Non-homogeneous rogue wave probability over a shoal

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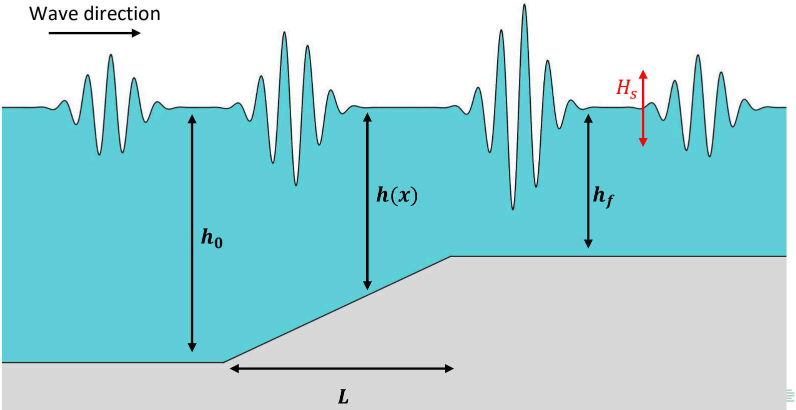
Non-Gaussian Statistics Paradox

- Rogue waves are less common for $kh \lesssim 1$ [Glukhovskii, 1966]
- RWs are strongly amplified within $kh \in [0.5, 1.5]$ [Raustøl, 2014]
- Is it possible to connect a PDE solution to probabilities?



Bathymetry effect on the probability distribution

- Initially Rayleigh distribution over a flat bottom:

$$\mathcal{R}_{\alpha}(H > \alpha H_s) = e^{-2\alpha^2} \quad .$$

- Non-homogeneous spectral analysis [Mendes et al., 2022]:

$$\Gamma(x) = \frac{\mathbb{E}[\zeta^2(x,t)](x)}{\mathscr{E}(x)} \approx \frac{\langle \zeta^2 \rangle_t(x)}{\mathscr{E}(x)} \quad .$$

- Evolving distribution over a shoal (universal amplification):

$$\mathcal{R}_{\alpha,\Gamma}(H > \alpha H_s) = e^{-2\alpha^2/\mathfrak{S}^2\Gamma} \quad \text{or} \quad \frac{\mathbb{P}_{\alpha\,,\Gamma}}{\mathbb{P}_{\alpha}} \approx e^{2\alpha^2\left(1-\frac{1}{\mathfrak{S}^2\Gamma}\right)} \; ,$$

where $1 \leqslant \mathfrak{S} \leqslant 2$ is the vertical asymmetry between crest height and trough depths.

Deriving probability distributions

- Generalized velocity potential Φ and surface elevation $\zeta(x,t)$:

$$\Phi = \sum_m \frac{\Omega_m(kh)}{mk} \cosh(m\varphi) \sin(m\phi) \, ; \, \zeta = \sum_m \tilde{\Omega}_m(kh) \cos(m\phi) \, .$$

- Energy computation [Dean and Dalrymple, 1984]:

$$\mathscr{E}(x) = \frac{1}{2g\lambda} \int_x^{x+\lambda} \left[g \left(\zeta + h \right)^2 + \int_{-h(x)}^{\zeta} \left(u_x^2 + u_z^2 \right) dz \right] dx - \frac{1}{2} h^2 \, .$$

- Airy's solution has ($m = 1$):

$$\mathscr{E}_{(1)} = \frac{1}{4}\left[a^2 + \left(\frac{agk}{\omega \cosh kh} \right)^2 \cdot \frac{\sinh(2kh)}{2gk} \right] = \frac{a^2}{2} \quad .$$

- Stokes' second-order solution ($m = 2$):

$$\mathscr{E}_{(2)} = \frac{a^2}{4}\left\{ 2 + \left(\frac{\pi\varepsilon}{4} \right)^2 \chi_1 + \left(\frac{\pi\varepsilon}{4} \right)^2 \tilde{\chi}_1 \right\} .$$

- Second-order non-homogeneous correction:

$$\Gamma = \frac{1 + \frac{\pi^2\varepsilon^2\mathfrak{S}^2\tilde{\chi}_1}{16}}{1 + \frac{\pi^2\varepsilon^2\mathfrak{S}^2(\tilde{\chi}_1+\chi_1)}{32}} \, , \, \tilde{\chi}_1 = \left(\frac{3 - \tanh^2(kh)}{\tanh^3(kh)} \right)^2 , \chi_1 = \frac{9\cosh(2kh)}{\sinh^6(kh)}$$

Deriving probability distributions - Correction

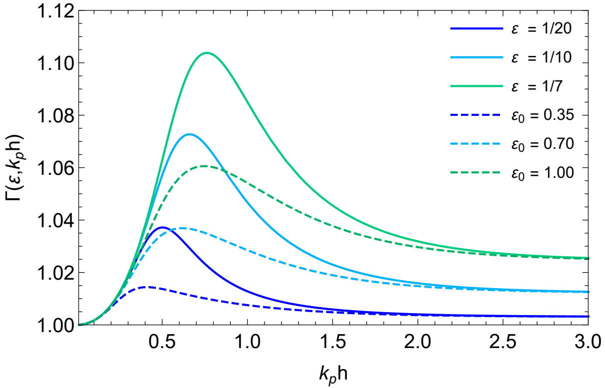
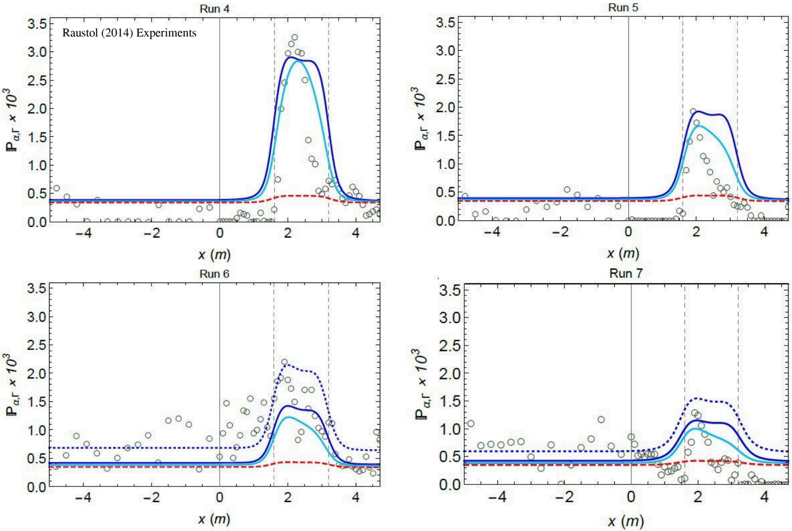


Figure: Γ correction for steep slopes $|\nabla h| \geqslant 1/20$ with the same initial steepness in deep water with (dashed) or without wave breaking (solid). Here we set $\varepsilon \leqslant (\varepsilon_0/7) \tanh kh$ with $0 \leqslant \varepsilon_0 \leqslant 1$ for the breaking condition.

Probability distributions (*Mendes et al. 2022, J. Fluid Mech., 939:A25*)



3. Summary

- 3.0 Connection between the PDE solution and Probability
- 3.1 Challenging homogeneity assumption changes statistics
- 3.2 Airy-Stokes transition universality
- 3.3 Slope constraint $1/20 \leqslant \partial h(x)/\partial x \ll 1$
- 3.4 Model depends on steepness ε and depth $k_p h$
- 3.6 Vertical asymmetry amplifies this effect ($\mathfrak{S}_0 > 1$)
- 3.7 Model fits well most experiments
- 3.8 Absence of free parameters
- 3.9 Amplification mostly at $0.5 \leqslant k_p h \leqslant 1.3$

Thank you!

R. Dean and R. Dalrymple. Water wave mechanics for engineers and scientists. *World Scientific*, 1984.

B. Glukhovskii. Investigation of sea wind waves (in russian). *Gidrometeoizdat*, 1966.

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A. Raustøl. Freake bølger over variabelt dyp. *Master's thesis, University of Oslo*, 2014.