Global gravity field modelling by the finite element method involving mapped infinite elements

M. Macák*, Z. Minarechová, R. Čunderlík, K. Mikula and L. Tomek

Department of Mathematics and Descriptive Geometry
Slovak University of Technology, Faculty of Civil Engineering in Bratislava, Slovakia

23. May, 2022

*marek.macak@stuba.sk

Abstract

- The numerical approach for solving the fixed gravimetric boundary value problem with an oblique derivative (FGBVP) based on the finite element method (FEM) with mapped infinite elements is presented.
- In this approach, the 3D semi-infinite domain outside the Earth is bounded by the triangular approximation of the Earth's surface and extends to infinity.
- As a numerical method, the FEM with finite and mapped infinite triangular prisms - pentahedral elements has been derived and implemented.
- The key idea of this FEM approach is based on a division of the computational domain into two centrical parts, where the lower one is meshed with several layers of finite elements and the upper one with one layer of infinite elements.
- The global gravity field modelling using EGM2008 data is performed and discussed.

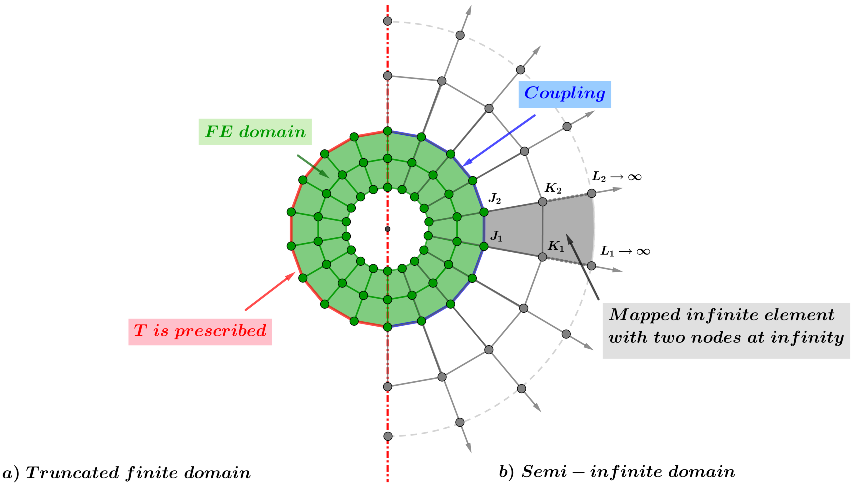
Formulation of the FGBVP

- Let us consider the FGBVP:

$$\begin{aligned} \Delta T(\mathrm{x}) &= 0, \quad \mathrm{x} \in \Omega, && (1) \\ \nabla T(\mathrm{x}) \cdot \mathrm{s}(\mathrm{x}) &= -\delta g(\mathrm{x}), \quad \mathrm{x} \in S, && (2) \\ T(\mathrm{x}) &\rightarrow 0, \quad \text{as} \quad |\mathrm{x}| \rightarrow \infty, && (3) \end{aligned}$$

where Ω is defined as $R^3 - S$, S is the Earth, $T(\mathrm{x})$ is the disturbing potential defined as a difference between the real and normal gravity potential at any point $\mathrm{x} = (x, y, z)$, $\delta g(\mathrm{x})$ is the gravity disturbance, and the vector $\mathrm{s}(\mathrm{x}) = -\nabla U(\mathrm{x})/|\nabla U(\mathrm{x})|$ is the unit vector normal to the equipotential surface of the normal potential $U(\mathrm{x})$ at any point x.

Formulation of the FGBVPs



Solution of the GBVP by the FEM with MIE

- In our approach, we follow the fundamental principles of FEM published in [1].
- We have obtained the weak formulation (1) - (3)

$$\int_{\Omega^e} \nabla T \cdot \nabla w \,\mathrm{d}x\mathrm{d}y\mathrm{d}z + \frac{c_2}{c_1} \int_{\Gamma^e} \frac{\partial T}{\partial \mathbf{t}_1} w \,\mathrm{d}\sigma + \frac{c_3}{c_1} \int_{\Gamma^e} \frac{\partial T}{\partial \mathbf{t}_2} w \,\mathrm{d}\sigma =$$

$$= \int_{\Gamma^e} \frac{-\delta g}{c_1} w \,\mathrm{d}\sigma + \int_{\partial\Omega^e \setminus \Gamma^e} \nabla T \cdot \mathbf{n}\, w \,\mathrm{d}\sigma. \tag{4}$$

[1] J.N. Reddy, An Introduction to the Finite Element Method, 3rd Edition, McGraw-Hill Education, New York, ISBN: 9780072466850 (2006)

Solution of the GBVP by the FEM with MIE

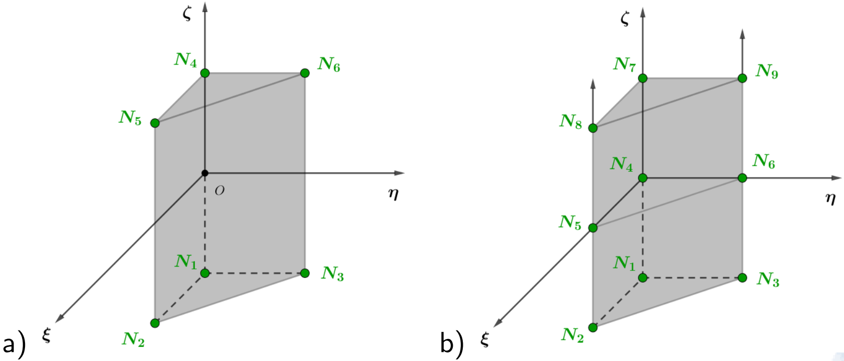


Figure 1: Types of elements used in our computations: a) The finite pentahedral elements with six nodes and b) mapped infinite pentahedral elements with nine nodes. Isoparametric coordinates are within intervals $0 \leq \xi \leq 1$, $0 \leq \eta \leq 1$ and $-1 \leq \zeta \leq 1$.

Solution of the GBVP by the FEM with MIE

- For a finite pentahedral element Ω^e with six nodes, see Fig. 1 a), we can write

$$T \approx T^e = \sum_{j=1}^{6} T_j^e \psi_j(x, y, z), \tag{5}$$

i. e. we take an approximation of the unknown value T as T^e, a linear combination of basis functions ψ_j with coefficients T_j^e, $j = 1, \ldots, 6$. Then we substitute it into the weak formulation (4), namely for elements Ω^e with indexes $k = 1, \ldots, n_1$, $l = 1, \ldots, n_2$ and $m = 2, \ldots, n_3 - 1$, and consider ψ_i for weight function w. We obtain the i^{th} equation in the form

$$\sum_{j=1}^{6} T_j^e \int_{\Omega^e} \frac{\partial \psi_j}{\partial x}\frac{\partial \psi_i}{\partial x} + \frac{\partial \psi_j}{\partial y}\frac{\partial \psi_i}{\partial y} + \frac{\partial \psi_j}{\partial z}\frac{\partial \psi_i}{\partial z} \, \mathrm{d}x\mathrm{d}y\mathrm{d}z = \sum_{j=1}^{6} \int_{\partial\Omega^e} q_n \, \psi_i \, \mathrm{d}x\mathrm{d}y, \tag{6}$$

where $q_n = \nabla T \cdot \mathbf{n}$ denotes the projection of the vector ∇T along unit normal $\mathbf{n}$.

Solution of the GBVP by the FEM with MIE

- For the row of elements Ω^e we obtain the i^{th} equation in the form

$$\sum_{j=1}^{6} T_j^e \left(\int\limits_{\Omega^e} \frac{\partial \psi_j}{\partial x}\frac{\partial \psi_i}{\partial x} + \frac{\partial \psi_j}{\partial y}\frac{\partial \psi_i}{\partial y} + \frac{\partial \psi_j}{\partial z}\frac{\partial \psi_i}{\partial z} \,\mathrm{d}x\mathrm{d}y\mathrm{d}z \right) +$$

$$+ \sum_{j=1}^{3} T_j^e \left(\frac{c_2}{c_1} \int\limits_{\Gamma^e} \frac{\partial \psi_j}{\partial \mathbf{t}_1} \psi_i \,\mathrm{d}x\mathrm{d}y + \frac{c_3}{c_1} \int\limits_{\Gamma^e} \frac{\partial \psi_j}{\partial \mathbf{t}_2} \psi_i \,\mathrm{d}x\mathrm{d}y \right) =$$

$$= \sum_{j=1}^{3} \int\limits_{\Gamma^e} \frac{-\delta g_j}{c_1} \psi_i \,\mathrm{d}x\mathrm{d}y + \sum_{j=1}^{6} \int\limits_{\partial\Omega^e \setminus \Gamma^e} q_n \, \psi_i \,\mathrm{d}x\mathrm{d}y, \qquad (7)$$

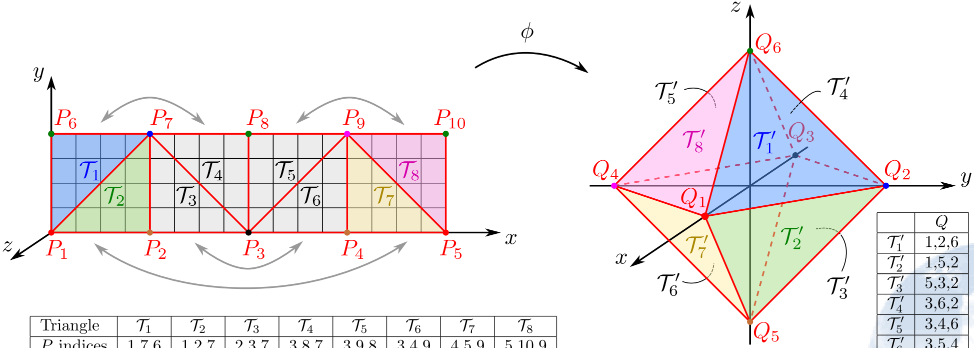
where index $j = 1, \ldots, 3$ refers to nodes of the element Ω^e that lie on the bottom boundary Γ of the computational domain Ω.

Solution of the GBVP by the FEM with MIE

- Finally, for the mapped infinite pentahedral element Ω^e with nine nodes we can write
$$T \approx T^e = \sum_{j=1}^{9} T_j^e \psi_j(x, y, z). \tag{8}$$
We substitute it for elements Ω^e with indexes $k = 1, \ldots, n_1$, $l = 1, \ldots, n_2$ and $m = n_3$ into the weak formulation (4), consider ψ_i for weight function w and we obtain the i^{th} equation in the form
$$\sum_{j=1}^{9} T_j^e \int_{\Omega^e} \frac{\partial \psi_j}{\partial x}\frac{\partial \psi_i}{\partial x} + \frac{\partial \psi_j}{\partial y}\frac{\partial \psi_i}{\partial y} + \frac{\partial \psi_j}{\partial z}\frac{\partial \psi_i}{\partial z} \,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \sum_{j=1}^{9} \int_{\partial\Omega^e} q_n \, \psi_i \,\mathrm{d}x\mathrm{d}y, \tag{9}$$
where $q_n = \nabla T \cdot \mathbf{n}$ denotes the projection of the vector ∇T along the unit normal $\mathbf{n}$.

Solution of the GBVP by the FEM with MIE

- The process of mesh generation is then performed in several steps.



Triangle	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}_4$	$\mathcal{T}_5$	$\mathcal{T}_6$	$\mathcal{T}_7$	$\mathcal{T}_8$
P indices	1,7,6	1,2,7	2,3,7	3,8,7	3,9,8	3,4,9	4,5,9	5,10,9

	Q
$\mathcal{T}'_1$	1,2,6
$\mathcal{T}'_2$	1,5,2
$\mathcal{T}'_3$	5,3,2
$\mathcal{T}'_4$	3,6,2
$\mathcal{T}'_5$	3,4,6
$\mathcal{T}'_6$	3,5,4
$\mathcal{T}'_7$	5,1,4
$\mathcal{T}'_8$	1,6,4

Figure 2: Piecewise affine mapping of rectangle (meshed with $n = 4$) onto regular octahedron.

Solution of the GBVP by the FEM with MIE

- In our numerical computations we used meshes with $n = 2^l$, where l is the level of refinement of the mesh. Examples of two coarse meshes (level $l = 2, 4$).

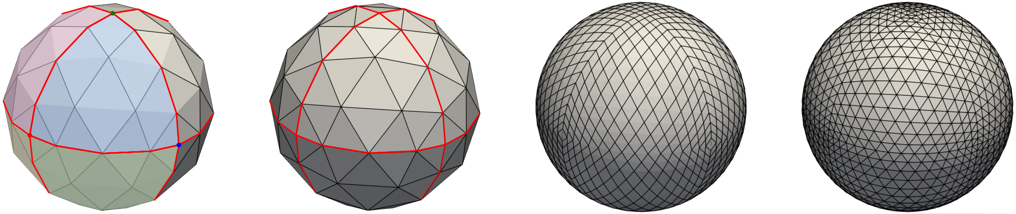


Figure 3: Quad and triangular meshes for levels $l = 2$ and $l = 4$.

Testing numerical experiments

- In the first testing experiment, the bottom boundary has been at level 6 371 [km], the height of the finite domain Ω_{FE} has been 500 [km], so the finite/infinite element interface has been at level 6 871 [km]. Then the center of the infinite elements has been at level 13 742 [km]. We have started with the mesh made up of $64 \times 16 \times 4$ nodes and then we performed four successive refinements.

No. of nodes	Min	Max	Mean	Median	STD	EOC
64x16x4	-0.5678	0.2036	-0.1057	-0.1239	0.1690	-
128x32x8	-0.1705	0.0460	-0.0282	-0.0317	0.0434	1.9597
256x64x16	-0.0490	0.0109	-0.0072	-0.0079	0.0112	1.9533
512x128x32	-0.0138	0.0027	-0.0018	-0.0019	0.0028	2.0134
1024x256x64	-0.0038	0.0007	-0.0005	-0.0005	0.0007	2.0076

Table 1: Statistics of residuals [m^2s^{-2}] on the bottom boundary S.

Testing numerical experiments

- For the second testing experiment we have chosen mesh consisting of $256 \times 64 \times 16$ nodes from *Testing experiment 1* and we have fixed the size of elements while redoubling the radius of the finite domain. It means that with a doubling of the height of the finite domain, the number of elements redoubled as well to remain the size of elements.

Height of Ω_{FE} $[km]$	No. of nodes	Min	Max	Mean	Median	STD
500	256x64x16	-0.0490	0.0109	-0.0072	-0.0078	0.0112
1000	256x64x32	-0.0487	0.0109	-0.0072	-0.0078	0.0111
2000	256x64x64	-0.0486	0.0108	-0.0072	-0.0079	0.0111
4000	256x64x128	-0.0485	0.0107	-0.0072	-0.0079	0.0110
8000	256x64x256	-0.0485	0.0106	-0.0072	-0.0079	0.0110

Table 2: Statistics of residuals $[m^2 s^{-2}]$ on the bottom boundary S.

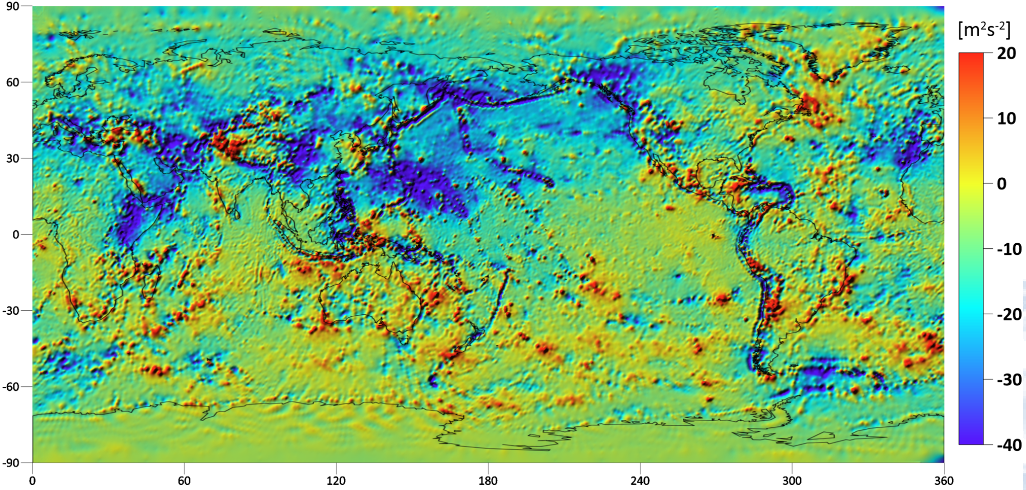
Global gravity field modelling

- The bottom boundary S has been the Earth's surface discretized by the series of triangles. The radial size of Ω_{FE} has been 5 000 $[km]$, so infinite elements started approximately at radius 11 371 $[km]$ and their center has been approximately at 22 742 $[km]$. Input surface gravity disturbances as BC (2) have been generated from Earth Gravitational Model 2008 (EGM2008).

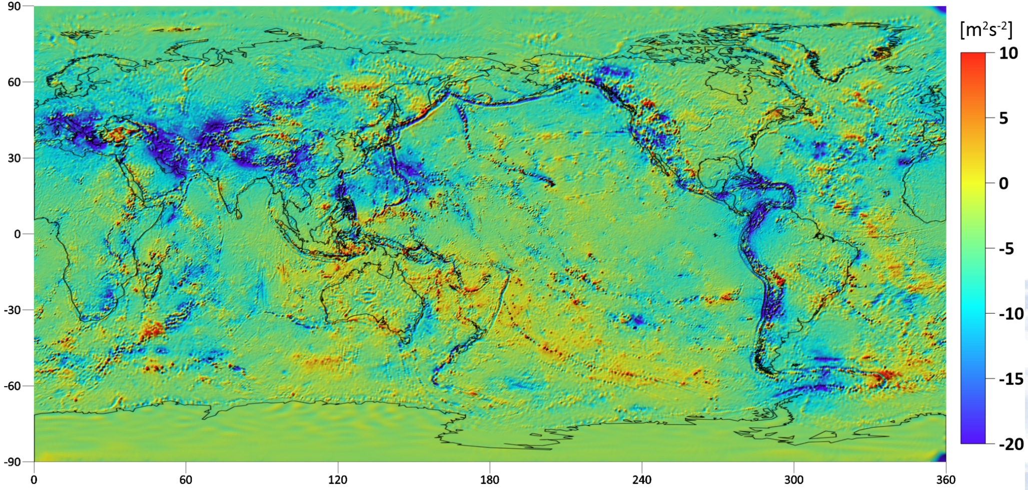
No. of nodes	Hmin	Min	Max	Mean	Median	STD
256x64x64	20000	-427.982	318.320	-24.785	-20.839	39.153
512x128x128	10000	-192.035	183.795	-9.951	-8.507	15.343
1024x256x256	5000	-77.456	82.134	-5.284	-4.930	5.200
2048x512x512	2500	-35.309	24.675	-1.358	-1.357	1.894

Table 3: *Experiment with gravity data*: Statistics of residuals $[m^2 s^{-2}]$ on the bottom boundary S.

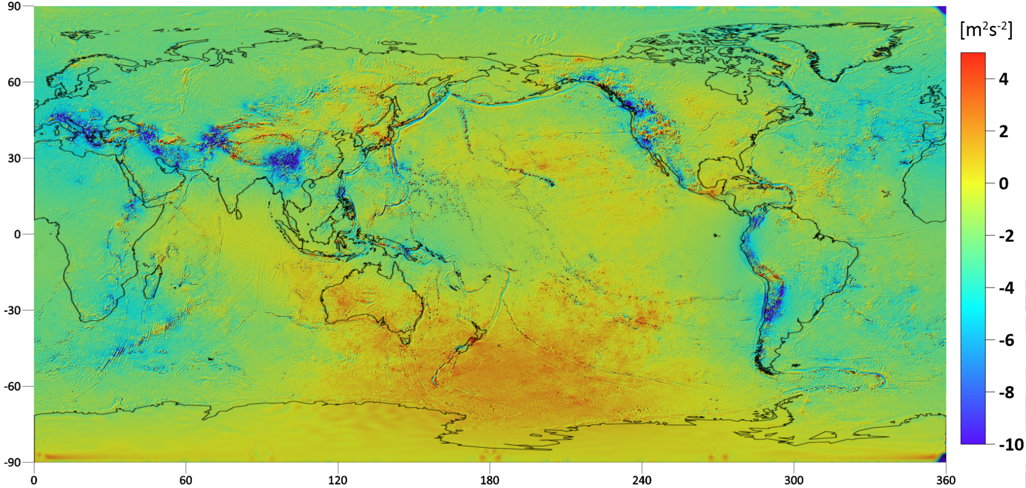
Global gravity field modelling



Global gravity field modelling



Global gravity field modelling



Conclusions

- We have presented an numerical approach for solving the fixed gravimetric boundary value problem with an oblique derivative (FGBVP) based on the finite element method (FEM) with mapped infinite elements.
- Reconstruction of EGM2008 as a harmonic function has shown that with a sufficient refinement of the discretization we are able to achieve high accuracy, even on such extremely complicated Earth's surface.