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The matrix model for the barotropic equation, connections to variational discretizations, and generalizations to the shallow water equations

Clauson Carvalho da Silva and Christian Lessig,
Otto-von-Guericke-Universität Magdeburg

From Modin and M. Viviani. Canonical scale separation in two-dimensional incompressible hydrodynamics, 2021.

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Moyal bracket from phase space formulation of quantum mechanics

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- Model is not derived from barotropic vorticity equation
  - Analogue to continuous model with weak convergence argument[1]
  - Disconnected from other approaches, e.g. Pavlov[2], ...

[1] M. Bordemann, J. Hoppe, P. Schaller, and M. Schlichenmaier. gl(∞) and geometric quantization. Communications in Mathematical Physics, 138(2):209–244, 1991.

[2] D. Pavlov, P. Mullen, Y. Tong, E. Kanso, J. E. Marsden, and M. Desbrun. Structure-preserving discretization of incompressible fluids. Physica D: Nonlinear Phenomena, 240(6):443–458, mar 2011.

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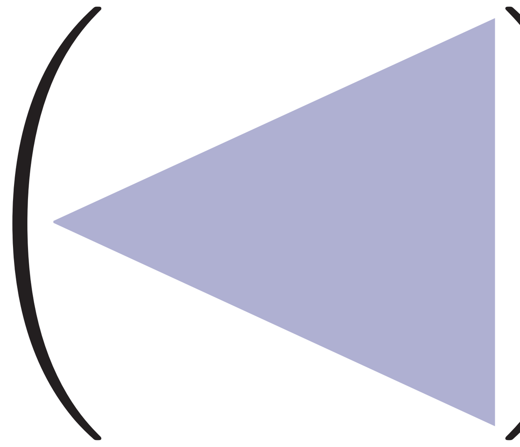
$\bar{\zeta} = \Big($  $\Big)$

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Distinguised orbits: $|\omega| \in \mathbb{Z}$

Geometry

F. A. Berezin. Some remarks about the associated envelope of a lie algebra. Functional Analysis and Its Applications, 1(2):91–102, 1967.

Irreducible representations (algebra/group)

- Group algebra: $\big(H_l(\mathrm{SO}(3)), *\big)$
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Approximation of $\mathfrak{X}_{\mathrm{div}}(S^2)$ in matrix model through uni-reps up to *l*. Discrete time evolution equation through *un-constrained* Euler-Poinaré reduced variation principles.

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Summary

- Matrix model can be understood through representation theory of $\mathfrak{so}(3)$
  - Matrix model is infinitesimal representation on $\mathbb{C}^n \otimes (\mathbb{C}^n)^*$
  - Connection to work by Pavlov, Gawlik, ...
  - Analogous situation for the matrix model for the torus: representation theory of (discrete) Heisenberg group
- Representation theoretic interpretation opens up avenue for extensions