The Framework of Relativistic Geodesy: What do we know? (beyond pN)

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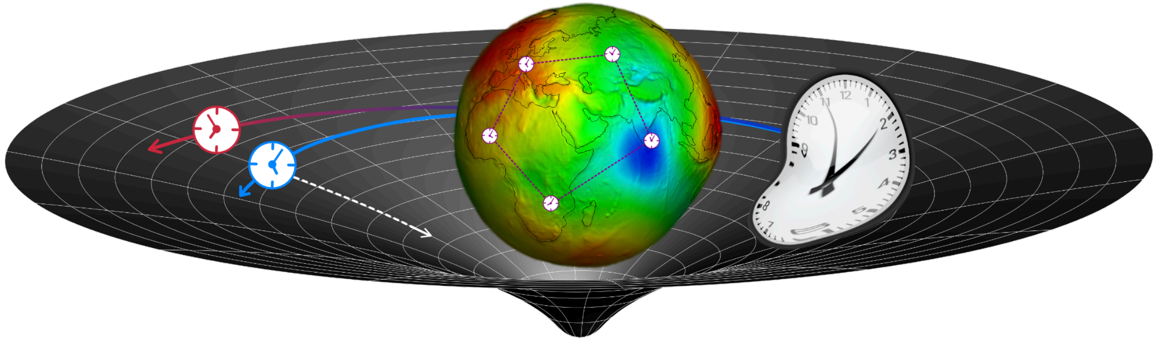

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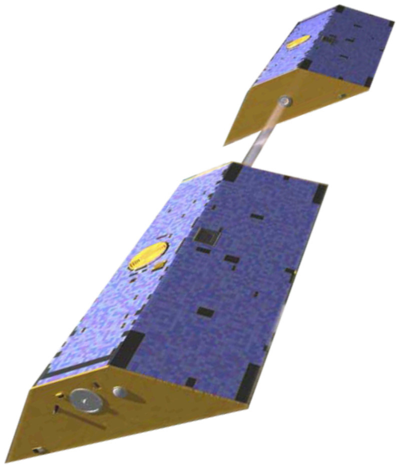
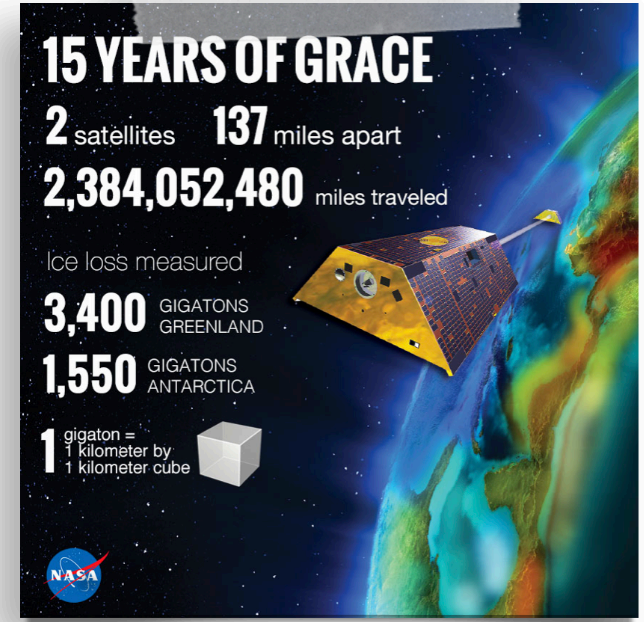
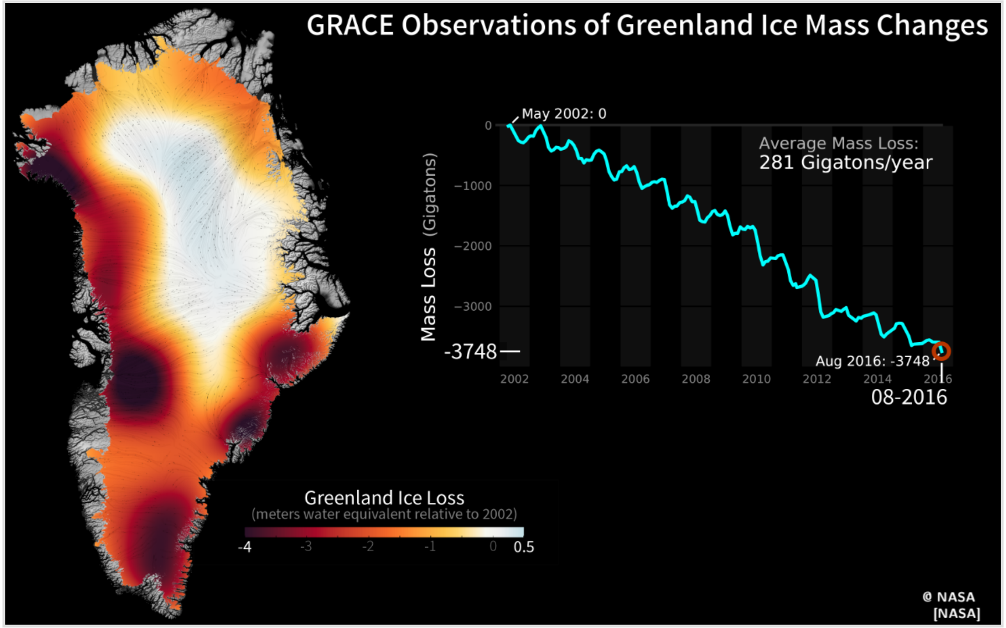
Terra

Global challenges

Introduction & Motivation







- Geodesy might be described as the science of the properties and gravity field of our Earth; engineering – physics
- There are important links to fields such as positioning, environmental and climate research.
- Conventional Geodesy builds upon Newtonian gravity and the gravitational potential is a central notion.

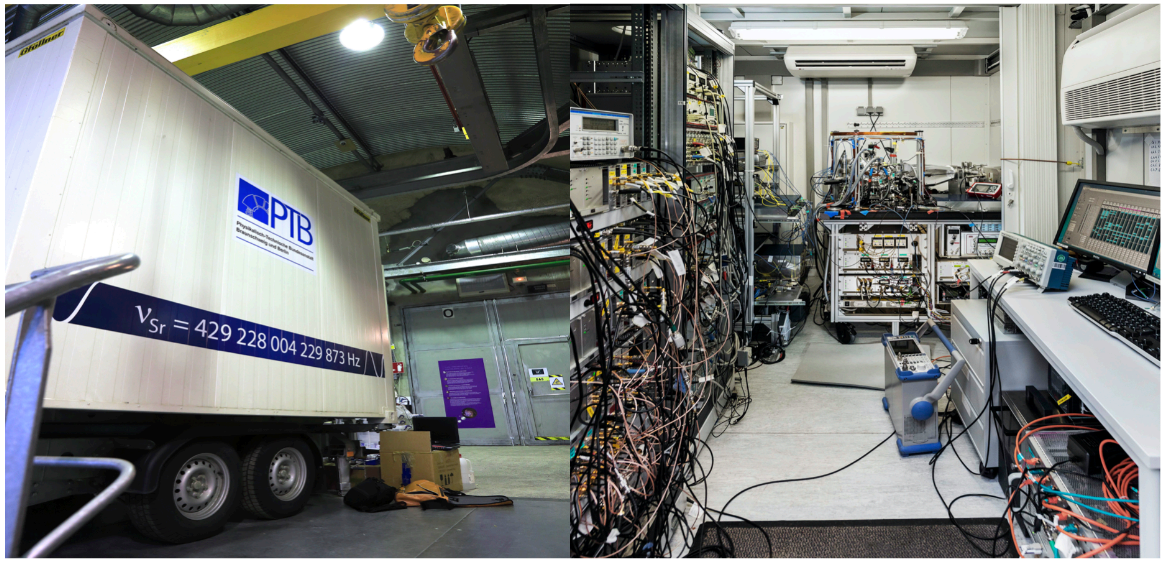
Introduction & Motivation

- For observations, reference systems are needed.
  Height references are of particular interest and intimately related to properties of the gravity field.

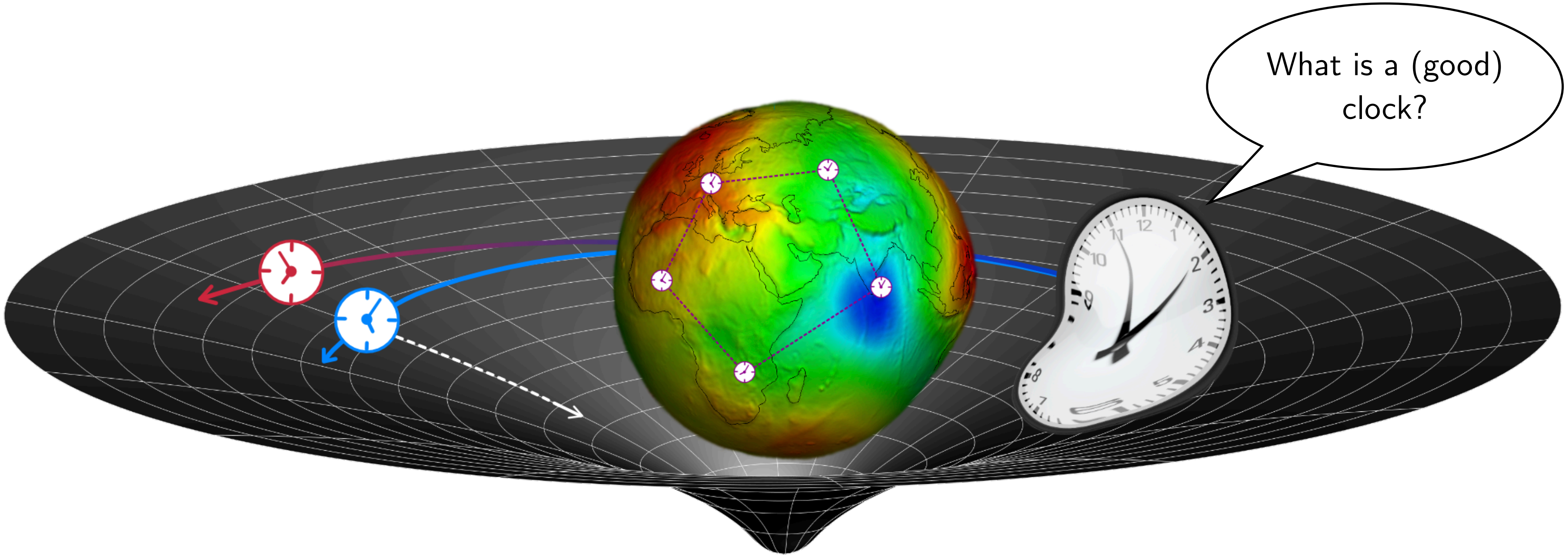
- Relativistic gravity: reformulate basic geodetic notions (dof) and develop a consistent theoretical framework - relativistic geodesy - to yield an undoubtedly correct interpretation of measurement results.

- Chronometric geodesy builds on the comparison of clocks and offers fundamental insight into the spacetime geometry if a solid theoretical formulation of observables is underlying modern high-precision measurements.

- Genuine relativistic def. of potential(s), geoid, ellipsoid, normal gravity, height notions, multipoles and more is possible.



Relativistic geodesy: physics of a timelike Killing vector field

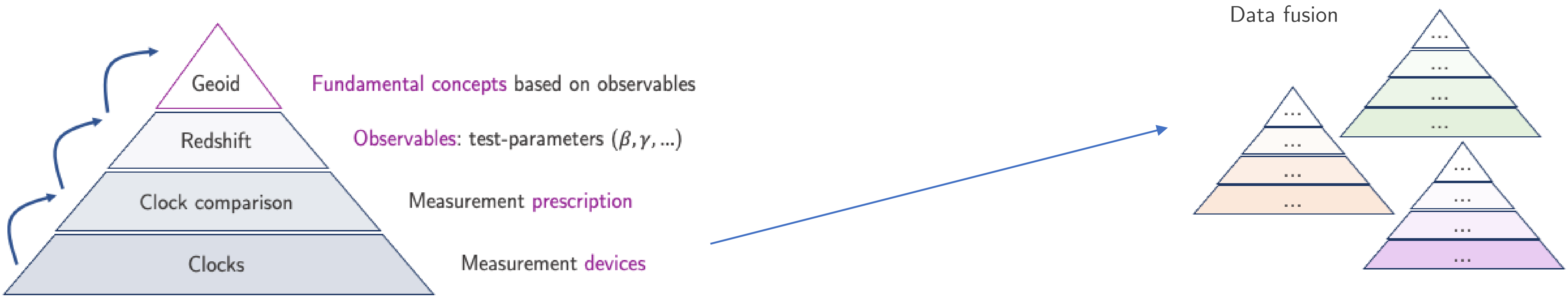


Make sense of geodetic concepts and notions in a relativistic theory of gravity!


$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu}$
A. EINSTEIN

Relativistic geodesy: physics of a timelike Killing vector field

- Set up a theoretical framework following a set of assumptions (e.g. stationarity, asymptotic flatness)
- Employ measurement devices and give a prescription of how to obtain observables —> define basic notions
- Constrain the set of model parameters by a multitude of measurements in data fusion



$(\mathcal{M}, g, \nabla, ...)$ Theoretical framework (GR, ppN, ...) + assumptions

- Geoid, normal gravity, ellipsoid, mass multipole moments, height notions, field equations
- spin moments, twist potential, relativistic effects on orbits & clocks (precession, Lense-Thirring, gravitomagnetism, ...)

Relativistic geodesy: physics of a timelike Killing vector field

- ∃ timelike Killing vector field ξ; the Earth is modeled by an isometric (Killing) congruence in GR.
- dof are contained in two potentials and the spatial metric.
- Rigidly co-rotating observers agree on a time-independent redshift potential ϕ; in the congruence: $1+z=\exp(\Delta\phi)$
- We can formulate a relativistic gravity potential —> Relativistic geoid: $U^* = U^*_0 = \text{const.}$

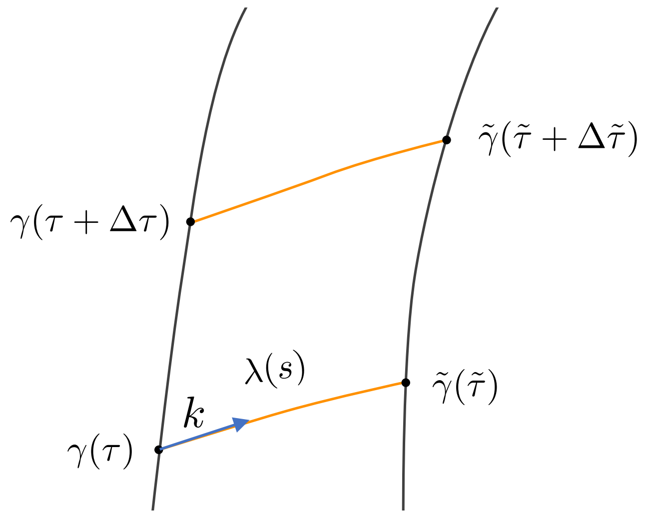
$$c^2 e^{2\phi} = g(\xi,\xi) \;\Rightarrow U^* = c^2\left(e^{\phi}-1\right)$$

$$g = -c^2 e^{2\phi}\mathrm{d}t^2 + 2g_{ti}\mathrm{d}t\mathrm{d}x^i + g_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

gravito-electric (norm of ξ)

gravito-magnetic (twist of ξ)

spatial metric



$$z+1 := \lim_{\Delta\tau\to 0}\frac{\Delta\tilde{\tau}}{\Delta\tau} = \frac{d\tilde{\tau}}{d\tau} = \frac{\nu}{\tilde{\nu}}$$

$$z+1 = \frac{k(u)\big|_p}{k(\tilde{u})\big|_{\tilde{p}}} = \frac{k(u)\big|_p}{k(\Lambda_*(u))\big|_{\Lambda(p)}}$$

$$a = -\mathrm{d}\phi$$

The post-Newtonian limit is

$$1+z \cong \left(1+\frac{\Delta W}{c^2}+\frac{v^2-\tilde{v}^2}{2c^2}\right)\frac{1+\vec{n}(x)\cdot\vec{v}/c}{1+\vec{n}(\tilde{x})\cdot\vec{\tilde{v}}/c}$$

See excellent work of Kopeikin et al. on pN developments,

DSX papers, Blanchet, Poncin-Lafitte, Petit, Wolf, Linet, Teyssandier et al.

Relativistic geodesy: physics of a timelike Killing vector field

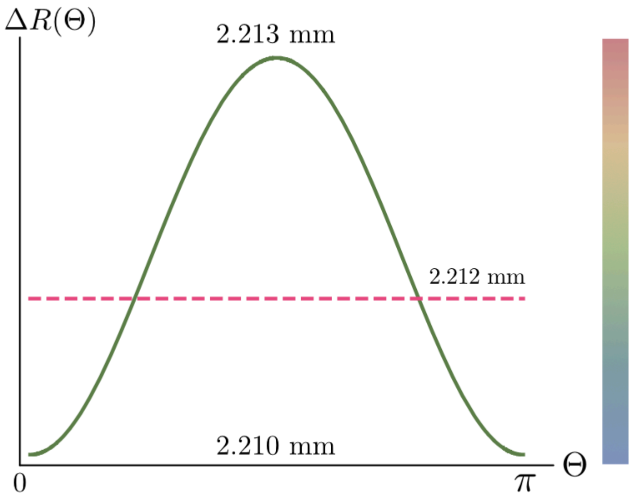
- For some exact solutions, we have

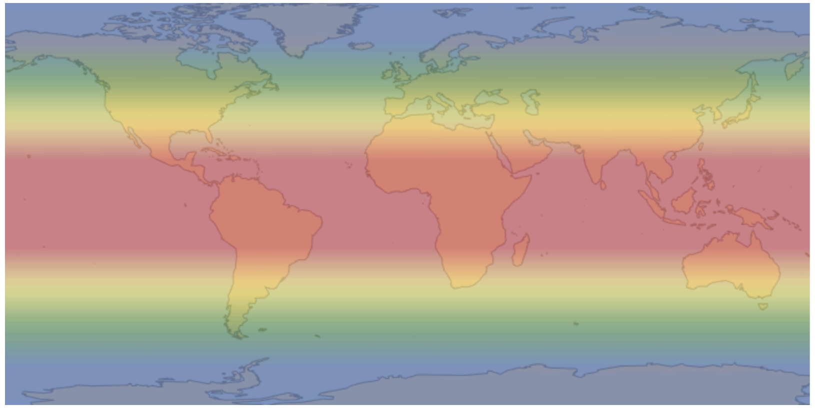
$$\frac{U^*_{\text{Schwarzschild}}}{c^2} = \sqrt{1 - \frac{2GM}{c^2 r} - \frac{\omega^2}{c^2} r^2 \sin^2\vartheta} - 1\,,$$

$$\frac{U^*_{\text{ER}}}{c^2} = \sqrt{e^{2\psi_{\text{ER}}(x,y)} - \frac{\omega^2}{c^2}(GM/c)^2(x^2-1)(1-y^2)e^{-2\psi_{\text{ER}}(x,y)}} - 1$$

with $\psi_{\text{ER}(x,y)} = \frac{1}{2}\log\left(\frac{x-1}{x+1}\right) + q_2 \frac{(3y^2-1)}{2}\left(\frac{(3x^2-1)}{4}\log\left(\frac{x-1}{x+1}\right) + \frac{3}{2}x\right)\,,$

$$\frac{U^*_{\text{Kerr}}}{c^2} = \sqrt{1 - \frac{2mr}{\rho(r,\vartheta)^2} + 4\,\frac{\omega}{c}\,\frac{amr\sin^2\vartheta}{\rho(r,\vartheta)^2} - \frac{\omega^2}{c^2}\,\sin^2\vartheta\left(r^2 + a^2 + \frac{2mra^2\sin^2\vartheta}{\rho(r,\vartheta)^2}\right)} - 1\,.$$





- Use the Newtonian relation for normal gravity

$$J_0 = 1\,, \quad J_{2l} = f(J_2, E, a, l)\,, \quad \forall\, l > 1\,.$$

- Model spacetime by a Weyl solution
  and investigate Newtonian limit [Ehlers, Quevedo]
- Exact expression for normal gravity spacetime,
  which reduces to the Newtonian notion in the limit.
- Should frame-dragging effects be included?
  —> Quevedo-spacetime (degenerate notion)

$$g_{\mu\nu}dx^\mu dx^\nu = -e^{2\psi}c^2dt^2 + m^2e^{-2\psi}(x^2-1)(1-y^2)d\varphi^2 + m^2e^{-2\psi}e^{2\gamma}(x^2-y^2)\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right)\,,$$

$$\text{where}\quad \psi = \sum_{l=0}^{\infty}(-1)^{l+1}q_l\,Q_l(x)\,P_l(y)$$

$$q_l = (-1)^l\frac{(2l+1)!!}{l!\,m^l}\,R^l_{\mathrm{ref}}J_l\,.$$

$$\begin{aligned}
\psi_N &= -\sum_{k=0}^{\infty}\frac{(4k+1)!!}{(2k)!}\left(\frac{R_{\mathrm{ref}}}{m}\right)^{2k} J_{2k}\,Q_{2k}(x)\,P_{2k}(y)\\
&= -\sum_{k=0}^{\infty}\left[\frac{(4k+1)!!}{(2k)!}\left(\frac{R_{\mathrm{ref}}}{m}\right)^{2k} J_{2k}\,P_{2k}(y)\times\left(\log\left(\frac{x+1}{x-1}\right)P_{2k}(x) - 2\sum_{i=0}^{k-1}\frac{4k-4i-1}{(2k-i)(2i+1)}P_{2k-2i-1}(x)\right)\right]\,,\\
&= -Q_0(x) - \frac{15}{2}\left(\frac{R_{\mathrm{ref}}}{m}\right)^2 J_2Q_2(x)P_2(y) - \sum_{k=2}^{\infty}\dots
\end{aligned}$$

Relativistic geodesy: physics of a timelike Killing vector field

- Multipole moment definitions [Simoin+Beig 1982; Geroch+Hansen 1970,74, Thorne 1980]
- Moments related to STF parts of

$$\Phi_M = \sum_{l=0}^{m-1} \frac{E_{a_1 \cdots a_i} x^{a_1} \cdots x^{a_l}}{l! r^{2l+1}} + O^\infty \left(r^{-(m+1)} \right)$$

$$\Phi_S = \sum_{l=0}^{m-1} \frac{F_{a_1 \cdots a_i} x^{a_1 \cdots x^{a_l}}}{l! r^{2l+1}} + O^\infty \left(r^{-(m+1)} \right)$$

gravito-electric (norm of ξ)

gravito-magnetic (twist of ξ)

- Field equation [Geroch+Hansen; Simon+Beig; Bäckdahl 2006]

$$\mathcal{M} = \mathbb{R} \times \mathcal{N} \qquad \gamma_{ij} = e^{2\phi} g_{ij} - \xi_i \xi_j$$ space

$$ds^2 = e^{2\phi} \left(dt + \sigma_i dx^i \right)^2 - e^{-2\phi} \gamma_{ij} dx^i dx^j$$ spacetime

$$R_{ij}(\gamma) = \frac{1}{2} e^{-4\phi} \left(\partial_i e^{2\phi} \partial_j e^{2\phi} + \partial_i \varpi \partial_j \varpi \right)$$ → curvature

$$\gamma^{ij} D_i D_j e^{2\phi} = e^{-2\phi} \gamma^{ij} \left(\partial_i e^{2\phi} \partial_j e^{2\phi} - \partial_i \varpi \partial_j \varphi \right)$$

$$\gamma^{ij} D_i D_j \varpi = 2 e^{-2\phi} \gamma^{ij} \partial_i e^{2\phi} \partial_j \varpi$$

$$\varpi_i = e^{\phi} \epsilon_i^{jk} D_j \sigma_k$$

- Thorne's definition in an ACMC coord. system

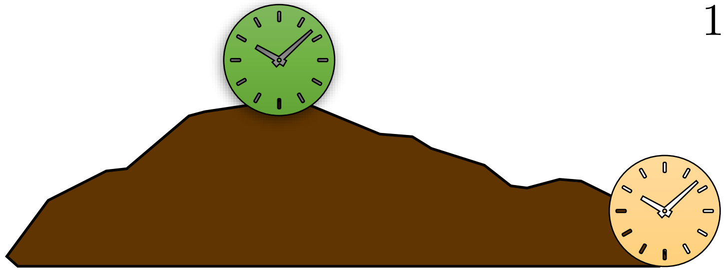
$$g_{00} = -1 + \frac{2g}{r} - \frac{2g^2}{r^2} + \sum_{i=2}^{\infty} \frac{1}{r^{l+1}} \left[\frac{2(2l-1)!!}{l!} g_{A_l} N_{A_l} + [(l-1) \text{ pole }] \quad + \cdots + [\text{ monopole }] \right]$$

$$g_{0j} = \sum_{l=1}^{\infty} \frac{1}{r^{l+1}} \left\{ -\frac{4l(2l-1)!!}{(l+1)!} \epsilon_{jka_l} S_{kA_{l-1}} N_{A_l} + [(l-1) \text{ pole }] + \cdots + [\text{ monopole }] \right\}$$

$$g_{ij} = \delta_{ij} \left[1 + \frac{2g}{r} \right] + \frac{g^2}{r^i} (\delta_{ij} + n_i n_j) + \sum_{l=2}^{\infty} \frac{1}{r^{l+1}} \left[\frac{2(2l-1)!!}{l!} g_{A_l} N_{A_l} \delta_{ij} + [(l-1) \text{ pole }] + \cdots + [\text{ monopole }] \right]$$

Relativistic geodesy: physics of a timelike Killing vector field

- Chronometric geodesy allows to measure heights by clock comparison



$$1 + z \approx \left(1 + \frac{\Delta W}{c^2}\right)$$

$$H_P = \frac{C_P}{\bar{g}} = \frac{W_P - W_{P_0}}{\bar{g}} = \frac{W_P + W_0}{\bar{g}}$$

$$H_P \approx \frac{c^2 z}{\bar{g}}$$

- This is a mixed concept! We can do better.
- We use the relativistic gravity potential U^* to define the notion of chronometric height.
- The average acceleration $\bar{a}$ is computed from the acceleration potential, which coincides with the redshift potential!
- Alternatively, GR introduces a unique acceleration: surface gravity κ at, e.g., the gravitational radius (of the monopole part)

$$C_P^* := U_P^* - U_0^* = z(c^2 + U_0^*)$$

$$H_P^* := \frac{C_P^*}{\bar{a}} = \frac{z(c^2 + U_0^*)}{\bar{a}}\,, \quad a = c^2 \mathrm{d}\phi$$

alternative

$$H_P^* = \frac{C_P^*}{\kappa} = \frac{z(c^2 + U_0^*)}{\kappa}$$