The most unstable modes in rotating magnetoconvection with anisotropic diffusion in the Earth's outer core

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Turbulence, anisotropy and magnetoconvection

Most unstable modes

Conclusions

anisotropy = anisotropic diffusion, anisotropic diffusive coefficients, anisotropic diffusivities

the most unstable modes have the max growth rate

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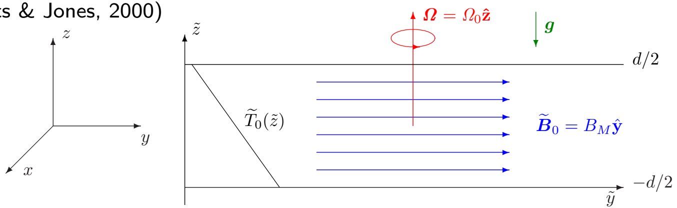
Anisotropy in the magnetoconvection models

Turbulence can cause anisotropy, therefore it is worth to introduce anisotropy in the rotating magnetoconvection models (see, e.g., Šoltis & Brestenský, 2010; Filippi et al., 2019; Filippi & Brestenský, 2020); the geometry of these models (Roberts & Jones, 2000)



Model of rotating magnetoconvection with homogeneous horizontal basic magnetic field in the infinite horizontal unstably stratified fluid layer with temperature profile, $\tilde{T}_0(\tilde{z})$.

governing equations

$$R_o \partial_t \mathbf{u} + \hat{\mathbf{z}} \times \mathbf{u} = -\boldsymbol{\nabla} P + \Lambda_z (\boldsymbol{\nabla} \times \mathbf{b}) \times \hat{\mathbf{y}} + R\vartheta \hat{\mathbf{z}} + E_z \nabla^2_\nu \mathbf{u}, \ \partial_t \mathbf{b} = \boldsymbol{\nabla} \times (\mathbf{u} \times \hat{\mathbf{y}}) + \nabla^2_\eta \mathbf{b}, \ q_z^{-1} \partial_t \vartheta = \hat{\mathbf{z}} \cdot \mathbf{u} + \nabla^2_\kappa \vartheta, \ \nabla \cdot \mathbf{u} = 0, \ \nabla \cdot \mathbf{b} = 0,$$

SA anisotropy $E_z \nabla^2_\nu \mathbf{u} = E_z[(1-\alpha_\nu)\partial_{zz} + \alpha_\nu \nabla^2]\mathbf{u}, \quad \nabla^2_\eta \mathbf{b} = [(1-\alpha_\eta)\partial_{zz} + \alpha_\eta \nabla^2]\mathbf{b}, \quad \nabla^2_\kappa \vartheta = [(1-\alpha_\kappa)\partial_{zz} + \alpha_\kappa \nabla^2]\vartheta$

where dimensionless parameters, $\Lambda_z = \frac{B_M^2}{2\Omega_0 \rho_0 \mu_0 \eta_{zz}}$, $E_z = \frac{\nu_{zz}}{2\Omega_0 d^2}$, $R = \frac{\alpha_T g \Delta T d}{2\Omega_0 \kappa_{zz}}$, $q_z = \frac{\kappa_{zz}}{\eta_{zz}}$, $R_o = \frac{\eta_{zz}}{2\Omega_0 d^2}$ are used;

$\alpha_\nu = \frac{\nu_{xx}}{\nu_{zz}} = \alpha_\kappa = \frac{\kappa_{xx}}{\kappa_{zz}} = \alpha_\eta = \frac{\eta_{xx}}{\eta_{zz}}$; analysis in term of normal modes, γ, orientation of rolls of convection

solution $f(x,y,z,t) = \Re e[F(z)\exp(ilx + imy)\exp(\lambda t)]$, $\Re e[\lambda] \equiv$ growth rate, $a = \sqrt{l^2 + m^2}$

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Most unstable modes in T and G cases

isotropic T-case, $\partial_t \vartheta \neq 0$, $\partial_t \mathbf{b} = \partial_t \mathbf{u} = 0$, $\alpha = 1$ (inspired by Braginsky & Meytlis (1990) and Roberts & King (2013))

$$R_i = \frac{K^2+\lambda/q}{K^2-\pi^2}\left(EK^4 + \Lambda m^2 + \frac{\pi^2 K^2}{EK^4+\Lambda m^2}\right), \quad \lambda_{max} = \frac{qR_i}{4\pi K}\left(K^2 - 3\pi^2\right), \quad K^2 = \pi^2 + a^2$$

isotropic most unstable modes

$$\Lambda = c^2\left(\frac{\pi K - EK^4}{K^2-\pi^2}\right) = c^2 \Lambda_T^\dagger \left(1 - \frac{E}{E_T^\dagger}\right), \quad c^2 = \frac{K^2-\pi^2}{m^2}, \quad \Lambda_T^\dagger(R_i) = \frac{\pi K}{K^2-\pi^2}, \quad E_T^\dagger(R_i) = \frac{\pi}{K^3}$$

isotropic G-case $\partial_t \vartheta \neq 0$, $\partial_t \mathbf{b} \neq 0$, $\partial_t \mathbf{u} \neq 0$, $\alpha = 1$, $p = R_o/E$

$$R_i = \frac{K^2\left(K^2+\lambda/q\right)}{(K^2+\lambda)(K^2-\pi^2)}\left[(K^2+\lambda)(K^2+p\lambda)E + \Lambda m^2 + \frac{\pi^2 (K^2+\lambda)^2/K^2}{(K^2+\lambda)(K^2+p\lambda)E+\Lambda m^2}\right], \quad \lambda_{max} = \frac{qR_i}{4\pi K}\left(K^2 - 3\pi^2\right)$$

$$\Lambda = c^2 \Lambda_G^\dagger\left(1 - \frac{E}{E_G^\dagger}\right), \quad E_G^\dagger(pq, R_i) = \frac{\pi}{K(K^2+p\lambda)} < E_T^\dagger, \quad \Lambda_G^\dagger(q, R_i) = \frac{\pi(K^2+\lambda)}{K(K^2-\pi^2)} = \frac{K^2+\lambda}{K^2}\Lambda_T^\dagger$$

Anisotropic T-case, $\partial_t \vartheta \neq 0$, $\partial_t \mathbf{b} = \partial_t \mathbf{u} = 0$, $\alpha \neq 1$

anisotropic most unstable modes

$$R = \frac{K^2(K_\alpha^2+\lambda/q_z)}{K^2-\pi^2}\left[E_z K_\alpha^2 + \frac{\Lambda_z m^2}{K_\alpha^2} + \frac{\pi^2}{K^2\left(E_z K_\alpha^2 + \frac{\Lambda_z m^2}{K_\alpha^2}\right)}\right], \quad \lambda_{max} = q_z\left[\frac{R}{4\pi K}(K^2 - 3\pi^2) + (\alpha-1)\pi^2\right],$$

$$K_\alpha^2 = \pi^2 + \alpha a^2, \quad \Lambda_z = c^2 \Lambda_T^\dagger\left(1 - \frac{E_z}{E_T^\dagger}\right), \quad \Lambda_T^\dagger(R,\alpha) = \frac{\pi[\alpha K^2 + (1-\alpha)\pi^2]}{K(K^2-\pi^2)}, \quad E_T^\dagger(R,\alpha) = \frac{\pi}{\alpha K^3 + (1-\alpha)\pi^2 K}$$

Anisotropic G-case, $\partial_t \vartheta \neq 0$, $\partial_t \mathbf{b} \neq 0$, $\partial_t \mathbf{u} \neq 0$, $\alpha \neq 1$, $p_z = R_o/E_z$

$$R = \frac{K^2\left(K_\alpha^2+\lambda/q_z\right)}{(K_\alpha^2+\lambda)(K^2-\pi^2)}\left[N_f + \frac{\pi^2(K_\alpha^2+\lambda)^2/K^2}{N_f}\right], \quad N_f = (K_\alpha^2+\lambda)(K_\alpha^2+p_z\lambda)E_z + \Lambda m^2,$$

$$\lambda_{max} = q_z\left[\frac{R}{4\pi K}(K^2 - 3\pi^2) + (\alpha-1)\pi^2\right]; \quad \Lambda_z = c^2\Lambda_G^\dagger\left(1 - \frac{E_z}{E_G^\dagger}\right),$$

$$E_G^\dagger(R, q_z p_z, \alpha) = \frac{\pi}{K[\alpha K^2 + (1-\alpha)\pi^2 + p_z\lambda]} < E_T^\dagger(R,\alpha), \quad \Lambda_G^\dagger(R, q_z, \alpha) = \frac{\pi[\alpha K^2+(1-\alpha)\pi^2+\lambda]}{K(K^2-\pi^2)}$$

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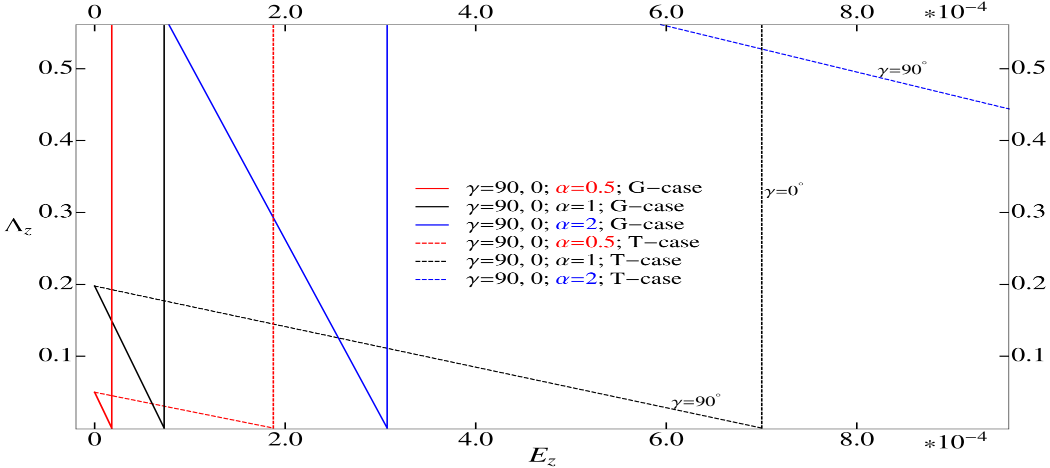
Regime diagrams in Sa ($\alpha < 1$) and So ($\alpha > 1$) anisotropy for most unstable modes in T and G cases

$\Lambda_z E_z$ diagrams for several T and G cases in Sa and So anisotropy compared with isotropic case.



...back to most unstable modes equations

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Conclusions

- In the isotropic most unstable modes, the modes with maximum growth rate, the convection is facilitated in the general case, G-modes, (see regime diagram) . Thus, not only heat and buoyancy, but all forces, including magnetic and Coriolis ones, contribute to great instability
- Anisotropy strongly influences the occurrence of these most unstable modes (in T and G cases) and even more significantly than steady ($\lambda = 0$) marginal modes of convection
- Like in steady marginal modes, also in the most unstable modes the Sa, atmospheric SA anisotropy, strongly facilitates the convection
- Future development: rederiving these anisotropic magnetoconvection models for different directions of magnetic fields, e.g., with vertical magnetic field. Moreover, a model with rotation and magnetic field in the horizontal plane, but orthogonal each other, will also be studied

anisotropy = anisotropic diffusion, anisotropic diffusive coefficients,anisotropic diffusivities

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There is strong belief that the Earth's Core is in turbulent state

The Earth's core is driven into motion by buoyancy forces so strong that the flow and field are turbulent, fluctuating on every length and time scale, as it is accepted by the most of geophysicists. Traditional approach to turbulence (see, e.g., Krause & Rädler, 1980; Sokoloff et al., 2014) splits each variables into mean and fluctuating ones, e.g., $\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}'$, $\mathbf{B} = \overline{\mathbf{B}} + \mathbf{B}'$, thus the mean IE is:

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \eta_0 \nabla^2 \overline{\mathbf{B}} + \boldsymbol{\nabla} \times (\overline{\mathbf{u}} \times \overline{\mathbf{B}}) + \boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}}, \quad \boldsymbol{\mathcal{E}} = \overline{\mathbf{u}' \times \mathbf{B}'}$$

within suitable approximations and conditions we can say that $\boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}} = -\boldsymbol{\nabla} \times (\beta \boldsymbol{\nabla} \times \overline{\mathbf{B}})$, the well known "$\beta$-effect". In general, due to turbulence the diffusive coefficients are anisotropic; for instance the buoyancy has a preferred direction $\Rightarrow$ local turbulence may be significantly anisotropic with respect to gravity direction

$$\boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}} = -\boldsymbol{\nabla} \times (\beta \boldsymbol{\nabla} \times \overline{\mathbf{B}}) \rightarrow \boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}} = -\boldsymbol{\nabla} \times (\boldsymbol{\beta} \boldsymbol{\nabla} \times \overline{\mathbf{B}}) = \boldsymbol{\nabla} \cdot (\boldsymbol{\beta} \boldsymbol{\nabla} \overline{\mathbf{B}})$$

where $\boldsymbol{\beta}$ is a tensor quantity, thus, in general we should speak about an "anisotropic beta-effect" or anisotropic magnetic diffusivity, anisotropic $\boldsymbol{\eta}$ tensor ($\boldsymbol{\eta} = \boldsymbol{\eta}_0 + \boldsymbol{\beta}$), which parameterizes mean electromotive force, $\boldsymbol{\mathcal{E}}$.
Analogous considerations can be made (see, e.g., Fearn & Roberts, 2007) for Navier-Stokes and heat equations

$$\rho \left[\frac{\partial \overline{\mathbf{u}}}{\partial t} + \overline{\mathbf{u}} \cdot \boldsymbol{\nabla} \overline{\mathbf{u}} + 2\Omega \hat{\mathbf{k}} \times \overline{\mathbf{u}} \right] = \boldsymbol{\nabla} \overline{p} + \nu \nabla^2 \overline{\mathbf{u}} + (1/\mu_0) \overline{\mathbf{B}} \cdot (\boldsymbol{\nabla} \overline{\mathbf{B}}) + \boldsymbol{\nabla} \cdot \overline{\mathbf{u}'\mathbf{u}' - \mathbf{B}'\mathbf{B}'}$$

where, within some approximations $\boldsymbol{\nabla} \cdot \overline{\mathbf{u}'\mathbf{u}' - \mathbf{B}'\mathbf{B}'} = \boldsymbol{\nabla} \cdot (\boldsymbol{\nu}_T \boldsymbol{\nabla} \overline{\mathbf{u}})$, tensorial turbulent viscosity, $\boldsymbol{\nu}_T$, parameterizes Reynolds and Maxwell stresses related to $\overline{\mathbf{u}'\mathbf{u}' - \mathbf{B}'\mathbf{B}'}$.

$$\frac{\partial \overline{\Theta}}{\partial t} + \overline{\mathbf{u}} \cdot \boldsymbol{\nabla} \overline{\Theta} = \kappa \nabla^2 \overline{\Theta} - \boldsymbol{\nabla} \cdot \overline{\mathbf{u}'\Theta'}$$

where, in some cases $-\boldsymbol{\nabla} \cdot \overline{\mathbf{u}'\Theta'} = \boldsymbol{\nabla} \cdot (\boldsymbol{\kappa}_T \boldsymbol{\nabla} \overline{\Theta})$, $\boldsymbol{\kappa}_T$ is the parameterized tensorial turbulent thermal diffusivity.

Thus, introducing anisotropy in the governing equations...

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Different anisotropic rotating magnetoconvection models studied

k	isotropic	anisotropic	name	reference
i	ν, κ, η	–	isotropy	Roberts & Jones (2000)
h	ν, η	κ	pure-h anisotropy	Donald & Roberts (2004)
m	ν, κ	η	pure-m anisotropy	Filippi & Brestenský (2020)
p	η	ν, κ	partial-p anisotropy	Šoltis & Brestenský (2010)
q	ν	κ, η	partial-q anisotropy	Filippi & Brestenský (2020)
f	–	ν, κ, η	full anisotropy	Filippi et al. (2019)

Table: Different anisotropy cases, $k = i,\ h,\ m, p, q$ and f, with related isotropic or/and anisotropic diffusivities, ν, κ and η. Cases i, p and f have been already studied and published, cases m and q are new (FB) and case h is not yet published.

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Anisotropic diffusive coefficients in SA and BM anisotropy (partial and full)

SA and BM anisotropic diffusions modelling

Partial anisotropy model by Šoltis & Brestenský (2010) with $\boldsymbol{\nu}$, $\boldsymbol{\kappa}$ anisotropic tensors and $\boldsymbol{\eta}_0$ isotropic tensor, where

$$\boldsymbol{\nu} = \begin{pmatrix} \nu_{xx} & 0 & 0 \\ 0 & \nu_{yy} & 0 \\ 0 & 0 & \nu_{zz} \end{pmatrix}, \quad \boldsymbol{\kappa} = \begin{pmatrix} \kappa_{xx} & 0 & 0 \\ 0 & \kappa_{yy} & 0 \\ 0 & 0 & \kappa_{zz} \end{pmatrix}, \quad \boldsymbol{\eta}_0 = \begin{pmatrix} \eta_0 & 0 & 0 \\ 0 & \eta_0 & 0 \\ 0 & 0 & \eta_0 \end{pmatrix}.$$

Magnetic diffusion also due to turbulence inspired a full anisotropy model (Filippi et al., 2019) with anisotropic $\boldsymbol{\nu}$, $\boldsymbol{\kappa}$, $\boldsymbol{\eta}$, where $\eta_{xx} = \eta_0 + \beta_{xx}$, $\eta_{yy} = \eta_0 + \beta_{yy}$, $\eta_{zz} = \eta_0 + \beta_{zz}$.

Later, we will speak about another kind of anisotropy, heat transport anisotropy, a partial anisotropy with $\boldsymbol{\nu}$ also isotropic.

SA, Stratification anisotropy:

$$\nu_{xx} = \nu_{yy} \neq \nu_{zz}, \ \kappa_{xx} = \kappa_{yy} \neq \kappa_{zz}, \eta_{xx} = \eta_{yy} = \eta_{zz} = \eta_0 \quad \text{and} \quad \eta_{xx} = \eta_{yy} \neq \eta_{zz}.$$

Gravity or/and the Archimedean buoyancy force are dominant

BM by Braginsky & Meytlis (1990):

$$\nu_{xx} < \nu_{yy} = \nu_{zz}, \ \kappa_{xx} < \kappa_{yy} = \kappa_{zz}, \ \eta_{xx} = \eta_{yy} = \eta_{zz} = \eta_0 \quad \text{and} \quad \eta_{xx} < \eta_{yy} = \eta_{zz}.$$

Rotation and magnetic field are dominant

There is "horizontal isotropy" in SA, but not in BM

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Dimensionless governing equations and parameters

In magnetoconvection problems, for simplicity we use $\tilde{\mathbf{u}}$, $\tilde{\mathbf{b}}$, $\tilde{\vartheta}$ instead of $\mathbf{u}'$, $\mathbf{B}'$, Θ' used when we spoke about turbulence and anisotropy. In the most general anisotropy model we get the following main dimensionless equations, after using standard procedures

$$R_o\partial_t\mathbf{u} + \hat{\mathbf{z}} \times \mathbf{u} = -\boldsymbol{\nabla} P + \Lambda_z(\boldsymbol{\nabla} \times \mathbf{b}) \times \hat{\mathbf{y}} + R\vartheta\hat{\mathbf{z}} + E_z\nabla_\nu^2\mathbf{u},$$

$$\partial_t\mathbf{b} = \boldsymbol{\nabla} \times (\mathbf{u} \times \hat{\mathbf{y}}) + \nabla_\eta^2\mathbf{b},$$

$$q_z^{-1}\partial_t\vartheta = \hat{\mathbf{z}} \cdot \mathbf{u} + \nabla_\kappa^2\vartheta,$$

$$\nabla \cdot \mathbf{u} = 0, \qquad \nabla \cdot \mathbf{b} = 0,$$

SA anisotropy: $E_z\nabla_\nu^2\mathbf{u} = E_z[(1-\alpha_\nu)\partial_{zz} + \alpha_\nu\nabla^2]\mathbf{u}, \quad \nabla_\kappa^2\vartheta = [(1-\alpha_\kappa)\partial_{zz} + \alpha_\kappa\nabla^2]\vartheta,$

$$\nabla_\eta^2\mathbf{b} = [(1-\alpha_\eta)\partial_{zz} + \alpha_\eta\nabla^2]\mathbf{b}$$

BM anisotropy: $E_z\nabla_\nu^2\mathbf{u} = E_z[(\alpha_\nu - 1)\partial_{xx} + \nabla^2]\mathbf{u}, \quad \nabla_\kappa^2\vartheta = [(\alpha_\kappa - 1)\partial_{xx} + \nabla^2]\vartheta,$

$$\nabla_\eta^2\mathbf{b} = [(\alpha_\eta - 1)\partial_{xx} + \nabla^2]\mathbf{b}$$

$\alpha_\nu = \frac{\nu_{xx}}{\nu_{zz}}, \quad \alpha_\kappa = \frac{\kappa_{xx}}{\kappa_{zz}}, \quad \alpha_\eta = \frac{\eta_{xx}}{\eta_{zz}}$ we use common dimensionless parameters and standard techniques to solve these equations

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Method of solution in the most general case

A solution can be found by some methods developed, e.g., in Chandrasekhar (1961):

$$\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{B} = 0 \Rightarrow \mathbf{u} = a^{-2}[\nabla \times (\nabla \times w\hat{\mathbf{z}}) + \nabla \times \omega\hat{\mathbf{z}}] \quad \text{and} \quad \mathbf{b} = a^{-2}[\nabla \times (\nabla \times b\hat{\mathbf{z}}) + \nabla \times j\hat{\mathbf{z}}]$$

All perturbations $(w, \omega, b, j$ and $\vartheta)$ have a form like $f(x, y, z, t) = \Re e[F(z)\exp(ilx + imy)\exp(\lambda t)]$ with horizontal wave number $a = \sqrt{l^2 + m^2}$, complex growth rate $\lambda = \lambda_R + i\lambda_I \in \mathbb{C}$ (λ_I is frequency), $F(z) = W(z)$, $\Omega(z)$, $B(z)$, $J(z)$, and $\Theta(z)$
If we take some curls of momentum, induction and heat equations, introduce inverted magnetic Prandtl number $p = \eta_{zz}/\nu_{zz} = R_o/E_z$ and we consider the simplest boundary conditions at $z = \pm 1/2$, i.e.:

- stress free boundaries: $W = D^2W = D\Omega = 0$
- perfectly thermally conducting boundaries: $\Theta = 0$
- perfectly electrically conducting boundaries: $B = DJ = 0$.

"dispersion relation"

$$Ra^2 \frac{q_z(K_\eta^2 + \lambda)}{(q_z K_\kappa^2 + \lambda)} = \frac{K^2 N_f^2 + \pi^2 \left(K_\eta^2 + \lambda\right)^2}{N_f} \tag{1}$$

where N_f

$$N_f = (K_\eta^2 + \lambda)(K_\nu^2 + p\lambda)E_z + \Lambda_z m^2, \tag{2}$$

$K^2 = \pi^2 + a^2$,
$K_\kappa^2 = \pi^2 + \alpha_\kappa a^2, \quad K_\nu^2 = \pi^2 + \alpha_\nu a^2, \quad K_\eta^2 = \pi^2 + \alpha_\eta a^2$ in SA
$K_\kappa^2 = \pi^2 + \alpha_\kappa l^2 + m^2, \quad K_\nu^2 = \pi^2 + \alpha_\nu l^2 + m^2, \quad K_\eta^2 = \pi^2 + \alpha_\eta l^2 + m^2$ in BM

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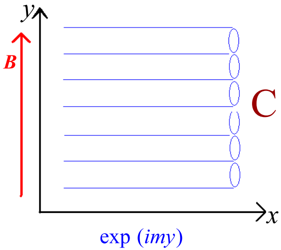
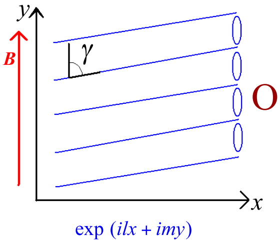
The orientation of the rolls of convection

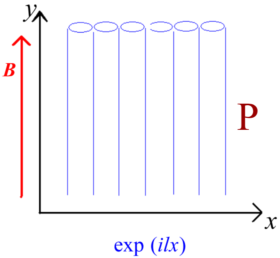
Roberts & Jones (2000)



Cross rolls (stationary convection SC mode)



Oblique roll (stationary convection SO mode)



Parallel rolls (stationary convection P mode)

$\gamma = \arctan\left(\frac{m}{l}\right)$ is the angle between the axis of rolls and the magnetic field $\boldsymbol{B}$.
$l = 0 \Rightarrow \gamma = \pi/2$ rolls $\perp$ $\boldsymbol{B}$, $m = 0 \Rightarrow \gamma = 0$ rolls $\parallel$ $\boldsymbol{B}$

preferred modes minimize R over l, and m. graphical results concerning the marginal modes $(\lambda = 0)$ about partial and full Sa, So and BM anisotropy; most unstable modes maximize λ over l and m

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Comparison of all anisotropic rotating magnetoconvection models studied

Several models of anisotropic magnetoconvection studied: isotropy (i), heat transport (h), partial-p (p), partial-q (q) and full (f) anisotropies; for simplicity $\alpha_\kappa = \alpha_\nu = \alpha_\eta = \alpha$

$$i - \text{case} \Rightarrow N_i = (K^2+\lambda)(K^2+p\lambda)E + \Lambda m^2, \quad R\frac{q_z(K^2+\lambda)}{(q_zK^2+\lambda)}a^2 = \frac{K^2N_i^2 + \pi^2\left(K^2+\lambda\right)^2}{N_i}$$

$$h - \text{case} \Rightarrow N_h = (K^2+\lambda)(K^2+p\lambda)E + \Lambda m^2, \quad R\frac{q_z(K^2+\lambda)}{(q_zK_\alpha^2+\lambda)}a^2 = \frac{K^2N_h^2 + \pi^2\left(K^2+\lambda\right)^2}{N_h}$$

$$m - \text{case} \Rightarrow N_m = (K_\alpha^2+\lambda)(K^2+p\lambda)E + \Lambda_z m^2, \quad R\frac{q_z(K_\alpha^2+\lambda)}{(q_zK^2+\lambda)}a^2 = \frac{K^2N_m^2 + \pi^2\left(K^2+\lambda\right)^2}{N_m}$$

$$p - \text{case} \Rightarrow N_p = (K^2+\lambda)(K_\alpha^2+p\lambda)E_z + \Lambda m^2, \quad R\frac{q_z(K^2+\lambda)}{(q_zK_\alpha^2+\lambda)}a^2 = \frac{K^2N_p^2 + \pi^2\left(K^2+\lambda\right)^2}{N_p}$$

$$q - \text{case} \Rightarrow N_q = (K_\alpha^2+\lambda)(K^2+p\lambda)E + \Lambda_z m^2, \quad R\frac{q_z(K_\alpha^2+\lambda)}{(q_zK_\alpha^2+\lambda)}a^2 = \frac{K^2N_q^2 + \pi^2\left(K_\alpha^2+\lambda\right)^2}{N_q}$$

$$f - \text{case} \Rightarrow N_f = (K_\alpha^2+\lambda)(K_\alpha^2+p\lambda)E_z + \Lambda_z m^2, \quad R\frac{q_z(K_\alpha^2+\lambda)}{(q_zK_\alpha^2+\lambda)}a^2 = \frac{K^2N_f^2 + \pi^2\left(K_\alpha^2+\lambda\right)^2}{N_f}$$

some regime diagrams about h, p and f anisotropy, m, q and f anisotropy and all cases

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$\Lambda_z - E_z$ Regime diagrams in SA anisotropy for the stationary marginal modes

For each SA anisotropies after getting the a_c, l_c, m_c for oblique modes it is possible to find the following relationships between Λ_z and E_z:

$$\Lambda_z = \Lambda_k^* \left(1 - \frac{E_z}{E_k^*}\right) \equiv (C/O), \quad E_z = E_k^* \equiv (O/P)$$

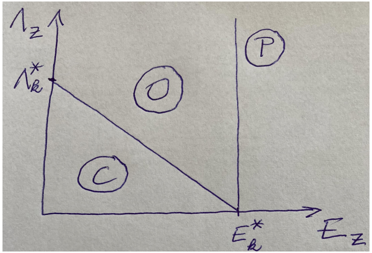
k	E	Λ	$\pi^2 E_k^*$	Λ_k^*
i	E	Λ	$3^{-1}3^{-1/2}$	$3^{1/2}2^{-1}$
h	E	Λ	$3_\alpha^{-1}3_\alpha^{-1/2}$	$3_\alpha^{1/2}2_\alpha^{-1}$
m	E	Λ_z	$3^{-1}3^{-1/2}$	$(1+2\alpha)2^{-1}3^{-1/2}$
p	E_z	Λ	$\alpha_3^{-1}3_\alpha^{-1/2}$	$3_\alpha^{1/2}2_\alpha^{-1}$
q	E	Λ_z	$3_\alpha^{-1}3_\alpha^{-1/2}$	$\alpha_3 3_\alpha^{-1/2}2_\alpha^{-1}$
f	E_z	Λ_z	$\alpha_3^{-1}3_\alpha^{-1/2}$	$\alpha_3 3_\alpha^{-1/2}2_\alpha^{-1}$

Table: Different variables and special dimensionless parameters

$$2_\alpha = \frac{1+\sqrt{1+8\alpha}}{2\alpha}, \quad 3_\alpha = 1 + 2_\alpha, \quad \alpha_3 = 1 + \alpha 2_\alpha$$

$\Lambda_z < \Lambda_k^* \left(1 - \frac{E_z}{E_k^*}\right) \Rightarrow$ preference of cross modes C

$E_z > E_k^* \Rightarrow$ preference of parallel modes P



Λ_z vs E_z is regime diagram in which all C, O, P modes are in their regions Ⓒ, Ⓞ, Ⓟ, respectively. The curves are in their (C/O) and (O/P) are the boundaries between Ⓒ and Ⓞ, and Ⓞ and Ⓟ regions, respectively.

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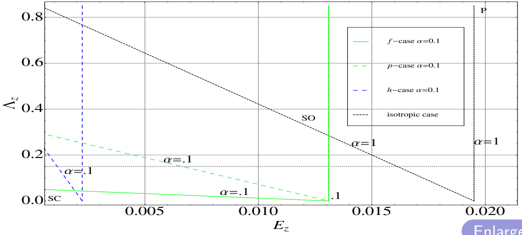
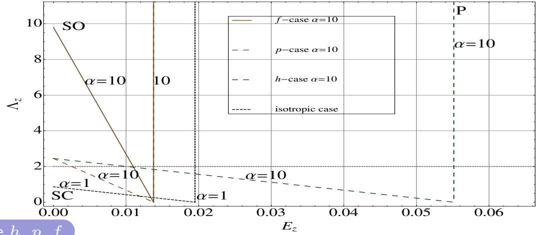
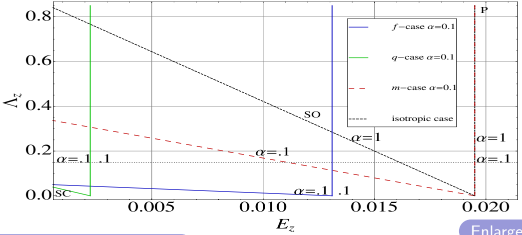
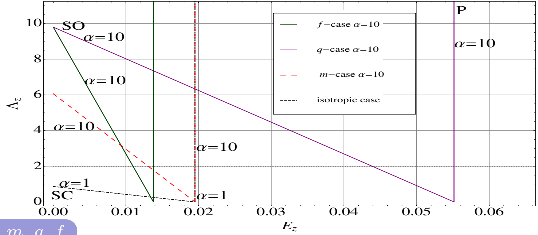
Some ΛE_z Regime diagrams in SA anisotropy with $\alpha = 0.1, 10$





Enlarge h, p, f





Enlarge m, q, f

these regime diagrams refers to...

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Anisotropic diffusive coefficients in SA and BM anisotropy (partial and full)

Assumption 1: following anisotropy in the viscosity and thermal diffusivity holds

$$\boldsymbol{\nu} = \begin{pmatrix} \nu_{xx} & 0 & 0 \\ 0 & \nu_{yy} & 0 \\ 0 & 0 & \nu_{zz} \end{pmatrix}, \qquad \boldsymbol{\kappa} = \begin{pmatrix} \kappa_{xx} & 0 & 0 \\ 0 & \kappa_{yy} & 0 \\ 0 & 0 & \kappa_{zz} \end{pmatrix}.$$

Assumption 2: same diffusion in two directions, but a different diffusion in the third one. This is called *partial anisotropy* for reasons which will be clearer later. However, in a more general case we need to deal also with the anisotropy in the magnetic diffusion also due to turbulence

$$\eta_{xx} = \eta_0 + \beta_{xx}, \; \eta_{yy} = \eta_0 + \beta_{yy}, \; \eta_{zz} = \eta_0 + \beta_{zz}$$

We call this situation *full anisotropy* (Filippi et al., 2019) considering magnetic diffusivity also anisotropic

SA, Stratification anisotropy:

$$\nu_{xx} = \nu_{yy} \neq \nu_{zz}, \; \kappa_{xx} = \kappa_{yy} \neq \kappa_{zz}, \eta_{xx} = \eta_{yy} = \eta_{zz} = \eta_0 \quad \text{and} \quad \eta_{xx} = \eta_{yy} \neq \eta_{zz}.$$

Gravity or/and the Archimedean buoyancy force lead the dynamics of turbulent eddies

BM by Braginsky & Meytlis (1990):

$$\nu_{xx} < \nu_{yy} = \nu_{zz}, \; \kappa_{xx} < \kappa_{yy} = \kappa_{zz}, \; \eta_{xx} = \eta_{yy} = \eta_{zz} = \eta_0 \quad \text{and} \quad \eta_{xx} < \eta_{yy} = \eta_{zz}.$$

Rotation and magnetic field lead the dynamics of turbulent eddies

There is "horizontal isotropy" in SA, but not in BM

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Heat transport anisotropy

Donald & Roberts (2004) developed a dynamo model which introduces anisotropy only in the thermal diffusivity. Inspired by them we began to study also another case of anisotropy in rotating magnetoconvection, we call it Heat transport anisotropy.

$$\boldsymbol{\nu} = \begin{pmatrix} \nu_0 & 0 & 0 \\ 0 & \nu_0 & 0 \\ 0 & 0 & \nu_0 \end{pmatrix}, \quad \boldsymbol{\kappa} = \begin{pmatrix} \kappa_{xx} & 0 & 0 \\ 0 & \kappa_{yy} & 0 \\ 0 & 0 & \kappa_{zz} \end{pmatrix}, \quad \boldsymbol{\eta}_0 = \begin{pmatrix} \eta_0 & 0 & 0 \\ 0 & \eta_0 & 0 \\ 0 & 0 & \eta_0 \end{pmatrix}.$$

This new simplified model is developed for wider chance to compare the effects of more anisotropic models.

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Dimensionless parameters

where $\tilde{z} = dz$, $\tilde{\mathbf{b}} = B_M \mathbf{b}$ and

$$\tilde{\mathbf{u}} = U\mathbf{u} = \frac{\eta_{zz}}{d}\mathbf{u}, \quad \tilde{t} = \frac{d}{U}t = \frac{d^2}{\eta_{zz}}t, \quad \tilde{P} = 2\Omega_0 \eta_{zz} \rho_0 P, \quad \tilde{\vartheta} = \frac{\eta_{zz}\Delta T}{\kappa_{zz}}\vartheta$$

$$R_o = \frac{\eta_{zz}}{2\Omega_0 d^2}, \; \Lambda_z = \frac{B_M^2}{2\Omega_0 \rho_0 \mu_0 \eta_{zz}}, \; E_z = \frac{\nu_{zz}}{2\Omega_0 d^2}, \; R = \frac{\alpha_T g \Delta T d}{2\Omega_0 \kappa_{zz}}, \; q_z = \frac{\kappa_{zz}}{\eta_{zz}}, \; p_z = \frac{R_o}{E_z}$$

$$\alpha_\nu = \frac{\nu_{xx}}{\nu_{zz}}, \quad \alpha_\kappa = \frac{\kappa_{xx}}{\kappa_{zz}}, \quad \alpha_\eta = \frac{\eta_{xx}}{\eta_{zz}}$$

the main equations can be solved by standard techniques

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Method of solution in the most general case

We look for a solution by using some methods developed, e.g., in Chandrasekhar (1961): $\mathbf{u}$ and $\mathbf{B}$ are divergenceless, therefore

$$\mathbf{u} = a^{-2}[\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times w\hat{\mathbf{z}}) + \boldsymbol{\nabla} \times \omega\hat{\mathbf{z}}] \quad \text{and} \quad \mathbf{b} = a^{-2}[\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times b\hat{\mathbf{z}}) + \boldsymbol{\nabla} \times j\hat{\mathbf{z}}]$$

All perturbations (w, ω, b, j and ϑ) have a form

$$f(x, y, z, t) = \Re e[F(z) \exp(ilx + imy) \exp(\lambda t)]$$

$a = \sqrt{l^2 + m^2}$, $\lambda = i\sigma \in \mathbb{C}$, $F(z) = W(z)$, $\Omega(z)$, $B(z)$, $J(z)$, and $\Theta(z)$

$$[E_z\mathcal{D}_\nu - R_o\lambda]\mathit{\Omega} + \mathrm{D}W + \mathrm{i}m\Lambda_z J = 0, \tag{1}$$

$$(\mathrm{D}^2 - a^2)[E_z\mathcal{D}_\nu - R_o\lambda]W - \mathrm{D}\mathit{\Omega} + \mathrm{i}m\Lambda_z(\mathrm{D}^2 - a^2)B = a^2 R\mathit{\Theta}, \tag{2}$$

$$(\mathcal{D}_\eta - \lambda)J + \mathrm{i}m\mathit{\Omega} = 0, \tag{3}$$

$$(\mathcal{D}_\eta - \lambda)B + \mathrm{i}mW = 0, \tag{4}$$

$$(\mathcal{D}_\kappa - \zeta\lambda)\mathit{\Theta} + W = 0, \tag{5}$$

$\zeta = q_z^{-1}$; $D = d/dz$; $\mathcal{D}_\kappa$, $\mathcal{D}_\nu$, $\mathcal{D}_\eta$ are equal to $D^2 - \alpha_\kappa l^2 - m^2$, $D^2 - \alpha_\nu l^2 - m^2$, $D^2 - \alpha_\eta l^2 - m^2$ and $D^2 - \alpha_\kappa a^2$, $D^2 - \alpha_\nu a^2$, $D^2 - \alpha_\eta a^2$ for BM and SA types of anisotropies, respectively.

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Critical Rayleigh numbers in stationary SA (partial, full) anisotropy

From now on, for simplicity we assume that in partial and full anisotropy:

$$\alpha_\kappa = \alpha_\nu = \alpha, \qquad \alpha_\kappa = \alpha_\nu = \alpha_\eta = \alpha$$

$\lambda = 0$ (marginal modes), $1_\alpha = (K_\alpha/K)^2$

$$R_p^s = \frac{\pi(\pi^2+a^2)^{1/2}K_\alpha^2}{a^2}\left(C_p + \frac{1}{C_p}\right), \; R^s = \frac{\pi(\pi^2+a^2)^{3/2}}{a^2}\left(C_\alpha + \frac{1_\alpha^2}{C_\alpha}\right).$$

where $C_p = \frac{E_z K^2 K_\alpha^2 + \Lambda m^2}{\pi K}$ and $C_\alpha = \frac{E_z K_\alpha^4 + \Lambda m^2}{\pi K}$
in SA $K_\alpha^2 = \pi^2 + \alpha a^2$ and

$$C_p = \frac{E_z(\pi^2+a^2)(\pi^2+\alpha a^2) + \Lambda m^2}{\pi(\pi^2+a^2)^{1/2}}, \; C_\alpha = \frac{E_z(\pi^2+\alpha a^2)^2 + \Lambda m^2}{\pi(\pi^2+a^2)^{1/2}} \qquad (6)$$

$$\frac{\partial R_p^s}{\partial C_p} = 0, \; \frac{\partial R_p^s}{\partial a^2} = 0 \qquad \text{and} \qquad \frac{\partial R^s}{\partial C_\alpha} = 0, \; \frac{\partial R^s}{\partial a^2} = 0$$

we have critical C_p, a^2 and C_α, a^2 in case of partial and full anisotropy respectively thus

$$C_{pc} = 1, \; a_c^2 = \pi^2 2_\alpha = \pi^2\frac{1+\sqrt{1+8\alpha}}{2\alpha} \qquad \text{and} \qquad C_{\alpha c} = 1_\alpha, a_c^2 = \pi^2 2_\alpha \qquad (7)$$

by the (6) and (7) it is possible to obtain m_c and l_c

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Critical Rayleigh numbers in stationary SA m-case anisotropy

$\alpha_\nu = \alpha_\kappa = 1, \ \alpha_\eta = \alpha$

$$R^s_m = \frac{\pi}{a^2} K^2 K_\alpha \left(\frac{C_m}{1_\alpha} + C_m^{-1} \right)$$

where

$$C_m = \frac{K_\alpha^2 K^2 E + \Lambda_z m^2}{\pi K_\alpha}$$

$$\frac{\partial R^s_m}{\partial C_m} = 0 \qquad \text{and} \qquad \frac{\partial R^s_m}{\partial a^2} = 0$$

$$C_{mc} = \sqrt{1_\alpha} \qquad \text{and} \qquad a_c^2 = 2\pi^2$$

a_c in pure-m anisotropy is identical to the isotropic one

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Critical Rayleigh numbers in stationary heat transport and partial-q SA anisotropy

From now on, for simplicity we assume that in heat transport and partial-q anisotropy:

$$\alpha_\kappa = \alpha, \alpha_\nu = \alpha_\eta = 1, \qquad \alpha_\nu = 1, \alpha_\kappa = \alpha_\eta = \alpha$$

$\lambda = 0,\ 1_\alpha = (K_\alpha/K)^2$

$$R_h^s = 1_\alpha \frac{\pi(\pi^2 + a^2)^{3/2}}{a^2}\left(C + \frac{1}{C}\right),\ R_q^s = \frac{\pi K_\alpha^3}{a^2}\left(\frac{C_q}{1_\alpha} + C_q^{-1}\right).$$

in SA $K_\alpha^2 = \pi^2 + \alpha a^2$ and

$$C = \frac{EK^4 + \Lambda m^2}{\pi K},\ C_q = \frac{K_\alpha^2 K^2 E + \Lambda_z m^2}{\pi K_\alpha}$$

$$\frac{\partial R_h^s}{\partial C} = 0,\ \frac{\partial R_h^s}{\partial a^2} = 0 \qquad \text{and} \qquad \frac{\partial R_q^s}{\partial C_q} = 0,\ \frac{\partial R_q^s}{\partial a^2} = 0$$

$$C_c = 1,\ a_c^2 = \pi^2 2_\alpha \quad \text{and} \quad C_{qc} = \sqrt{1_\alpha},\ a_c^2 = \pi^2 2_\alpha$$

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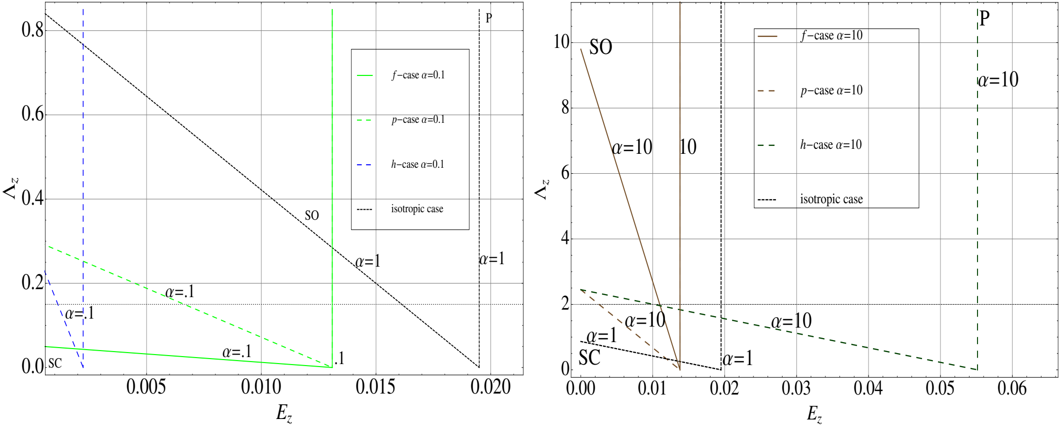
ΛE_z Regime diagrams in SA anisotropy (h, p, f with $\alpha = 0.1, 10$)



The ΛE_z regime diagrams for steady convection in SA anisotropy in h, p and f cases compared with isotropy. In the left figure atmospheric anisotropy (Sa), in the right figure oceanic anisotropy (So).

These regime diagrams...

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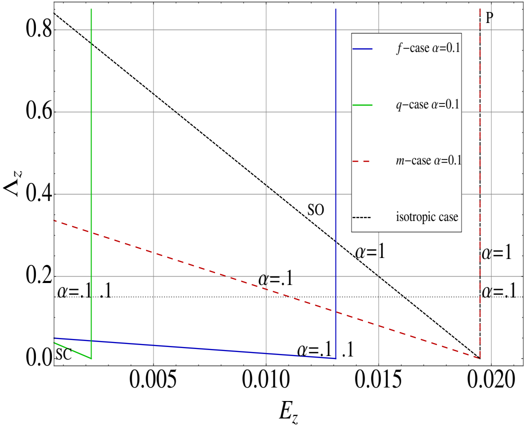
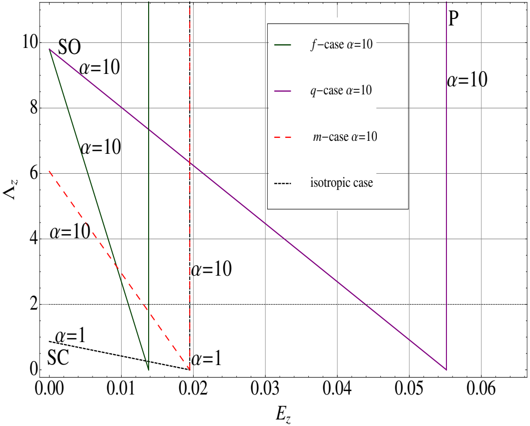
ΛE_z Regime diagrams in SA anisotropy (m, q, f with $\alpha = 0.1, 10$)





The ΛE_z regime diagrams for steady convection in SA anisotropy in m, q and f cases compared with isotropy. In the left figure atmospheric anisotropy (Sa), in the right figure oceanic anisotropy (So).

Regime diagrams...

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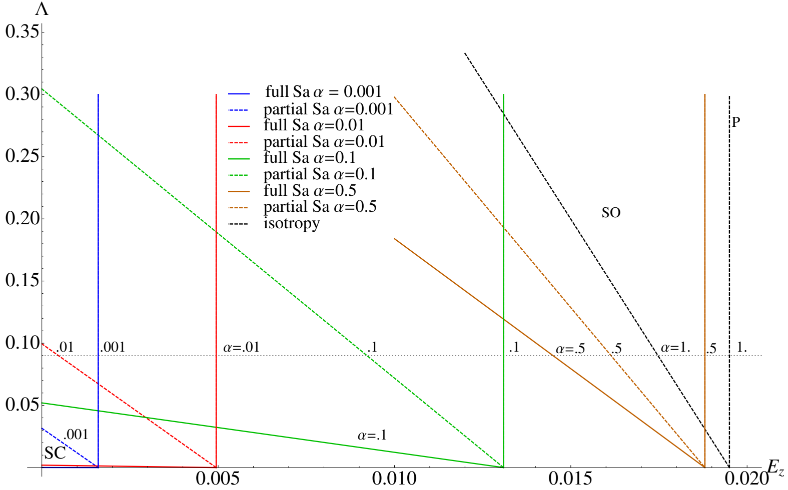
Regime diagrams in Sa anisotropy, $\alpha < 1$

ΛE_z diagrams for several cases of Sa anisotropy compared with isotropic case.



...back to rolls

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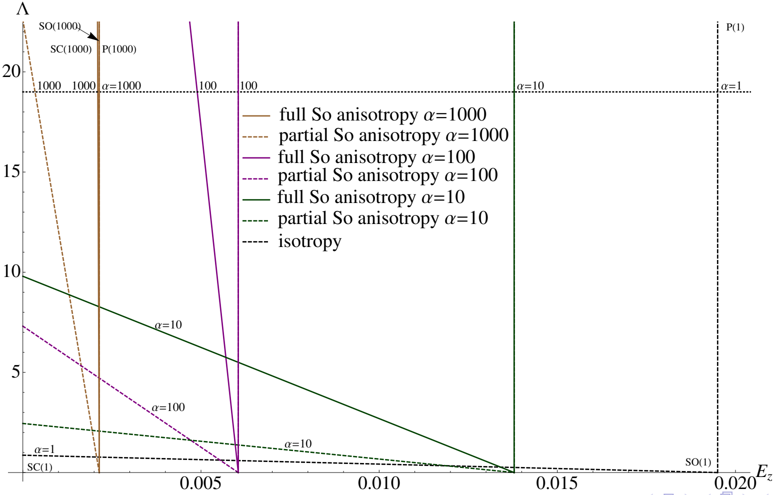
Regime diagrams in So anisotropy, $\alpha > 1$

ΛE_z diagrams for several cases of So anisotropy compared with isotropic case.



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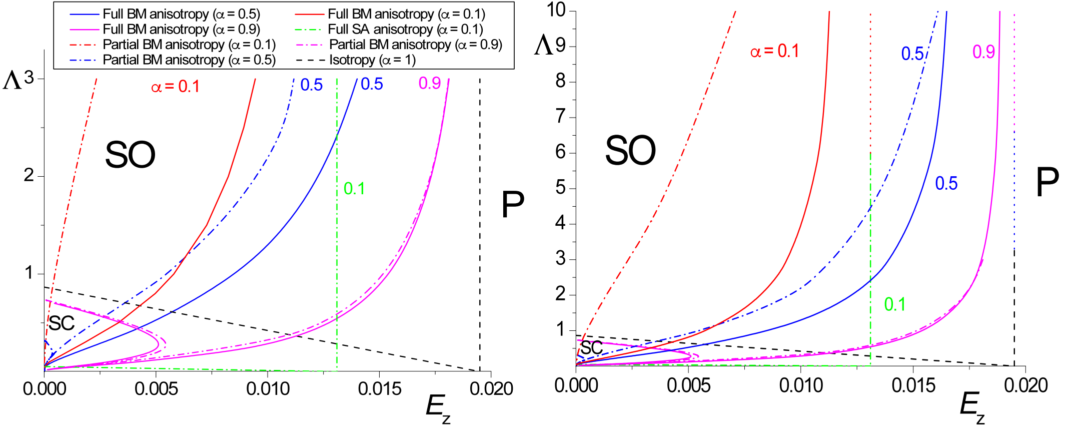
ΛE_z-Regime diagrams in BM anisotropy



The regime diagrams for steady convection in various cases of partial and full BM anisotropy in two different Λ-axis scales. For comparison cases of isotropy, $\alpha = 1$, and of strong Sa, $\alpha = 0.1$, are added; there are three asymptotes at $\Lambda \to \infty$, represented by the dotted vertical lines, for six SO/P lines for α = 0.1, 0.5, 0.9.

...back to rolls

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The most unstable modes in isotropic case

Roberts & King (2013);

$\lambda \neq 0$, $R_i = (\alpha_T g \Delta T d)/(2\Omega_0 \kappa)$, $R = (\alpha_T g \Delta T d)/(2\Omega_0 \kappa_{zz})$, $E = \nu/(2\Omega_0 d^2)$, $\Lambda = \frac{B_M^2}{2\Omega_0 \rho_0 \mu_0 \eta_0}$

the most unstable modes are the modes with maximum growth rate ($\partial_l \lambda = \partial_m \lambda = 0$)

Braginsky-Meytlis approximation (T-case) (see, e.g., Braginsky & Meytlis, 1990; Roberts & King, 2013):

$$R_i = \frac{K^2+\lambda/q}{K^2-\pi^2}\left(EK^4 + \Lambda m^2 + \frac{\pi^2 K^2}{EK^4+\Lambda m^2}\right), \lambda = -qK^2 + q\frac{R_i}{R_o}\frac{K^2-\pi^2}{K^2}\frac{X}{X^2+Y^2},$$

$$X = \frac{K^2}{p} + \frac{\Lambda}{R_o}\left(\frac{m}{K}\right)^2, Y = 2\Omega_0\left(\frac{\pi}{K}\right); \lambda_{max} = \frac{qR_i}{4\pi K}\left(K^2 - 3\pi^2\right) \text{ with } K:$$

$$EK^4 + \Lambda m^2 - \pi K = 0, \quad K^3 - \frac{R_i}{4\pi}K^2 - \frac{R_i}{4}\pi = 0, \quad \Lambda = c^2\left(\frac{\pi K - EK^4}{K^2-\pi^2}\right), \quad c^2 = (K^2-\pi^2)/m^2,$$

O/P boundary $E = \pi/K^3$

General case (G-case):

$$R_i = \frac{K^2(K^2+\lambda/q)}{(K^2+\lambda)(K^2-\pi^2)}\left[N_f + \frac{\pi^2(K^2+\lambda)^2/K^2}{N_f}\right], \quad N_f = E(K^2+\lambda)(K^2+p\lambda) + \Lambda m^2$$

$$\lambda = -qK^2 + q\frac{R_i}{R_o}\frac{K^2-\pi^2}{K^2}\frac{X}{X^2+Y^2}, X = \lambda + \frac{K^2}{p} + \frac{\Lambda}{R_o}\left(\frac{m^2}{K^2+\lambda}\right), Y = \frac{1}{R_o}\left(\frac{\pi}{K}\right); \lambda_{max} = \frac{qR_i}{4\pi K}\left(K^2-3\pi^2\right) \text{ with}$$

$$K: R_o K\lambda(K^2+\lambda) + EK^3(K^2+\lambda) + \Lambda m^2 K - \pi(K^2+\lambda) = 0, \quad \Lambda = c^2\left(\frac{K^2+\lambda}{K^2-\pi^2}\right)\left[\frac{\pi}{K} - E\left(K^2+p\lambda\right)\right],$$

O/P boundary $E = \pi/[K(K^2+p\lambda)]$, $K^3 - \frac{R_i}{4\pi}K^2 - \frac{R_i}{4}\pi = 0$,

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The most unstable modes in anisotropic case

$E_z = \nu_{zz}/(2\Omega_0 d^2), \Lambda_z = \frac{B_M^2}{2\Omega_0\rho_0\mu_0\eta_{zz}},$

anisotropic T-case:

$R = \frac{(K_\alpha^2+\lambda/q_z)K^2}{(K^2-\pi^2)K_\alpha^2}\left(N_f + \frac{\pi^2 K_\alpha^4}{N_f K^2}\right), N_f = E_z K_\alpha^4 + \Lambda_z m^2, \lambda = q\left(-K_\alpha^2 + \frac{R}{R_o}\frac{K^2-\pi^2}{K^2}\frac{X}{X^2+Y^2}\right),$

$X = \frac{K_\alpha^2}{p_z} + \frac{\Lambda_z}{R_o}\left(\frac{m}{K_\alpha}\right)^2, Y = \left(\frac{\pi}{R_o K}\right); \lambda_{max} = q_z\left[\frac{R}{4\pi K}(K^2-3\pi^2) + (\alpha-1)\pi^2\right]$, with K:

$\alpha K^3 - \frac{R_i}{4\pi}K^2 - \frac{R_i}{4}\pi = 0,$ $\Lambda_z = c^2\left(\frac{K_\alpha^2}{K^2-\pi^2}\right)\left(\frac{\pi}{K} - E_z K_\alpha^2\right),$ $c^2 = (K^2-\pi^2)/m^2,$

O/P boundary $E_z = \pi/(K K_\alpha^2)$

anisotropic G-case:

$R = \frac{K^2(K_\alpha^2+\lambda/q_z)}{(K_\alpha^2+\lambda)(K^2-\pi^2)}\left[N_f + \frac{\pi^2(K_\alpha^2+\lambda)^2/K^2}{N_f}\right], N_f = E_z(K_\alpha^2+\lambda)(K_\alpha^2+p_z\lambda) + \Lambda_z m^2,$

$\lambda = q_z\left[-K_\alpha^2 + \frac{R}{R_o}\frac{K^2-\pi^2}{K^2}\frac{X}{X^2+Y^2}\right], X = \lambda + \frac{K_\alpha^2}{p_z} + \frac{\Lambda_z}{R_o}\left(\frac{m^2}{K_\alpha^2+\lambda}\right), Y = \left(\frac{\pi}{R_o K}\right); \lambda_{max} =$

$q_z\left[\frac{R}{4\pi K}(K^2-3\pi^2) + (\alpha-1)\pi^2\right]$, with K:

$\alpha K^3 - \frac{R_i}{4\pi}K^2 - \frac{R_i}{4}\pi = 0,$ $\Lambda_z = c^2\left[\frac{K_\alpha^2+\lambda}{K^2-\pi^2}\right]\left\{\frac{\pi}{K} - E_z\left[K_\alpha^2 + p_z\lambda\right]\right\},$

O/P boundary $E_z = \pi/[K(K_\alpha^2 + p_z\lambda)]$

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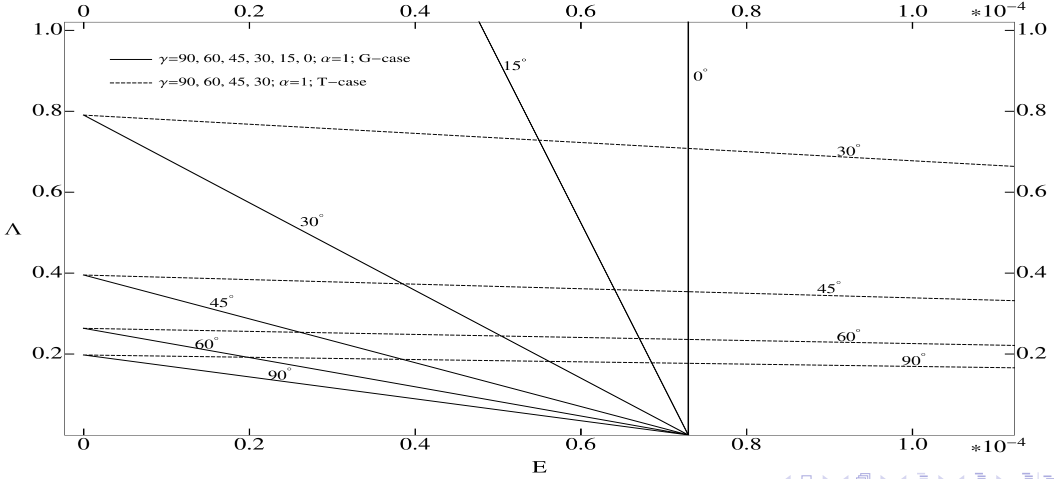
Regime diagrams for isotropic most unstable modes

Comparison among ΛE diagrams for different γ in isotropic T and G most unstable modes.



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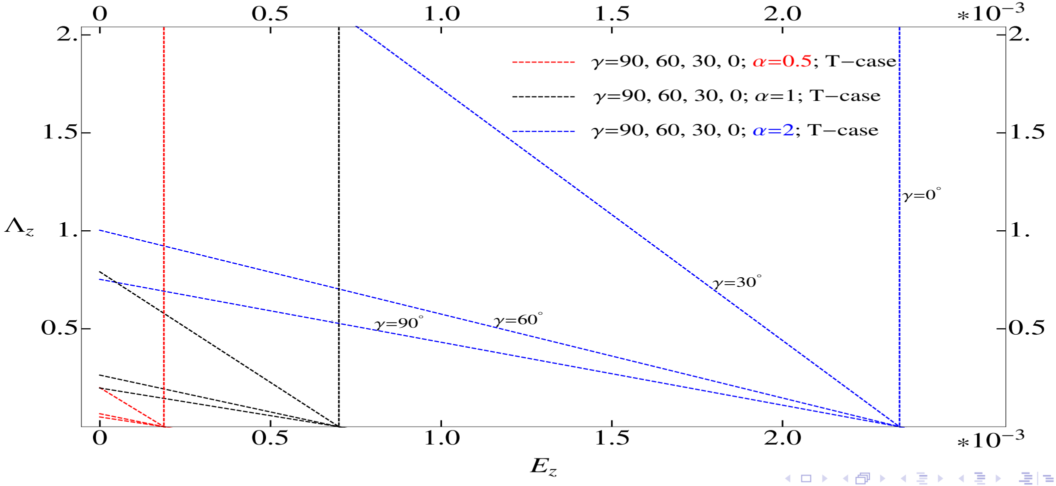
Regime diagrams for T most unstable modes in Sa and So anisotropy

Comparison among ΛE_z diagrams in T-cases for $\gamma=0^\circ$, 30°, 60°, 90° compared with isotropic cases.



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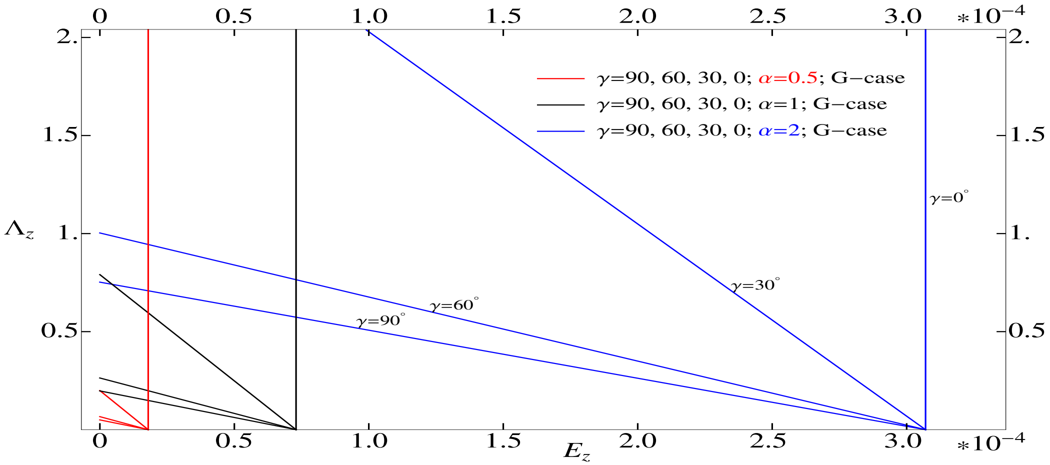
Regime diagrams for G most unstable modes in Sa and So anisotropy

Comparison among ΛE_z diagrams in G-cases for γ=0°, 30°, 60°, 90° compared with isotropic cases.



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Comparison between steady marginal and most unstable modes

α	$E_T^\dagger(\alpha)$	$E_G^\dagger(\alpha)$	$E^*(\alpha)$
0.5	$1.875 \cdot 10^{-4}$	$0.1797 \cdot 10^{-4}$	0.01880
1	$7.003 \cdot 10^{-4}$	$0.7295 \cdot 10^{-4}$	0.01950
2.	$23.43 \cdot 10^{-4}$	$3.070 \cdot 10^{-4}$	0.01884

Some values of isotropic and anisotropic $E^\dagger$ in T and G cases with $R = 200$, $q_z = 10^{-5}$, $p_z = 10^6$ compared with isotropic and anisotropic E^* for the steady modes.

α	$\Lambda_T^\dagger(\alpha)$	$\Lambda_G^\dagger(\alpha)$	$\Lambda^*(\alpha)$
0.5	0.0498242	0.0498247	0.393076
1	0.197652	0.197654	0.866025
2.	0.752291	0.752296	1.8413

Some values of isotropic and anisotropic $\Lambda^\dagger$ in T and G cases with $R = 200$, $q_z = 10^{-5}$ compared with isotropic and anisotropic Λ^* for the steady modes.