Long Time Steps for Advection: MPDATA with implicit time stepping

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Motivation

- Atmospheric models can have severe time-step restrictions due to advection
  - poles of the latitude-longitude grid
  - large updrafts and small Δz
  - unusually strong wind
- Solved using semi-Lagrangian advection at the expense of conservation
- Can advective time-step restrictions be removed using implicit time-stepping?

Requirements

Motivated by the knowledge that the Courant number (c) is only high in small regions.

- Accuracy and cost comparable with explicit time stepping where $c \leq 1$
- Stable and at least first-order accurate for $c > 1$
- Monotone $\forall\ c$
- Arbitrary meshes
- Exact local conservation
- Multi-tracer efficient
- Good parallel scaling
- Can implicit time stepping be used locally?

Order Barrier

Gottlieb et al. [2001] – no implicit method exists with order > 1 which is monotonic $\forall\, \Delta t$.

Proposed Solution

Revert to first-order accuracy locally where the Courant number is high.

Question

Consider the time-stepping scheme for advection with off-centering $\theta \in [0,1]$ $(\nabla \cdot \mathbf{u} = 0)$:

$$\psi^{n+1} = \psi^n - \Delta t\,(1-\theta)\,\nabla \cdot (\mathbf{u}\psi)^n - \Delta t\,\theta \nabla \cdot (\mathbf{u}\psi)^{n+1} \tag{1}$$

If θ varies in space, can we maintain conservation and monotonicity?

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If θ varies in space, can we maintain conservation and monotonicity?
For conservation, θ must be inside the divergence:

$$\psi^{n+1} = \psi^n - \Delta t\nabla \cdot ((1-\theta)\,\mathbf{u}\psi^n) - \Delta t\nabla \cdot \left(\theta\mathbf{u}\psi^{n+1}\right) \tag{2}$$

But now the advecting velocities, $(1-\theta)\,\mathbf{u}$ and $\theta\mathbf{u}$ are divergent.
Does this generate spurious oscillations?

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But now the advecting velocities, $(1-\theta)\,\mathbf{u}$ and $\theta\mathbf{u}$ are divergent.
Does this generate spurious oscillations?
If the spatial discretisation is first-order accurate, it is straightforward to prove that (2) is monotonic when

$$\theta \geq 1 - \frac{1}{c} \tag{3}$$

Infinite Guage MPDATA extension for implicit/explicit ($\theta \in [0,1]$)

$$\theta = \max\left(1 - \frac{1}{c + 0.25},\ 0\right) \tag{4}$$

Diffusive implicit/explicit update:

$$\psi^d = \psi^n - \Delta t \nabla\cdot\left((1-\theta)\,\mathbf{u}\psi^n_{\text{up}}\right) - \Delta t \nabla\cdot\left(\theta\mathbf{u}\psi^d_{\text{up}}\right) \tag{5}$$

Space and time second-order explicit correction:

$$\psi^{n+1} \;=\; \psi^d + \Delta t \nabla\cdot\left\{\frac{\Delta x}{2}\nabla\psi\cdot\hat{\mathbf{n}} - \chi\frac{\Delta t}{2}\mathbf{u}\cdot\nabla\psi\right\}\mathbf{u}\cdot\hat{\mathbf{n}} \tag{6}$$

where $\chi \;=\; 1 - 2\theta$ for second-order accuracy

$\chi \;\geq\; 0$ for stability

the divergence is finite volume Gauss's divergence theorem

gradients use centred differences

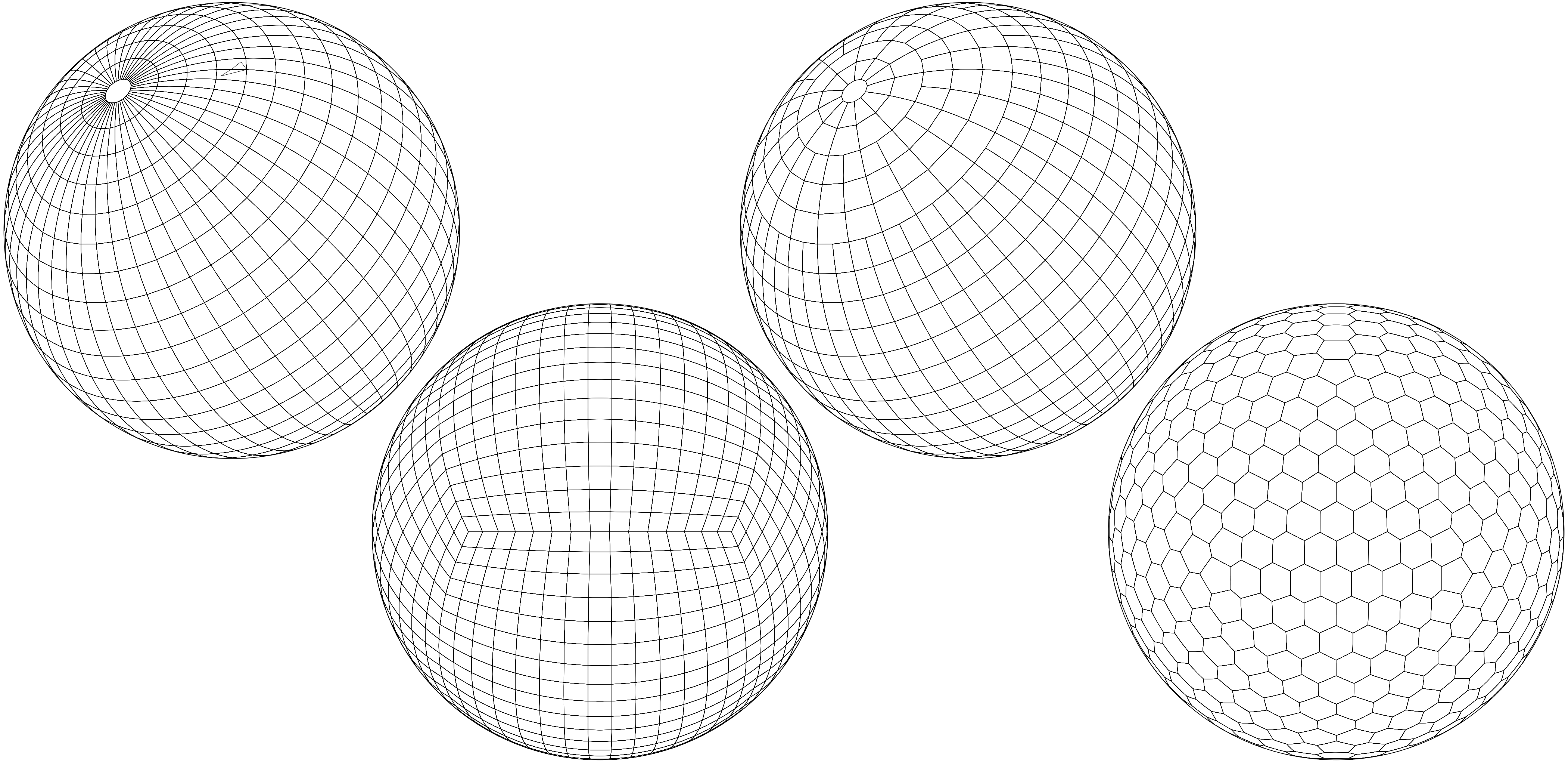
$\hat{\mathbf{n}}$ is the normal to each cell face

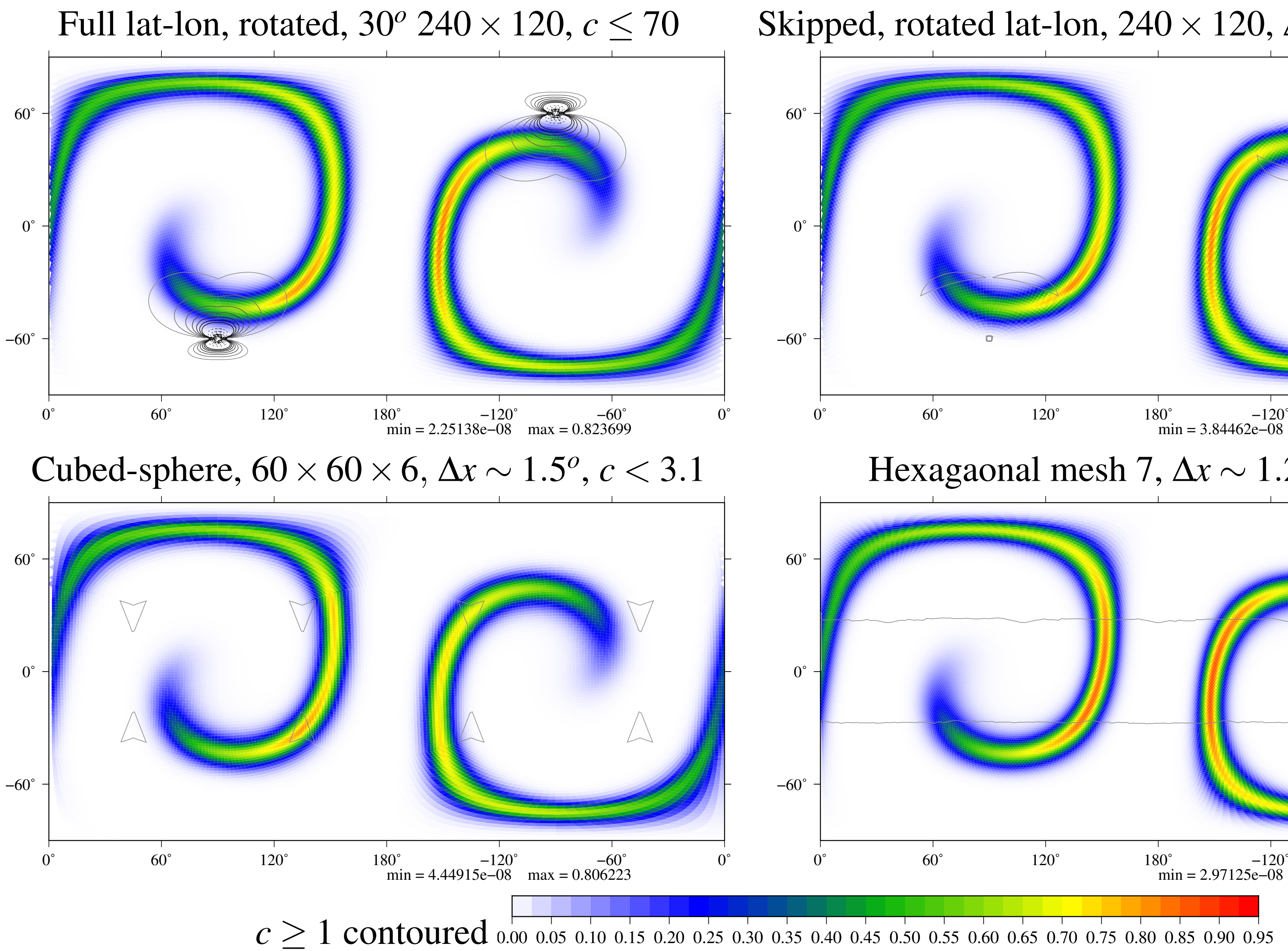
Δx is the cell centre to cell centre distance

Flux Corrected Transport for Monotonicity

Zalesak [1979] is applied but with a crucial difference:
The old time step solution cannot be used to provide local bounds on the solution because, with large Courant numbers, the old time step solution is not local.
Only the diffusive solution can be used to provide local bounds.

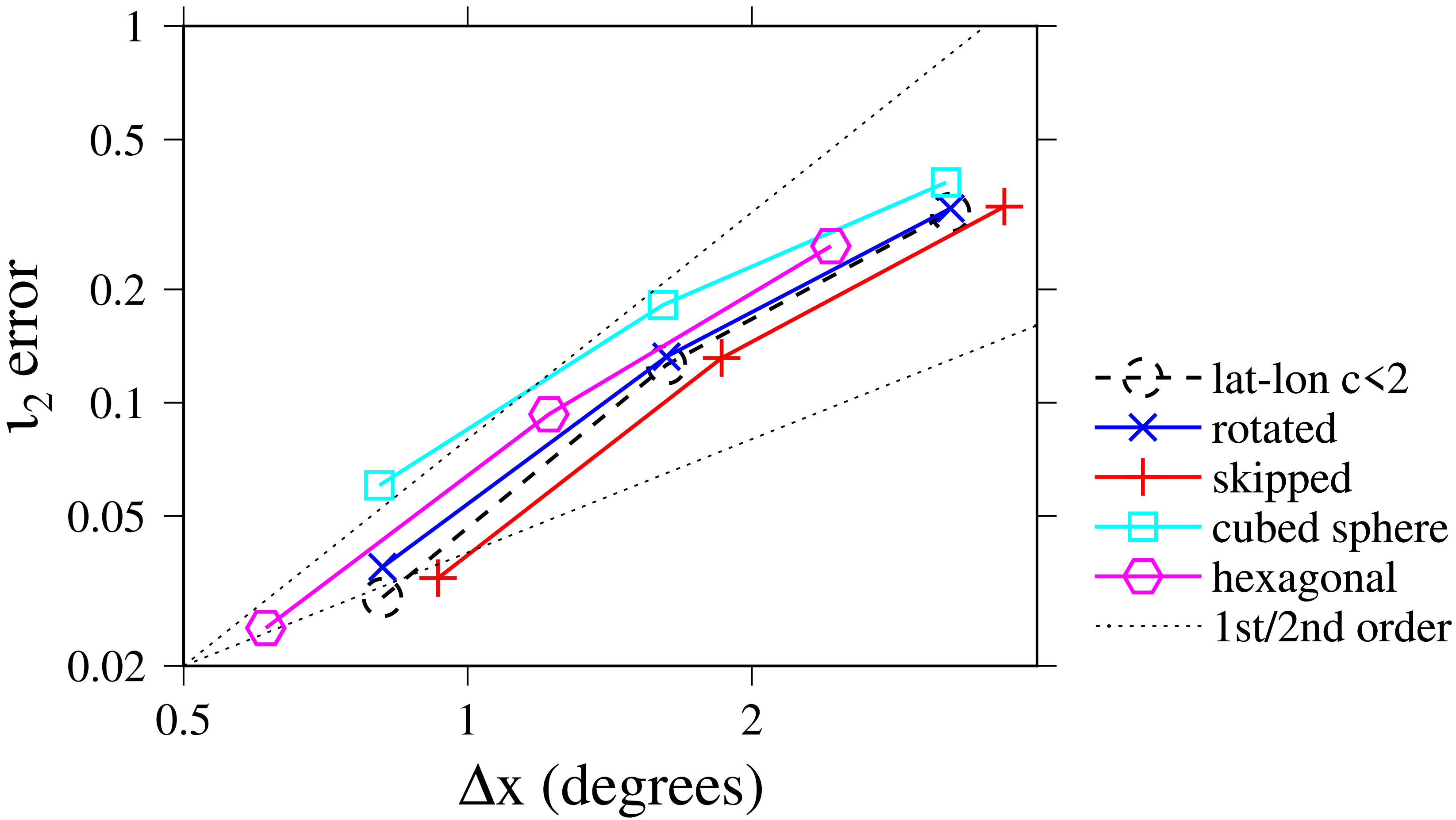
Meshes of the Sphere



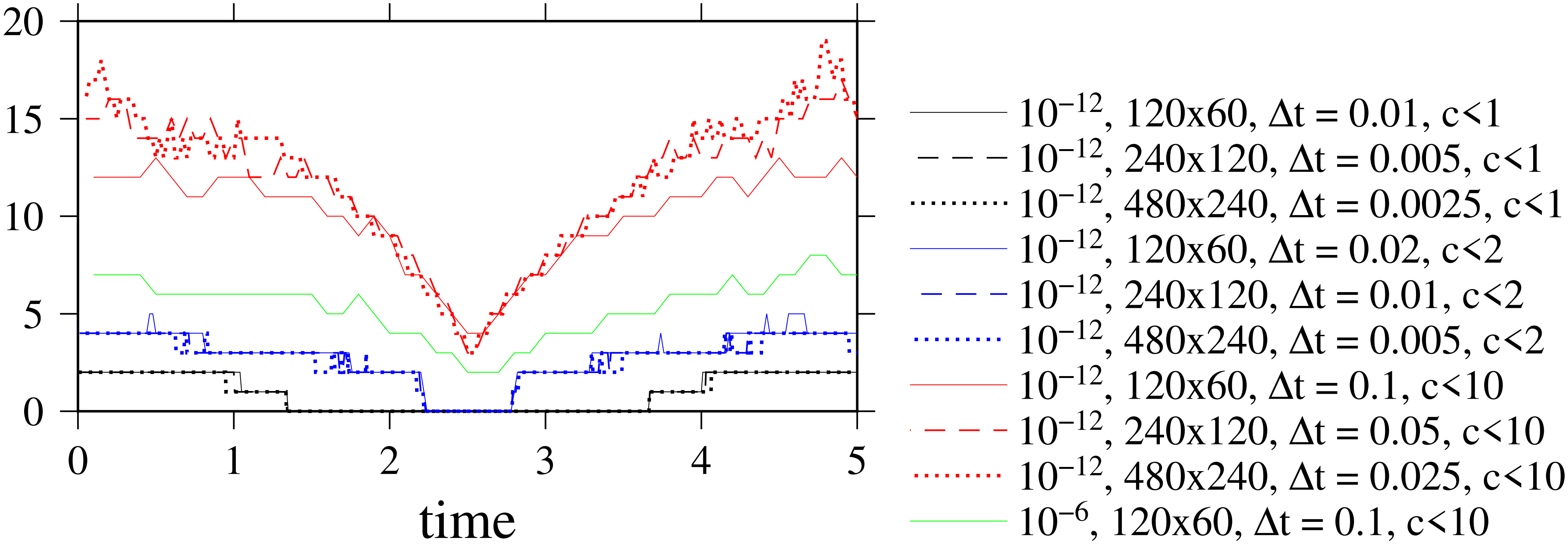


Full lat-lon, rotated, 30^o 240×120, $c \leq 70$
min = 2.25138e−08 max = 0.823699
Skipped, rotated lat-lon, 240×120, $\Delta x \geq 1.5^o$, $c \leq 2$
min = 3.84462e−08 max = 0.821629
Cubed-sphere, $60 \times 60 \times 6$, $\Delta x \sim 1.5^o$, $c < 3.1$
min = 4.44915e−08 max = 0.806223
Hexagaonal mesh 7, $\Delta x \sim 1.2^o$, $c \leq 2$
min = 2.97125e−08 max = 0.869226
60°
0°
−60°
0°
60°
120°
180°
−120°
−60°
0°
$c \geq 1$ contoured
0.00 0.05 0.10 0.15 0.20 0.25 0.30 0.35 0.40 0.45 0.50 0.55 0.60 0.65 0.70 0.75 0.80 0.85 0.90 0.95

Convergence of ℓ_2 error norm with resolution



Number of solver iterations per time step on the unrotated lat-lon meshes



Conclusions – review list of requirements

- Accuracy and cost comparable with explicit time stepping where $c \leq 1$ ✓
- Stable and at least first-order accurate for $c > 1$ ✓
- Monotone $\forall\ c$ ✓
- Arbitrary meshes ✓
- Exact local conservation ✓
- Multi-tracer efficient – same preconditioner for all advected quantities
- Good parallel scaling?
- Can implicit time stepping be used locally? ✓

Also

- Can increase resolution without reducing Δt and improve accuracy ✓
- 2nd order for $c \leq 2$. 1st order for $c > 2$ ✓

References

S. Gottlieb, C.-W. Shu, and E. Tadmor. Strong stability-preserving high-order time discretization methods. *SIAM review*, 43(1):89–112, 2001. (document)

S. Zalesak. Fully multidimensional flux-corrected transport algorithms for fluids. *J. Comput. Phys.*, 31(3): 335–362, 1979. (document)