Self-modulation of oceanic waves on an ice-covered surface

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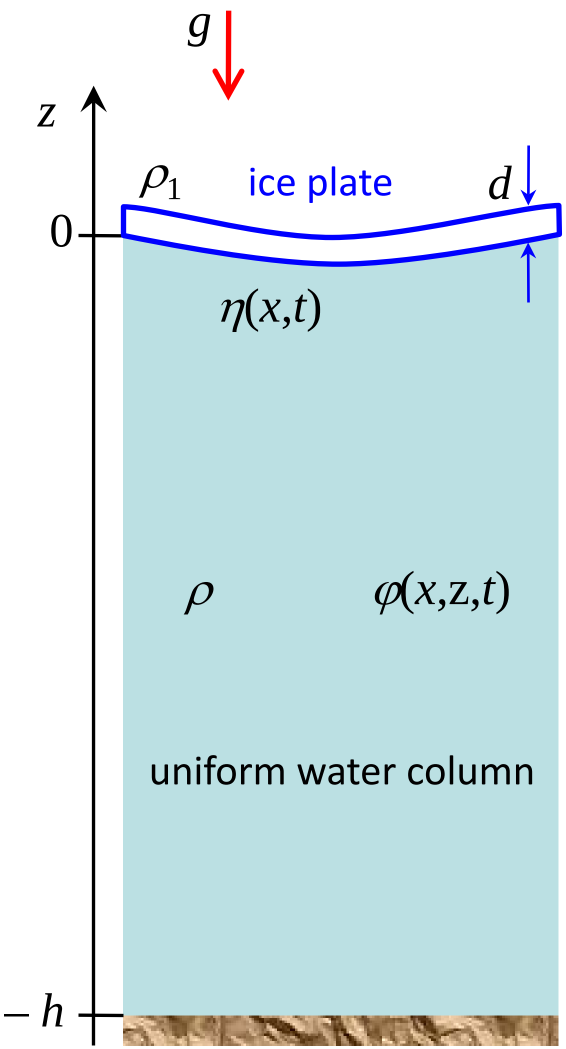
Problem setup

Ideal irrotational fluid with a solid ice cover is considered in planar geometry.
The standard set of hydrodynamic equations is supplemented with a modified dynamic boundary condition with four parameters: *d, M, D, Q*.



$$\frac{\partial\varphi}{\partial z} = \frac{\partial\eta}{\partial t} + \frac{\partial\varphi}{\partial x}\frac{\partial\eta}{\partial x}$$

$$\frac{\partial\varphi}{\partial t} + g\eta + \frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right] = -\frac{1}{\rho}\left[D\frac{\partial^2}{\partial x^2}K(\eta) + QK(\eta) + M\frac{\partial^2\eta}{\partial t^2}\right]$$

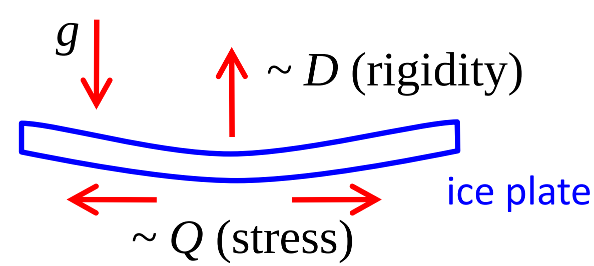
$$K(\eta) = \frac{\eta_{xx}}{\left[1+(\eta_x)^2\right]^{3/2} - d\eta_{xx}/2}$$

– ice-plate curvature [Forbes, 1986; Il'ichev, 2016, 2021]

$h > 0$ – is the water depth
$d \geq 0$ – is the ice thickness
$M = \rho_1 d \geq 0$ – is the additional mass
$D \geq 0$ – is the coefficient of ice rigidity/elasticity,
Q – is the coefficient of longitudinal stress. $Q > 0$ corresponds to the compression, and $Q < 0$ – to the stretching. For capillary waves $Q < 0$.

$$\Delta\varphi = 0, \quad -h \leq z \leq \eta$$

$$\frac{\partial\varphi}{\partial z} = 0, \quad z = -h$$



Nonlinear evolution equation for modulations

The desired evolution equation on the complex amplitude $B(x,t)$ which takes into account the leading-order effects of nonlinearity and dispersion has the form of the classic nonlinear Schrodinger equation with the coefficients of nonlinearity α and dispersion β:

$$i\left(\frac{\partial B}{\partial t}+V\frac{\partial B}{\partial x}\right)+\beta\frac{\partial^2 B}{\partial x^2}+\alpha|B|^2 B=0$$

The coefficients depend explicitly on the water depth (kh), the ice rigidity/elasticity ($k^4D/g/\rho$), the coefficient of the longitudinal stress ($k^2Q/g/\rho$), the geometrical ice thickness effect (kd), and the ice inertia (kM/ρ, $M = \rho_{ice}d$).
The parameter of the ice thickness d enters explicitly the solution only through the nonlinear term of the ice-plate curvature $K(\eta)$. Consequently, it does not affect linear properties of flexural-gravity waves, and does not enter the dispersion relation $\omega(k)$, group velocity V or the coefficient of dispersion β. However, the ice thickness influences the solution through the coefficient of ice rigidity, and via the added mass.

Obviously, the most interesting cases correspond to the situations when the ice thickness is relatively small compared to the wave length. Then the small parameter $kd << 1$ appears which leads to significant simplification of the nonlinear coefficient α. However, the coefficient as the function of k, h, Q and D still remains complicated for the analytic analysis. Here, we will focus on nonlinear wave properties.

Nonlinear coefficient of the NLSE

Under the assumption of a thin ice plate, $kd << 1$, and infinitely deep water, $kh >> 1$, the nonlinear coefficient α simplifies to a relatively simple form:

$$\alpha_\infty = \frac{\omega_\infty k^2}{32\left(1+k\hat{M}\right)} \frac{8\omega_\infty^4\left(2+3k\hat{M}\right)+k^3\omega_\infty^2\left(6\left(5\hat{Q}-21k^2\hat{D}\right)+18k\hat{M}\left(\hat{Q}-k^2\hat{D}\right)\right)+18k^6\left(\hat{Q}-k^2\hat{D}\right)\left(\hat{Q}-5k^2\hat{D}\right)}{\omega_\infty^2\left(\omega_\infty^2\left(1+3k\hat{M}\right)+3k^3\left(\hat{Q}-5k^2\hat{D}\right)\right)}$$

where $\omega_\infty^2 = \frac{k\left(g-k^2\hat{Q}+k^4\hat{D}\right)}{1+k\hat{M}}$ and $\hat{M}=\frac{M}{\rho}$ $\hat{Q}=\frac{Q}{\rho}$ $\hat{D}=\frac{D}{\rho}$

Under some conditions the nonlinear coefficient α may turn to zero or become singular. The dispersion coefficient β may vanish for some combination of parameters as well.

Singular values of α correspond to two types of synchronisms:

- between the second nonlinear harmonic and short-scale flexural-gravity waves;
- and between the difference nonlinear harmonic and long waves, $V^2 = gh$.

It is very well known that the modulations of waves described by the NLSE may be unstable or stable depending on the Lighthill (1965) criterion: $\alpha\beta > 0$ or $\alpha\beta < 0$ respectively (modulational or Benjamin – Feir instability). In the case of purely gravity waves (no ice cover) shallow-water wave modulations are stable, whereas long modulations of nonlinear waves in sufficiently deep water, $kh > 1.363$ are modulationally unstable.

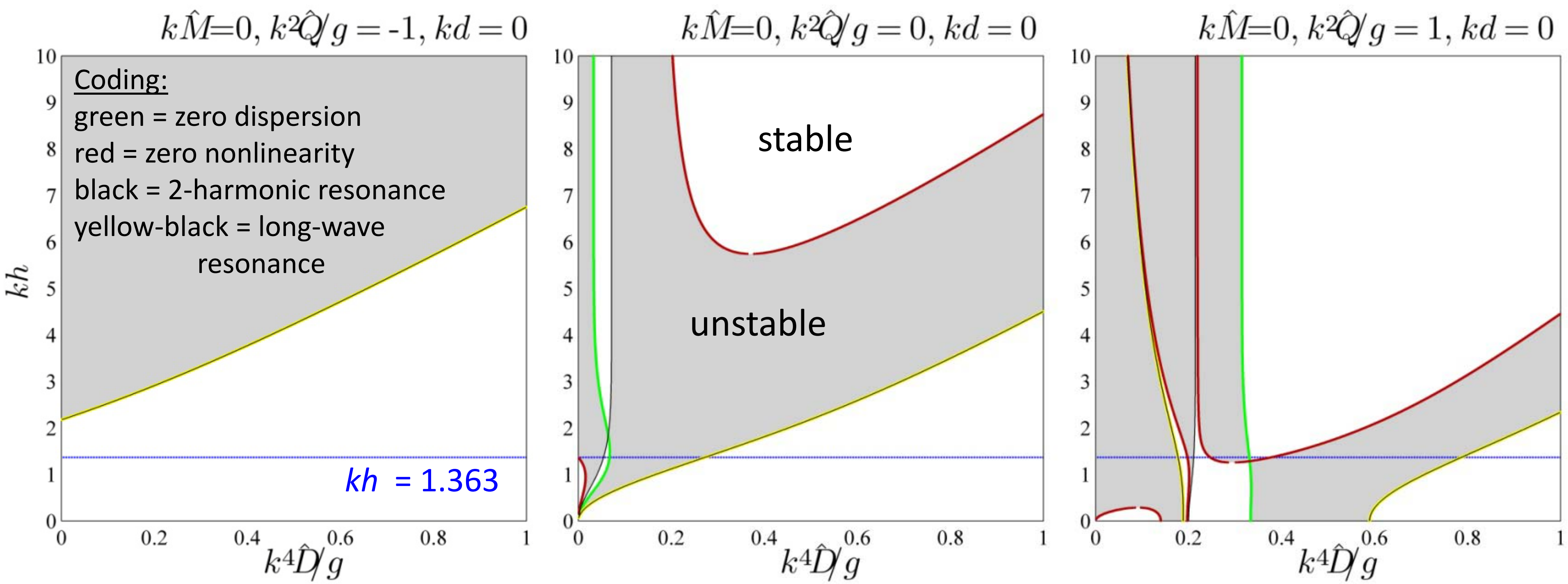
Domains of the modulational instability

It is convenient to analyze the stability diagrams as functions of scaled water depth kh and scaled rigidity parameter $k^4D/g/\rho$ for given values of M, Q and d.

Due to typically small values of kd, the geometrical effect of the ice thickness is not strong. The effect of added mass is also moderate, one may put $M = 0$.

The effect of stretching or compression changes the instability diagram drastically. The diagram may look very much complicated, it may exhibit several domains of instability in the parametric space.

$Q < 0$ (stretching) | $Q = 0$ | $Q > 0$ (compression)



Findings

Some interesting observations can be highlighted, and conclusions drawn:

- Waves under shallower conditions than $kh = 1.363$ may become unstable when $Q \geq 0$ (compressed ice).
- On the contrary, the instability threshold $kh = 1.363$ shifts toward deeper water when $Q < 0$ (stretched ice).
- Waves in deep water may become modulationally stable for sufficiently large value of the rigidity parameter D.
- We reproduce the coefficients of the gravity-capillary wave theory by Ablowitz & Segur (1979) when a few misprints are corrected.
- We report on inconsistency of the theory by Liu & Mollo-Christensen (1988), where a similar setup was considered in the limit of infinitely deep water.
- In terms of the number of waves in unstable groups and of the number of periods when the modulation gets amplified, the gravity-flexural waves may be even more unstable than deep water gravity waves.
- The derived theory obviously becomes invalid when the nonlinear coefficient tends to infinity. Collapsing curves of zero and singular nonlinear coefficients should probably correspond to cancellation of the singularity, physically.

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