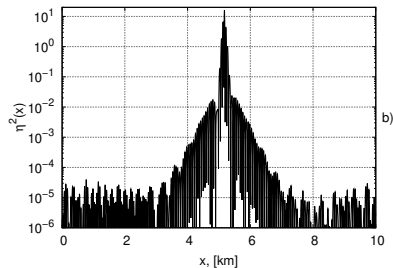
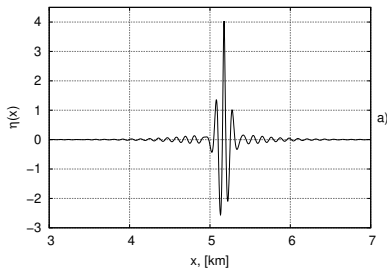
Procedure for setting initial conditions for RV-equations:

$$c(x, t), c^*(x, t) \rightarrow \eta(x, t), \psi(x, t) \rightarrow R, V$$

After completing the calculation in the supercompact equation the resulted structure was substituted into exact RV-equations where it also propagated stably for at least 10^6 s., the radiation level did not change significantly.

The bound structure in RV-equations

The structure can also be obtained by using the NLS bi-solitons or two single breathers. The example of the resulting structure with $\mu \approx 0.25$ and amplitude ratio $\kappa = 1.5$ is presented below.



The structure stably propagates for at least 10^6 s., the radiation level does not change significantly.

The obtained bound structures may turn out to be new solutions to the equations used.

Results:

- In the model of supercompact equation and in the full system of nonlinear equations written in conformal variables bound structures stably propagating on the free surface of the fluid for at least hundred of thousand of wave periods were obtained.
- To specify the initial conditions for obtaining such structures two single breathers found by using the Petviashvili method and placed at one point of the domain were used, as well as well-known bi-soliton solutions of the NLS.
- The structures were obtained in wide range of determining parameters — steepness and amplitude ratio. Also, the conditions under which it is impossible to obtain a bound structure have been identified.
- The results are included in the article. More information, including evolution films of the structure, can be found there.

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