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G1.5 Recent Developments in Geodetic Theory

Modelling the local gravity field by rectangular harmonics with numerical validations

Georgios Panou, Romylos Korakitis

geopanou@survey.ntua.gr

School of Rural, Surveying and Geoinformatics Engineering
National Technical University of Athens
15780 Zografou Campus, Greece

Introduction

For the representation of the Earth's global (geomagnetic or gravity) field, spherical harmonics are widely used, which are solutions of Laplace's equation found by the method of separation of variables.

On the other hand, for the representation of a local or regional field, **Spherical Cap Harmonic Analysis (SCHA)** (e.g. Haines 1985) and **Rectangular Harmonic Analysis (RHA)** (e.g. Alldredge 1981) are the preferred alternative techniques over the last decades. In comparison to classical spherical harmonic models, the resulting harmonic models from the above two techniques have two important advantages:

1. The local or regional field represented by an analytical model needs much less parameters (coefficients) and has less memory requirements in computation and storage than a spherical harmonic model.

2. These models can be easily enhanced using new data of different kinds (e.g. gravity anomalies, geoid undulations), leading to a further refinement of the local field.

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In this work, a global spherical geopotential model, over different small areas, is transformed to rectangular harmonic expansions by a least squares' adjustment process, using as input data values of the disturbing potential of the Earth's gravity field.

Alldredge LR (1981) Rectangular harmonic analysis applied to the geomagnetic field. Journal of Geophysical Research, 86: 3021-3026.

Haines GV (1985) Spherical cap harmonic analysis. Journal of Geophysical Research, 90: 2583-2591.
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Global spherical harmonic models

The Earth's gravitational potential V at any point, with spherical coordinates (r,ϕ,λ) on or above the Earth's surface, is conveniently expressed on a global scale by summing up over degree n and order m of a spherical harmonic expansion as follows (e.g. Barthelmes, 2013):

$$V(r,\phi,\lambda)=\frac{GM}{r}\sum_{n=0}^{n_{max}}\sum_{m=0}^{n}\left(\frac{R}{r}\right)^{n}P_{nm}(\sin\phi)(C_{nm}\cos m\lambda+S_{nm}\sin m\lambda)$$

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On the other hand, the normal gravitational potential $\bar{V}$ at any point with oblate spheroidal coordinates (u,β,λ) outside the level spheroid, is given by the following formula (e.g. Heiskanen and Moritz, 1967):

$$\bar{V}(u,\beta)=\frac{GM}{E}\tan^{-1}\frac{E}{u}+\frac{1}{2}\omega^2a^2\frac{q}{q_0}\left(\sin^2\beta-\frac{1}{3}\right)$$

where

$$q=\frac{1}{2}\left[\left(1+3\frac{u^2}{E^2}\right)\tan^{-1}\frac{E}{u}-3\frac{u}{E}\right] \quad \text{and} \quad q_0=\frac{1}{2}\left[\left(1+3\frac{b^2}{E^2}\right)\tan^{-1}\frac{E}{b}-3\frac{b}{E}\right]$$

Finally, the disturbing potential T at any point with global rectangular coordinates (X,Y,Z) outside the masses is given by:

$$T(X,Y,Z)=V(X,Y,Z)-\bar{V}(X,Y,Z)$$

Harmonic function

Barthelmes F (2013) Definition of functionals of the geopotential and their calculation from spherical harmonic models. Scientific Technical Report STR09/02, Revised Edition
Heiskanen WA and Moritz H (1967) Physical Geodesy. W.H. Freeman and Company, San Francisco.
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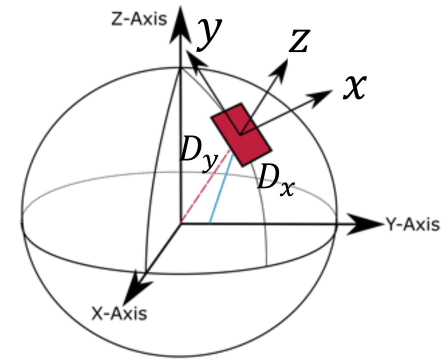
Laplace's equation in local rectangular coordinates and its solutions

The disturbing potential may also be expanded into a series of rectangular harmonics in local rectangular coordinates x, y and z.



Laplace's equation:

$$\Delta T \equiv \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

We shall attempt to solve this equation by separating the variables x, y, z by means of the trial substitution:

$$T(x,y,z) = f(x)g(y)h(z)$$

Thus, the most general solution for the disturbing potential, as linear combination, in terms of truncated series at maximum degree N and order M, is expressed in the form:

$$T(x,y,z) = T_{00}(x,y,z) + T_{n0}(x,y,z) + T_{0m}(x,y,z) + T_{nm}(x,y,z)$$

where

$$T_{00}(x,y,z) = T_0 + T_x x + T_y y + T_z z + T_{xy} xy + T_{xz} xz + T_{yz} yz + T_{xyz} xyz$$

$$T_{n0}(x,y,z) = \sum_{n=1}^{N} [C_{n0}\cos(p_n x) + S_{n0}\sin(p_n x) + c_{n0} y\cos(p_n x) + s_{n0} y\sin(p_n x)]\, e^{-z p_n}$$

$$T_{0m}(x,y,z) = \sum_{m=1}^{M} [C_{0m}\cos(q_m y) + S_{0m}\sin(q_m y) + c_{0m} x\cos(q_m y) + s_{0m} x\sin(q_m y)]\, e^{-z q_m}$$

$$T_{nm}(x,y,z) = \sum_{n=1}^{N}\sum_{m=1}^{M} [CC_{nm}\cos(p_n x)\cos(q_m y) + CS_{nm}\cos(p_n x)\sin(q_m y) + SC_{nm}\sin(p_n x)\cos(q_m y) + SS_{nm}\sin(p_n x)\sin(q_m y)] e^{-z\sqrt{p_n^2 + q_m^2}}$$

$$p_n = n\frac{2\pi}{D_x} \quad \text{and} \quad q_m = m\frac{2\pi}{D_y}$$

Adjustment process of the transformation

Model	Functional form	Number of RHC (n_{RHC})
M0	$T = T_{00} + T_{n0} + T_{0m} + T_{nm}$	$8 + 4N + 4M + 4NM$
M1	$T = T_{n0} + T_{0m} + T_{nm}$	$4N + 4M + 4NM$
M2	$T = T_{00} + T_{n0} + T_{0m}$	$8 + 4N + 4M$
M3	$T = T_{n0} + T_{0m}$	$4N + 4M$
M4	$T = T_{00} + T_{nm}$	$8 + 4NM$
M5	$T = T_{nm}$	$4NM$

Table 1 Functional forms of the mathematical models, along with the number of Rectangular Harmonic Coefficients

Using the method of least squares we can estimate the RHC of the models of Table 1 and, therefore, to transform a global spherical to an adjusted local rectangular harmonic model. We set up the linear observation equation e.g. for the model M0:

$$T_{00}(x_i, y_i, z_i) + T_{n0}(x_i, y_i, z_i) + T_{0m}(x_i, y_i, z_i) + T_{nm}(x_i, y_i, z_i) = T_i + \upsilon_i \,, \qquad i = 1, \dots, k$$

Hence, the system of k linear equations is represented in matrix form as

$$\mathbf{A}\hat{\mathbf{x}} = \mathbf{y} + \boldsymbol{\upsilon}$$

$$\mathbf{x} = [T_0 \quad T_x \quad T_y \quad T_z \quad T_{xy} \quad T_{xz} \quad T_{yz} \quad T_{xyz} \quad C_{10} \quad S_{10} \quad c_{10} \quad s_{10} \quad \cdots \quad C_{01} \quad S_{01} \quad c_{01} \quad s_{01} \quad \cdots]^{\mathrm{T}}$$

The solution of the adjustment is given by

$$\hat{\mathbf{x}} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{y}$$

$$\mathrm{RMS} = \sqrt{\frac{\boldsymbol{\upsilon}^{\mathrm{T}}\mathbf{P}\boldsymbol{\upsilon}}{k}}$$

$$\hat{\mathbf{V}}_{\hat{\mathbf{x}}} = \hat{\sigma}_0^2\left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-1}$$

Validation of different adjusted local rectangular harmonic models

Initially, we select the best mathematical model among the different functional forms presented in Table 1. We assume $N = M$ and extend the sizes D_x and D_y of the data domain by the distance d. So, we have to examine the values of the two parameters N and d which minimize the RMS of the residuals of the disturbing potential.

A 3D grid of $k = 6724$ points in Greece, with spatial resolution $3' \times 3'$, is generated over a $2° \times 2°$ area from $(\varphi, \lambda) = (37°, 22°)$ to $(\varphi, \lambda) = (39°, 24°)$. The values of the disturbing potential generated from the geopotential model EGM2008 (Pavlis et al. 2012) are computed at four ellipsoidal height levels (0 m, 100 m, 1000 m and 2000 m) via the ICGEM online platform (Ince et al. 2019).

Model	$N = M$	n_{RHC}	d (km)	RMS ($\mathrm{m}^2/\mathrm{s}^2$)	Time (min)/adj
M0	20	1768	125	$1.60 \cdot 10^{-5}$	4.17
M1	20	1760	150	$3.91 \cdot 10^{-3}$	5.23
M2	22	184	150	2.36	0.03
M3	22	176	150	2.38	0.03
M4	22	1944	150	$4.42 \cdot 10^{-5}$	5.12
M5	22	1936	175	$1.31 \cdot 10^{-2}$	6.44

Concluding, the numerical results confirm that the functional form M0 is able to provide a reliable and precise adjusted local model for the representation of a local gravity field, as we demonstrate and in the following case studies.

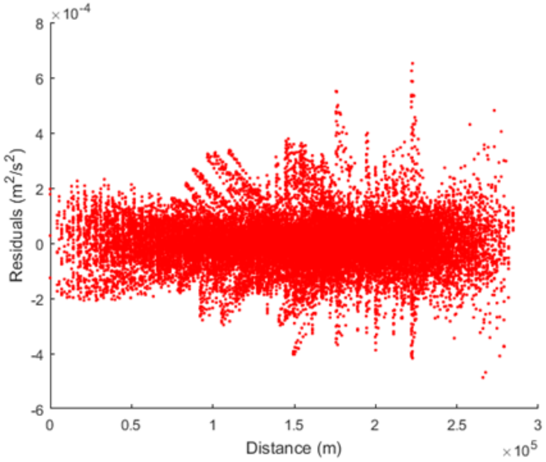
Ince ES, Barthelmes F, Reißland S, Elger K, Förste C, Flechtner F and Schuh H (2019) ICGEM – 15 years of successful collection and distribution of global gravitational models, associated services and future plans. Earth System Science Data, 11:647-674.
Pavlis NK, Holmes SA, Kenyon SC and Factor JK (2012) The development and evaluation of the Earth Gravitational Model 2008 (EGM2008). Journal of Geophysical Research: Solid Earth 117: B04406. http://dx.doi.org/10.1029/2011JB008916

Case studies and numerical results

Local rectangular harmonic model in Greece

Having selected the model M0, we extended the previous area in Greece and a 3D grid of $k = 26244$ points, with spatial resolution $3' \times 3'$, was generated over a $4° \times 4°$ area, from $(\varphi, \lambda) = (36°, 21°)$ to $(\varphi, \lambda) = (40°, 25°)$. The values of the disturbing potential were again computed at four ellipsoidal height levels (0 m, 100 m, 1000 m and 2000 m) via the ICGEM online platform.

In order to validate the resulting local model, a new grid of 6400 evaluation points was created in the same area. These were located intermediate to the adjustment points and had a single height value, corresponding to the actual height of the topography (for points in the sea, we used the corresponding height of the geoid).



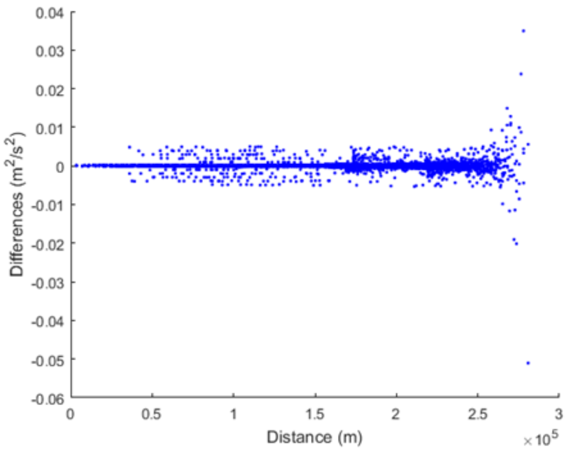


Fig. 1 From the adjustment of model M0 with expansion degree $N = M = 35$ and extension parameter $d = 125$ km in $4° \times 4°$ area in Greece: Residuals of the disturbing potential (Left) and differences of disturbing potential between the values from the global and local model at the validation points (Right), as a function of the distance from the origin

Case studies and numerical results

Local rectangular harmonic model in China

For a comparison with another mathematical model of RH, in this case study we generated the grid which was used in the work of Jiang et al. (2014). Thus, a 3D grid of $k = 14641$ points, with spatial resolution $2.5' \times 2.5'$, is generated over a $5° \times 5°$ area in China, from $(\varphi, \lambda) = (29°, 108°)$ to $(\varphi, \lambda) = (34°, 113°)$.

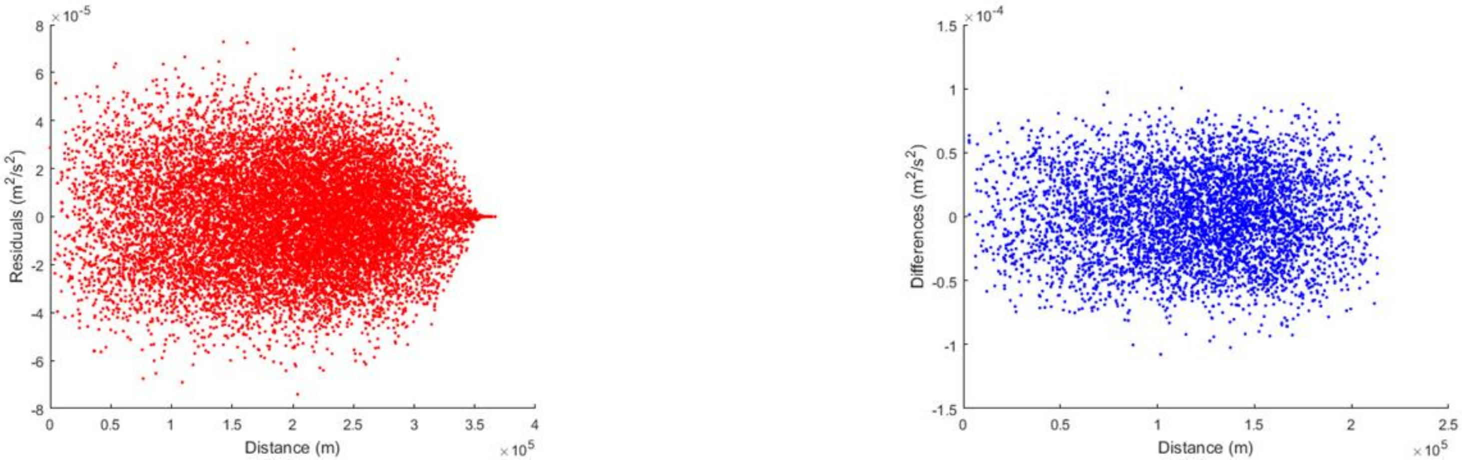


Fig. 2 From the adjustment of model M0 with expansion degree $N = M = 40$ and extension parameter $d = 75$ km in $5° \times 5°$ area in China: Residuals of the disturbing potential (Left) and differences of disturbing potential between the values from the global and local model at the validation points in a $3° \times 3°$ area (Right), as a function of the distance from the origin

Jiang T, Li J, Dang Y, Zhang C, Wang Z and Ke B (2014) Regional gravity field modeling based on rectangular harmonic analysis. Sci. China Earth Sci., 57: 1637-1644. https://doi.org/10.1007/s11430-013-4784-1

Case studies and numerical results

Local rectangular harmonic model in Germany

For a comparison with the method of ASCH by a least squares' estimation of coefficients of the mathematical model, in this case study we generated almost the same grid which was used in the work of Younis (2013). Thus, a 3D grid of $k = 11163$ points, with spatial resolution $3' \times 3'$, is generated over a $3° \times 3°$ area in Germany, from $(\varphi, \lambda) = (47°, 7.5°)$ to $(\varphi, \lambda) = (50°, 10.5°)$.

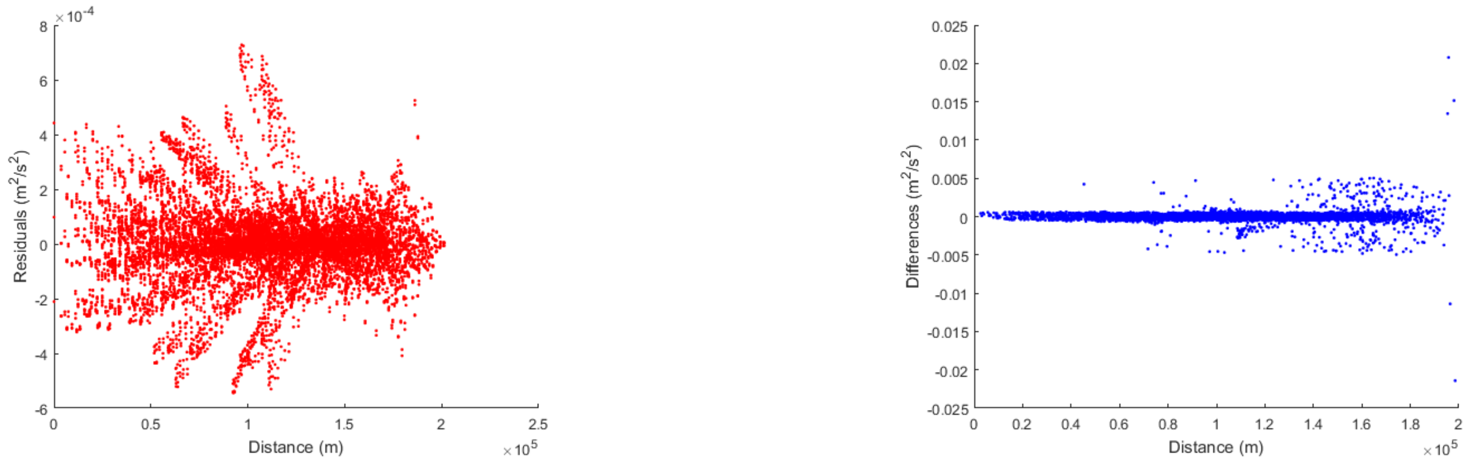


Fig. 3 From the adjustment of model M0 with expansion degree $N = M = 25$ and extension parameter $d = 125$ km in $3° \times 3°$ area in Germany: Residuals of the disturbing potential (Left) and differences of disturbing potential between the values from the global and local model at the validation points (Right), as a function of the distance from the origin

Younis G (2013) Regional Gravity Field Modeling with Adjusted Spherical Cap Harmonics in an Integrated Approach. Schriftenreihe Fachrichtung Geodäsie der Technischen Universität Darmstadt (39). Darmstadt. ISBN 978-3-935631-28-0

Functionals of the disturbing potential

We can also derive the functionals of the disturbing potential in terms of rectangular harmonics. The gravity disturbance is expressed by

$$\delta g = -\frac{\partial T}{\partial r} \approx -\frac{\partial T}{\partial z}$$

where r is the geocentric distance. Similarly, the gravity anomaly is expressed by

$$\Delta g = -\frac{\partial T}{\partial r} - \frac{2}{r}T \approx +\delta g - \frac{2}{r}T$$

Furthermore, the geoid undulation is expressed by

$$N = \frac{T}{\gamma_0}$$

where γ_0 is the normal gravity. Finally, the components ξ and η of the deflection of the vertical in north-south and east-west direction, respectively, are

$$\xi = -\frac{\partial N}{\partial y} \quad \text{and} \quad \eta = -\frac{\partial N}{\partial x}$$

Conclusions

A **global spherical harmonic model** requires a very high number of coefficients to represent the gravitational (or disturbing) potential, even in a local area. In contrast, a **local rectangular harmonic model** requires less coefficients and hence memory requirements in computation and storage.

Example:

The global model EGM2008, with maximum degree 2190, includes 4800481 coefficients, while the corresponding local model, for instance in a 4°×4° area in Greece with maximum degree 35, includes only 5188 coefficients and the disturbing potential is accurately reproduced.

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The results in the various case studies demonstrate the reliability and accuracy of the successfully transformed local model. The derived model of the potential is simpler to implement than other analyses (e.g. SCH) and, in addition, one can also calculate its functionals, such as gravity anomaly and disturbance, geoid undulation and deflection of the vertical.

On the other hand, a limitation of the rectangular harmonic model is that it cannot be extrapolated, as evidenced by the increased errors in the edge of the region constrained by the data.
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