Accelerating climate- and weather-forecasts with faster multigrid solvers

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Introduction

Current:

- ENDGame dynamical core
- Latitude-longitude (structured) grid
- Semi-implicit semi-Lagrangian **finite difference** method
- **10km** horizontal resolution, 70 vertical levels
- Approx. **400 million** degrees of freedom per diagnostic
- Require 5 day forecast in an hour, **~20 prognostics**

Next Gen:

- GungHo dynamical core
- Cube sphere (unstructured) grid
- Semi-implicit Eulerian **finite element** method
- Goal of **1km** horizontal resolution, 70 vertical levels
- Still require 5 day forecast in an hour
- Approx. **35 billion** unknowns per field

Current solver implementation

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} \mathrm{Id} & -A_{11}^{-1}A_{12} \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & S^{-1} \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ -A_{21}A_{11}^{-1} & \mathrm{Id} \end{pmatrix}$$

$$S^{-1} = A_{22} - A_{21}A_{11}^{-1}A_{12}$$

- Huge system → Iterative method
- Difficult to precondition as it is **not elliptic** → Schur complement.

$$\begin{pmatrix} M_u & -g\frac{\Delta t}{2}D^T \\ H\frac{\Delta t}{2}D & M_\phi \end{pmatrix} \begin{pmatrix} \delta u \\ \delta \phi \end{pmatrix} = \begin{pmatrix} R_u \\ R_\phi \end{pmatrix} \Leftrightarrow S\delta\phi = R$$

$$S \equiv M_\phi + \frac{gH\Delta t^2}{4} D M_u^{-1} D^T$$

$$R \equiv R_\phi - \frac{H\Delta t}{2} D M_u^{-1} R_u$$

$$\delta u = M_u^{-1}\left(R_u + g\frac{\Delta t}{2} D^T \delta\phi\right)$$

- Schur complement is elliptic, but M_u^{-1} is **dense** → Approximate Schur complement.

$$\tilde{S}\delta\phi = \tilde{R}$$

$$\tilde{S} \equiv M_\phi + \frac{gH\Delta t^2}{4} D \tilde{M}_u^{-1} D^T$$

$$\tilde{R} \equiv R_\phi - \frac{H\Delta t}{2} D \tilde{M}_u^{-1} R_u$$

$$\delta u \approx \tilde{M}_u^{-1}\left(R_u + g\frac{\Delta t}{2} D^T \delta\phi\right)$$

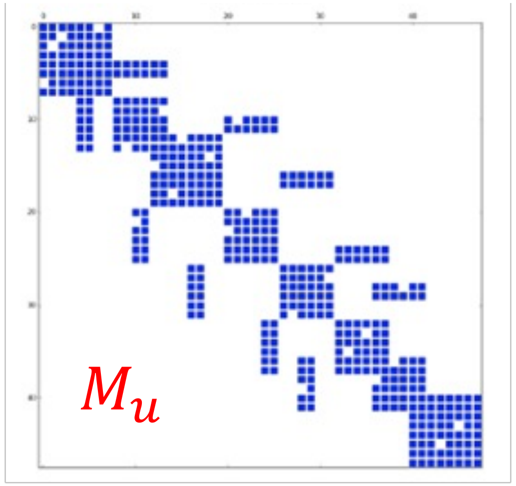


Image Source: Thomas Gibson

  - This means more iterations of the (expensive) outer solver.
  - Motivates looking for different methods.

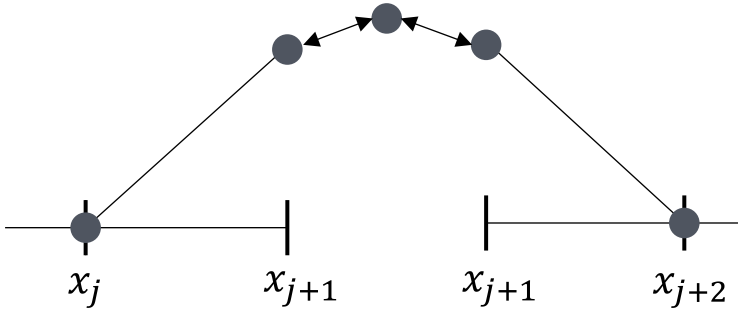
Hybridised solver implementation

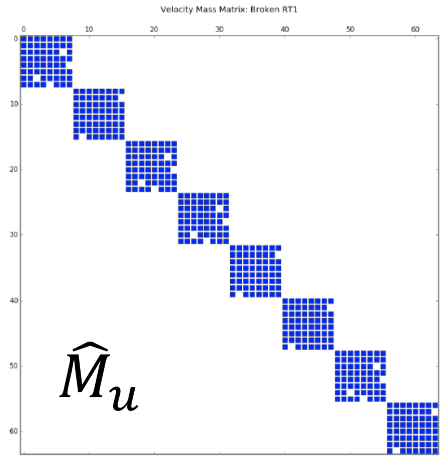
- Break the basis functions → cell-local
- Bigger system, but $\widehat{M}_u$ **block diagonal**!
- This means A is block diagonal → **Schur complement**:

$$\begin{pmatrix} A & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} X \\ \Lambda \end{pmatrix} = \begin{pmatrix} \hat{R}_X \\ 0 \end{pmatrix} \Leftrightarrow S\Lambda = R$$

$$S \equiv CA^{-1}C^T$$
$$R \equiv CA^{-1}\hat{R}_X$$
$$X = A^{-1}(\hat{R}_X - C^T\Lambda)$$

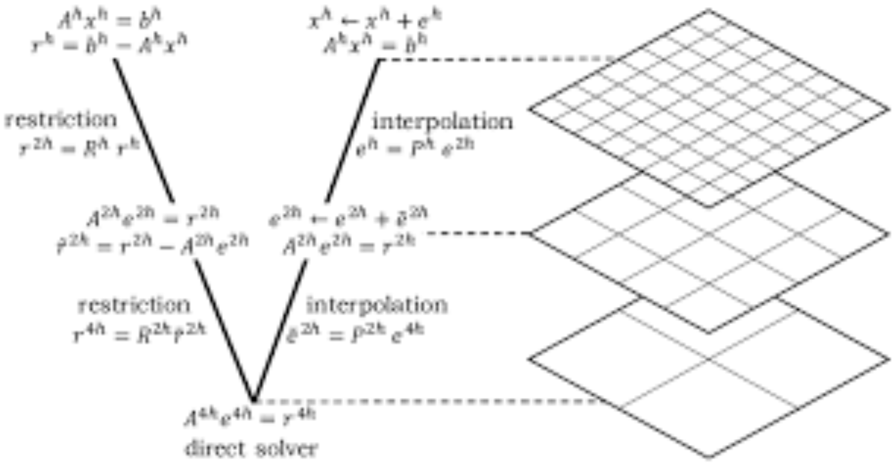
- A can be **inverted exactly**, S is **block sparse**.
- We need only **solve** for the Lagrange multipliers Λ.
- No approximations are made!





Standard multigrid

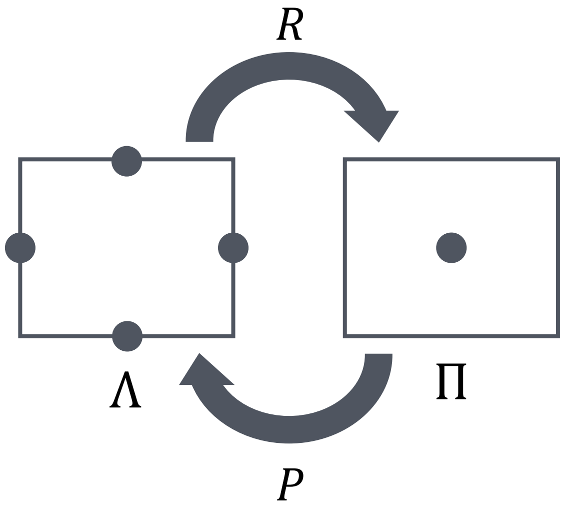
- Pre-smooth + Residual computation
- Restriction
- Solve coarse system
- Prolongate
- Correction + Post-smooth



Non-nested multigrid

- Modify the restriction and prolongation maps to account for the **non-nested** grids:

  - Pre-smooth
  - Restrict to pressure space
  - Solve Helmholtz (pressure) equation (e.g. standard multigrid)
  - Prolongate to trace space (Lagrange multiplier space)
  - Post-smooth



[1] Gopalakrishnan, J. and Tan, S., 2009. A convergent multigrid cycle for the hybridized mixed method. Numerical Linear Algebra with Applications, 16(9), pp.689-714.

The different solver configurations

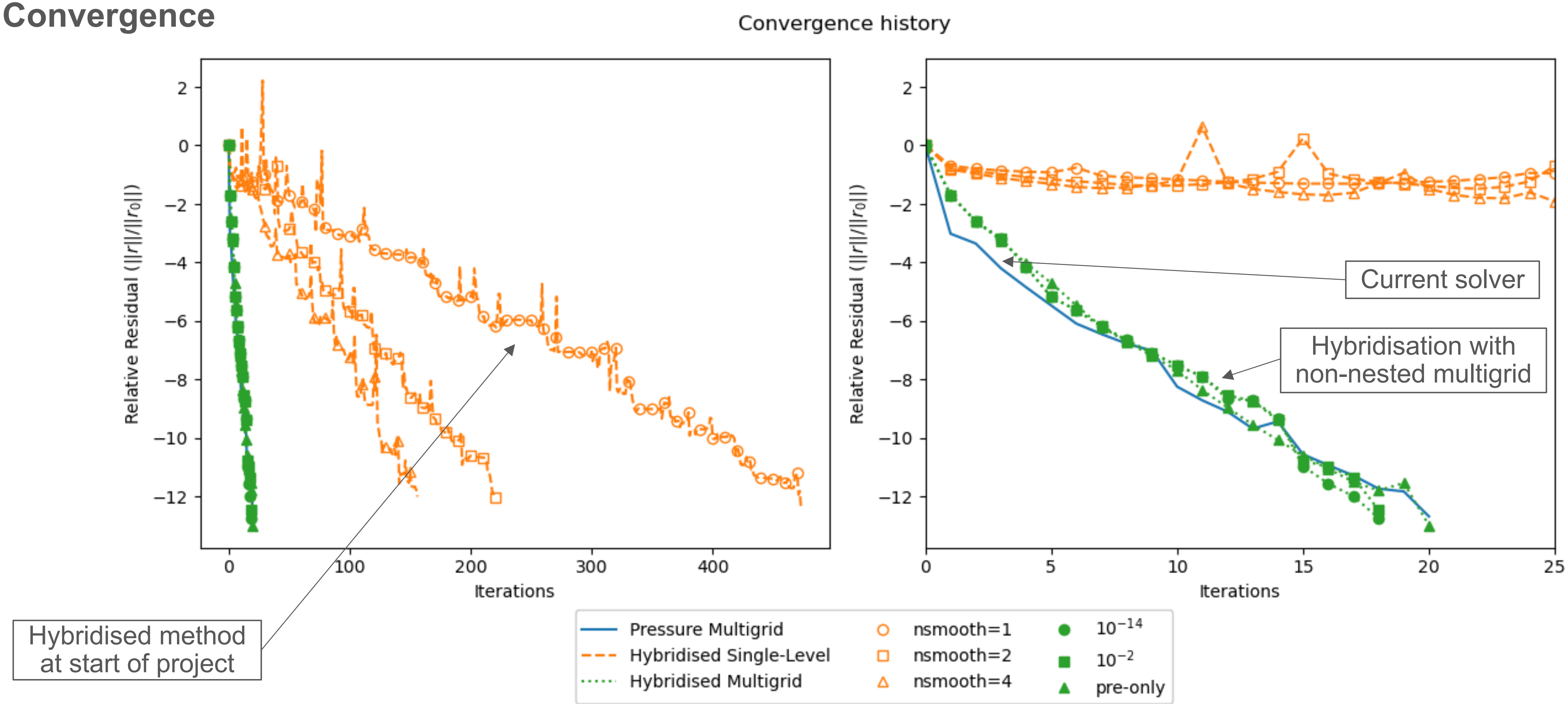
Configuration 1 – Multigrid with approximate Schur complement (current method)

Configuration 2 – Hybridisation with linesmoother only (before project)

Configuration 3 – Hybridisation with non-nested multigrid (this project)

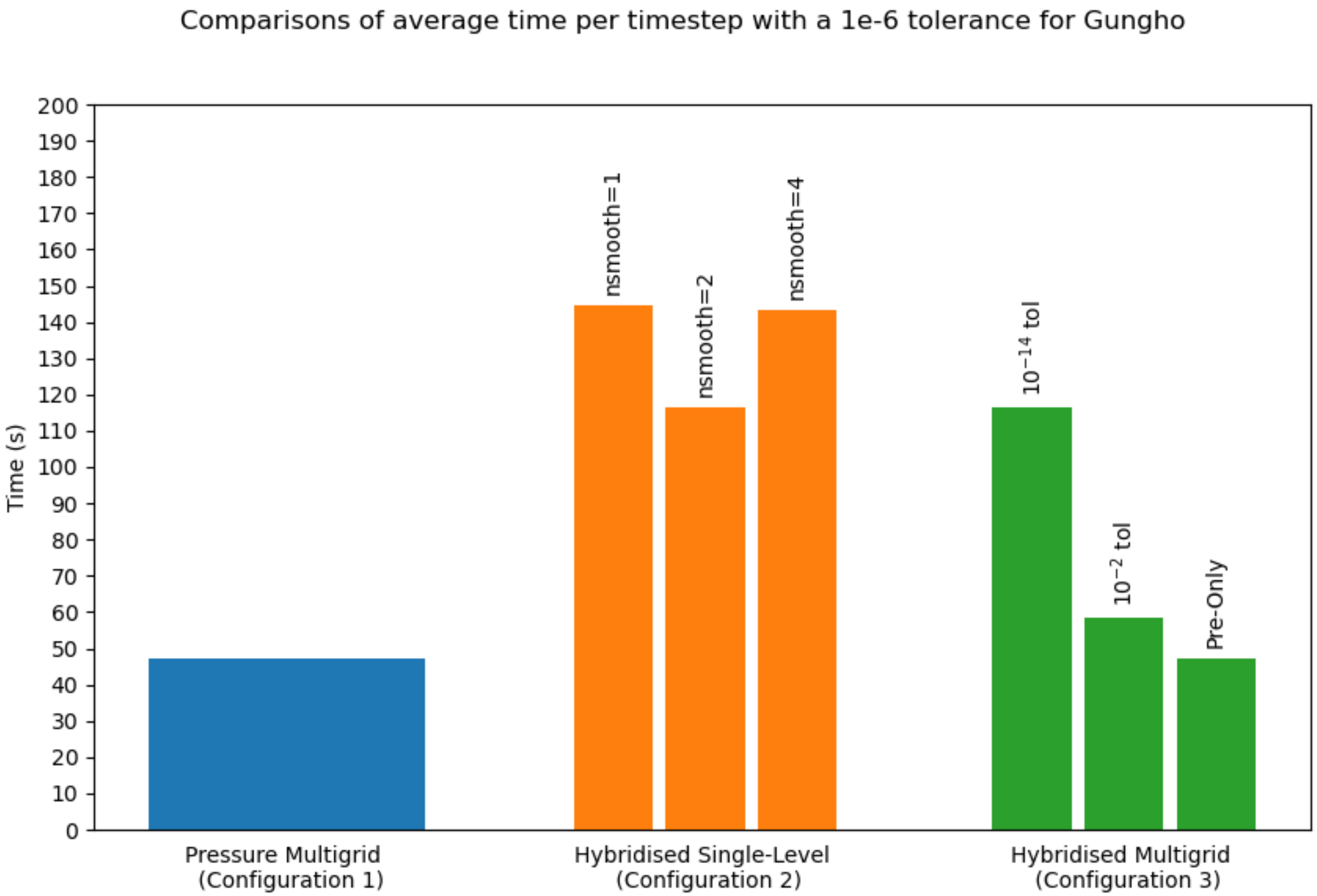
Results

Convergence



Results

Time to solution



- Close to current solver algorithm time with room for optimisation

Conclusions

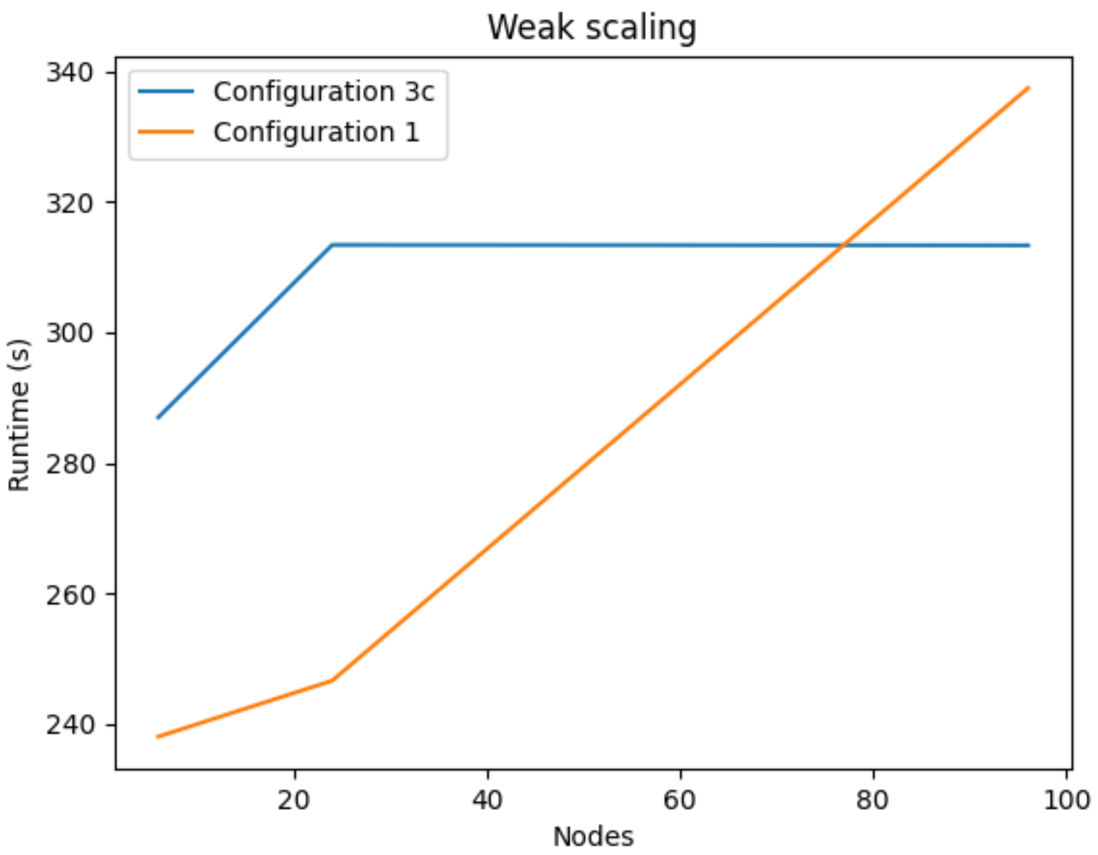
- Implemented a hybridised solver algorithm
- Combined this with the pre-existing multigrid preconditioner
- Implemented a non-nested multigrid solver algorithm
- The new solver algorithm performs comparably to the current solver with room for optimisation

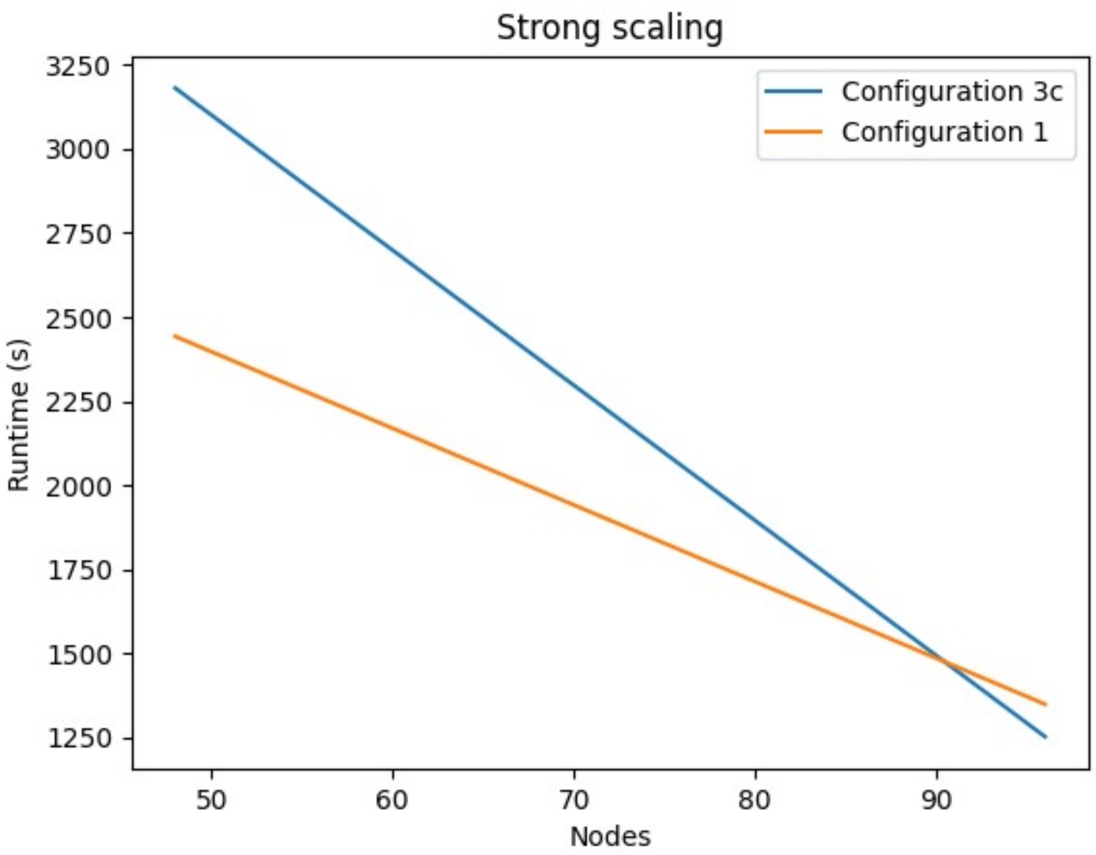
Future Work

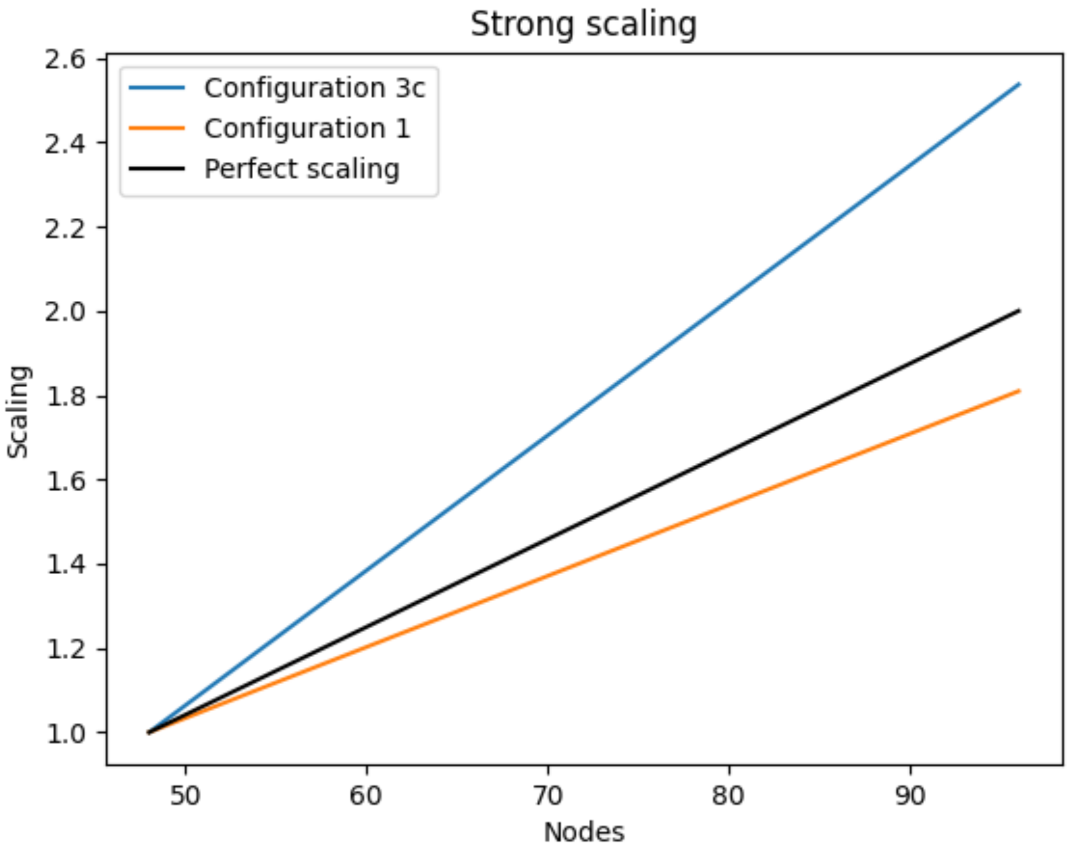
- Higher order finite elements → favours hybridisation
- Test parallel scaling of method
- Try an algebraic multigrid preconditioner

degree	number of unknowns per cell						cell-local matrix size	
p	N_p	+	N_u	=	N_{mixed}	N_λ	mixed	hybridised
0	1	+	3	=	4	3	7×7	6×6
1	8	+	24	=	32	12	44×44	24×24
2	27	+	81	=	108	27	135×135	54×54
3	64	+	192	=	256	48	304×304	96×96

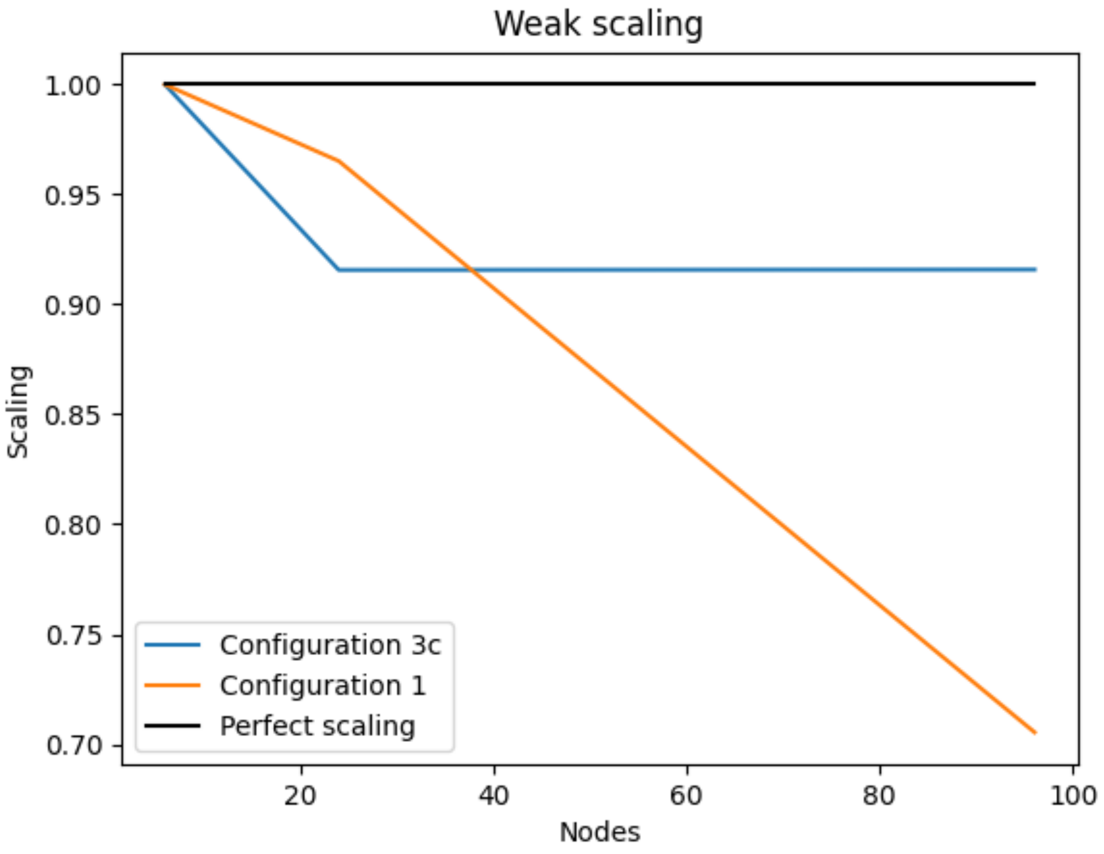
Thank you for listening!
Any questions?


Weak scaling
Configuration 3c
Configuration 1
Runtime (s)
Nodes
340
320
300
280
260
240
20
40
60
80
100

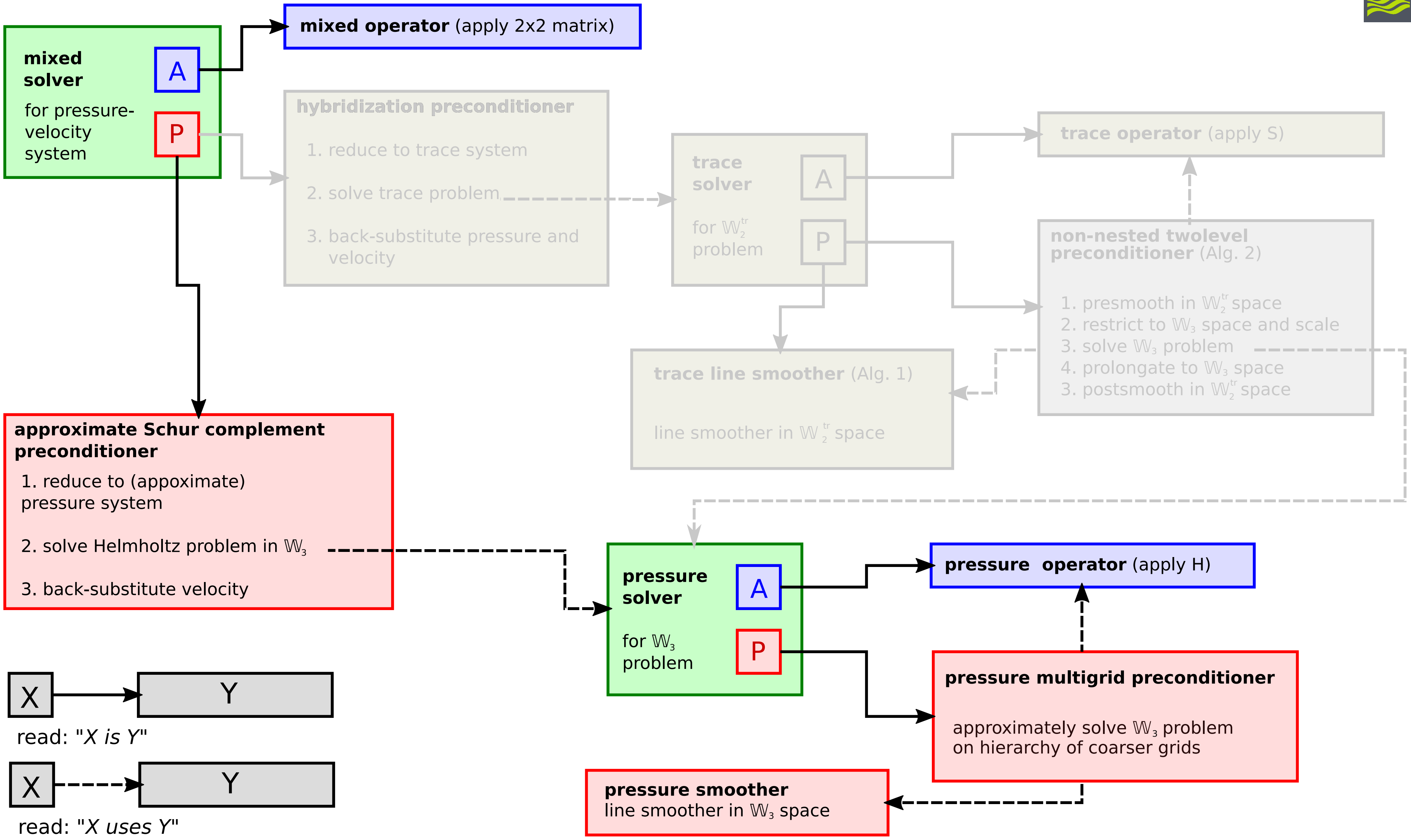

Strong scaling
Configuration 3c
Configuration 1
Runtime (s)
Nodes
3250
3000
2750
2500
2250
2000
1750
1500
1250
50
60
70
80
90

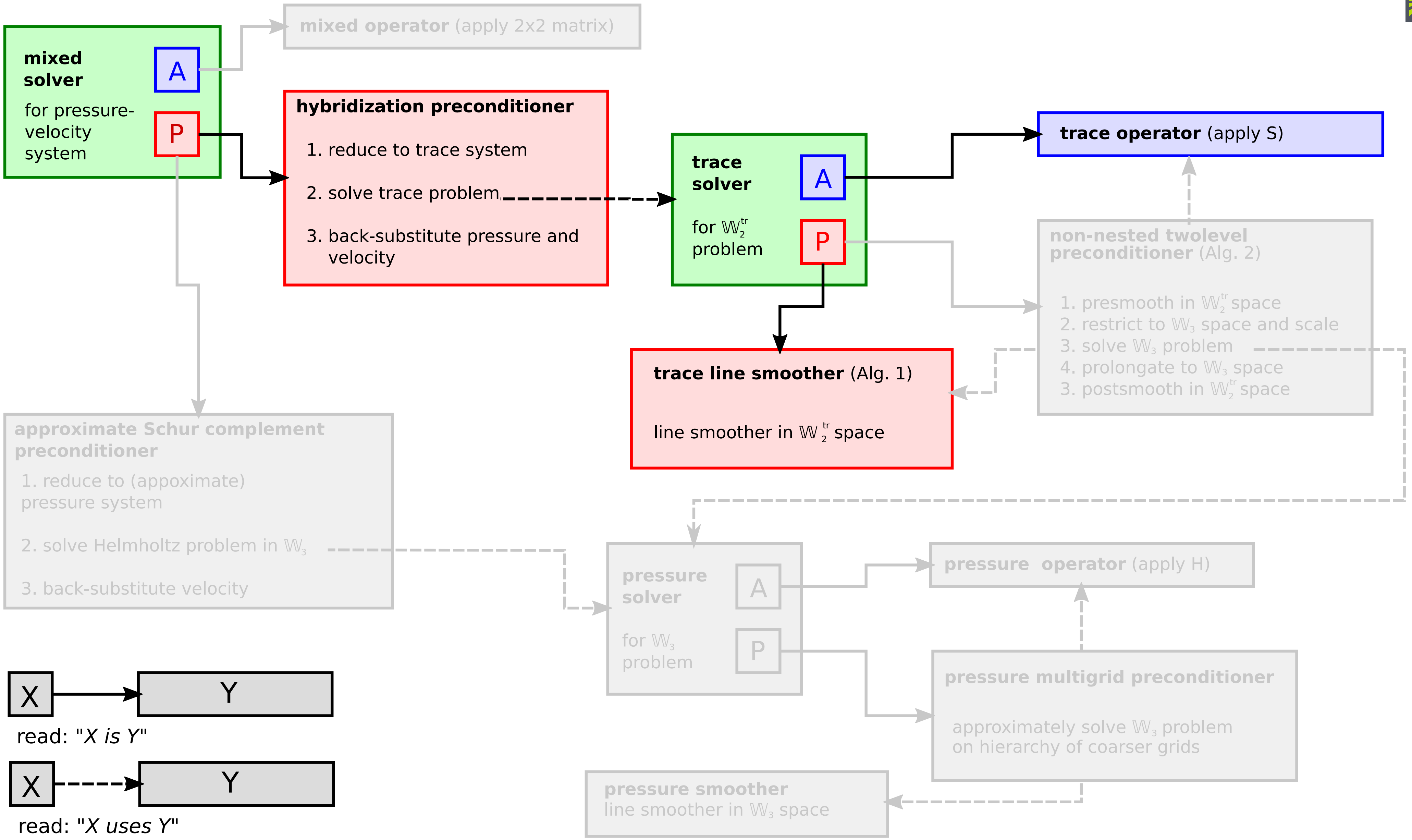


Strong scaling
Configuration 3c
Configuration 1
Perfect scaling
Scaling
Nodes
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
50
60
70
80
90



Weak scaling
Configuration 3c
Configuration 1
Perfect scaling
Scaling
Nodes
1.00
0.95
0.90
0.85
0.80
0.75
0.70
20
40
60
80
100


mixed solver
for pressure-velocity system
A
P
mixed operator (apply 2x2 matrix)
hybridization preconditioner
1. reduce to trace system
2. solve trace problem
3. back-substitute pressure and velocity
trace solver
for $\mathbb{W}_2^{tr}$ problem
A
P
trace operator (apply S)
non-nested twolevel preconditioner (Alg. 2)
1. presmooth in $\mathbb{W}_2^{tr}$ space
2. restrict to $\mathbb{W}_3$ space and scale
3. solve $\mathbb{W}_3$ problem
4. prolongate to $\mathbb{W}_3$ space
3. postsmooth in $\mathbb{W}_2^{tr}$ space
trace line smoother (Alg. 1)
line smoother in $\mathbb{W}_2^{tr}$ space
approximate Schur complement preconditioner
1. reduce to (appoximate) pressure system
2. solve Helmholtz problem in $\mathbb{W}_3$
3. back-substitute velocity
pressure solver
for $\mathbb{W}_3$ problem
A
P
pressure operator (apply H)
pressure multigrid preconditioner
approximately solve $\mathbb{W}_3$ problem on hierarchy of coarser grids
pressure smoother
line smoother in $\mathbb{W}_3$ space
X
Y
read: "*X is Y*"
X
Y
read: "*X uses Y*"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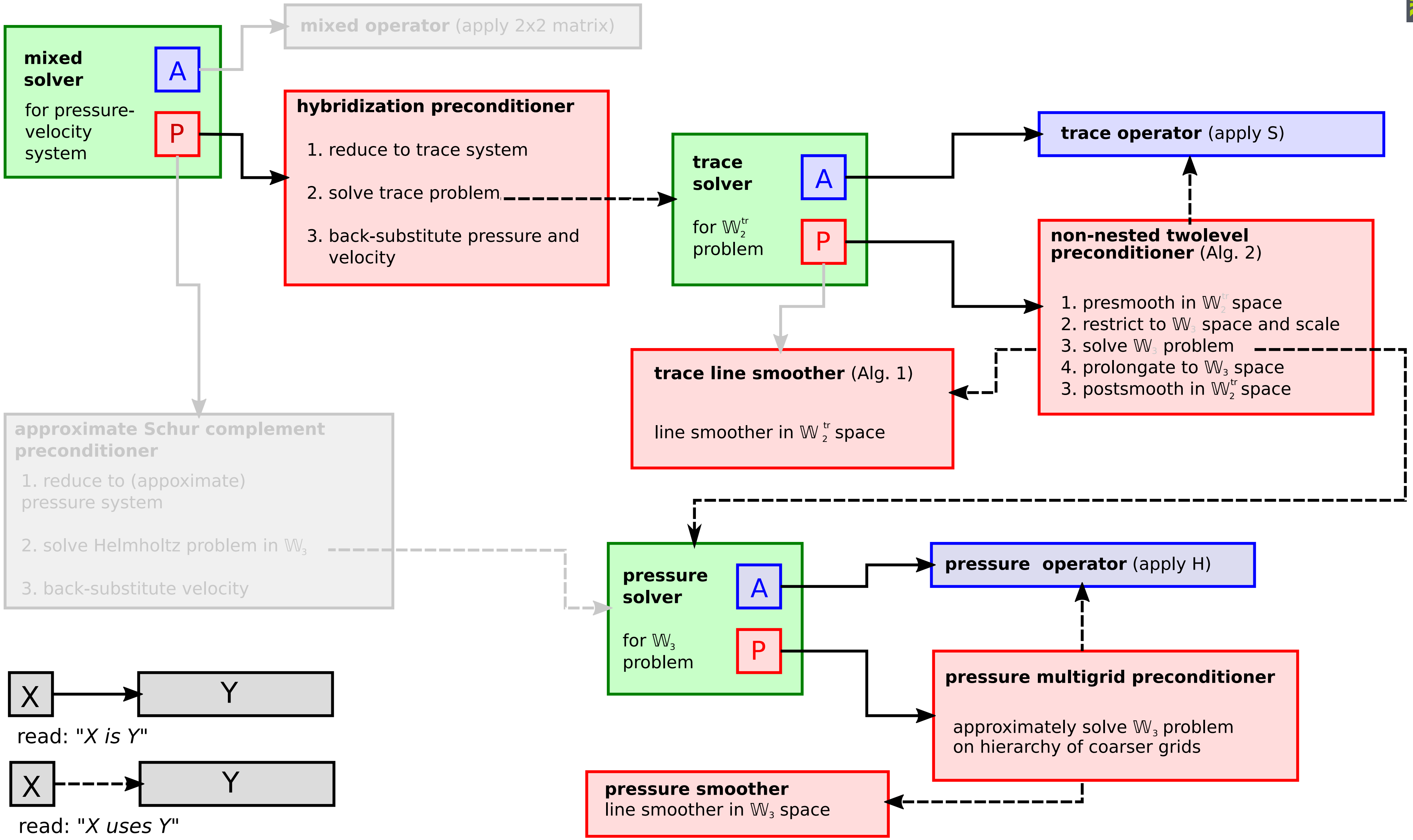
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