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Inertia-gravity wave diffusion by geostrophic turbulence: the impact of flow time dependence

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Action diffusion equation in wavevector space

Atmospheric and oceanic IGWs propagate in approximately geostrophic turbulent flow. As a result, they are scattered and their energy is redistributed in wavevector space. For IGWs in a random geostrophic flow of much larger spatial scales, wave-action evolves according to the diffusion equation

$$\partial_t a + \boldsymbol{c} \cdot \nabla_{\boldsymbol{x}} a = \nabla_{\boldsymbol{k}} \cdot (\mathbf{D} \cdot \nabla_{\boldsymbol{k}} a) + F, \tag{1}$$

with group velocity $\boldsymbol{c} = \nabla_{\boldsymbol{k}}\omega$, diffusivity $\mathbf{D}$ and forcing F [see 3; 4]. The intrinsic wave frequency – dependent on the Coriolis (f) and buoyancy (N) frequencies, and the angle θ between the wavevector and the vertical – is given by

$$\omega = (f^2\cos^2\theta + N^2\sin^2\theta)^{1/2}. \tag{2}$$

For time-independent flow, the diffusivity has two non-zero components in $\boldsymbol{k}$ space spherical polar coordinates (k, ϕ, θ),

$$\mathsf{D}_{kk} = \boldsymbol{e}_k \cdot \mathbf{D} \cdot \boldsymbol{e}_k \quad \text{and} \quad \mathsf{D}_{\phi\phi} = \boldsymbol{e}_\phi \cdot \mathbf{D} \cdot \boldsymbol{e}_\phi, \tag{3}$$

with explicit forms given in [3] and computed via the flow kinetic energy spectrum. The constant frequency surfaces in wavevector space are cones. As $\boldsymbol{e}_\theta \cdot \mathbf{D} = 0$, no diffusion occurs in the θ-direction across these cones. Numerical simulation of the Boussinesq equations confirms this holds approximately.

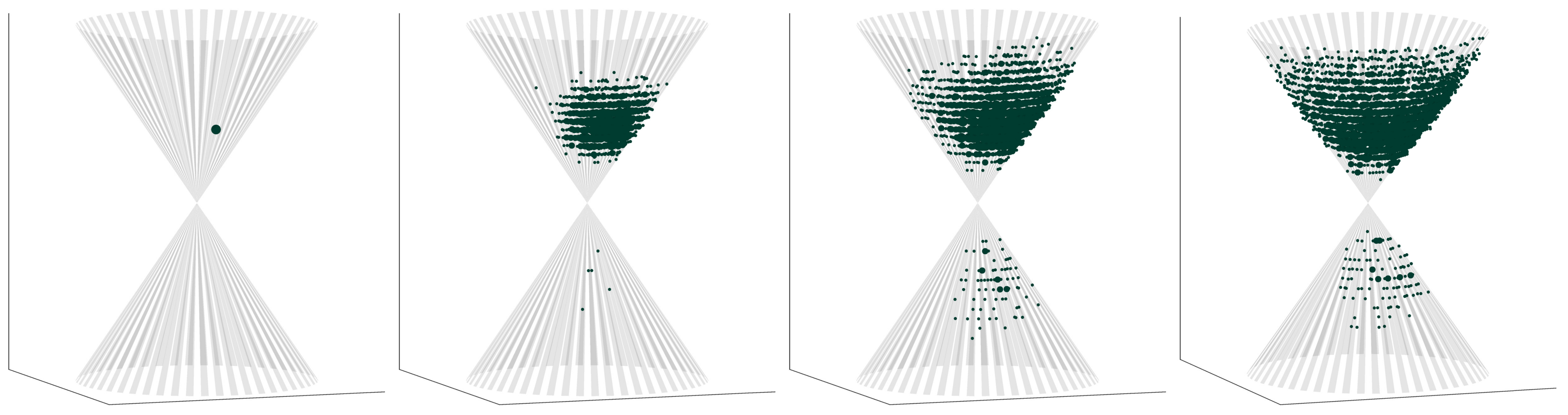

Figure 1. Boussinesq simulation of small-Rossby number, quasigeostrophic flow with energy packet initialised at a single point in wavevector space.

Time-dependent flow

Slow flow time dependence causes diffusion of wave action across the cone of constant frequency and it has been suggested that this results in significant wave action transfer from low to high frequency [2]. This work addresses the open question:

What is the effect of flow time dependence on IGW diffusion?

Diffusivity with time-dependent flow

Using that the ratio ϵ between IGW and flow time scales is small, we have derived an approximation for the diffusivity,

$$\mathbf{D} = \mathbf{D}^{(0)} + \epsilon^2 \mathbf{D}^{(1)} + O(\epsilon^4), \tag{4}$$

where $\mathbf{D}^{(0)}$ is the diffusivity with time-independent flow and $\mathbf{D}^{(1)}$ the correction term. This correction has one significant component,

$$\mathsf{D}^{(1)}_{\theta\theta} = \frac{4\omega^3 k^5}{(N^2 - f^2)^3 |\cos^5\theta|} \int_0^\infty \int_\theta^{\pi-\theta} K(\cot^2\theta - \cot^2\Theta)^{1/2} A(K, \theta)\, \mathrm{d}K\, \mathrm{d}\Theta \tag{5}$$

where the lowercase symbols (k, ϕ, θ) parameterise the IGW wavevector $\boldsymbol{k}$ and the uppercase symbols (K, Φ, Θ) parameterise the flow wavevector $\boldsymbol{K}$, with $A(\boldsymbol{K})$ the flow acceleration spectrum. (We assume horizontal isotropy such that $A(\boldsymbol{K}) = A(K, \theta)$.)

This correction corresponds to across-cone diffusion.

Equilibrium spectrum

We look for a local steady state solution to the diffusion equation (1) with time-dependent flow and

- forcing of the form $\delta(k - k_*)\delta(\theta - \theta_*)$ (which can be generalised to any k and θ-dependent forcing),
- horizontally isotropic flow,
- wave statistics independent of ϕ such that $\partial_\phi a = 0$.

Under these simplifications, the wave energy density – a more easily interpreted quantity than action – takes the exact (scaled) form

$$e(k, \sigma) = \frac{1}{2k_*^{5/2} k^{1/2}} \mathrm{Q}_{3/2}\left(\frac{k_*^2 + k^2 + \sigma^2}{2k_* k}\right) \tag{6}$$

where Q is the Legendre function of the second kind and

$$\sigma = \frac{\theta - \theta_*}{\epsilon} = O(1), \tag{7}$$

quantifies the small angular deviation from the forcing frequency θ_*. This is the main result of our work.

Equilibrium spectrum cont.

$\mathrm{Q}_{3/2}$ decays rapidly as its argument increases (see figures 2 and 4).

Crucially, energy is localised within an $O(\epsilon)$ layer around the constant frequency cone $\theta = \theta_*$.

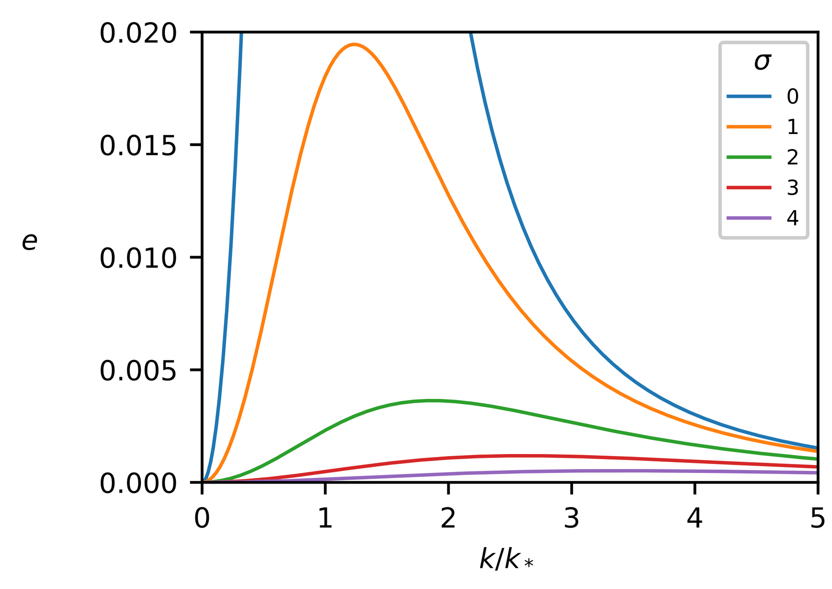

Figure 2. Scaled energy distribution $e(k, \sigma)$ against k/k_* for different σ values.

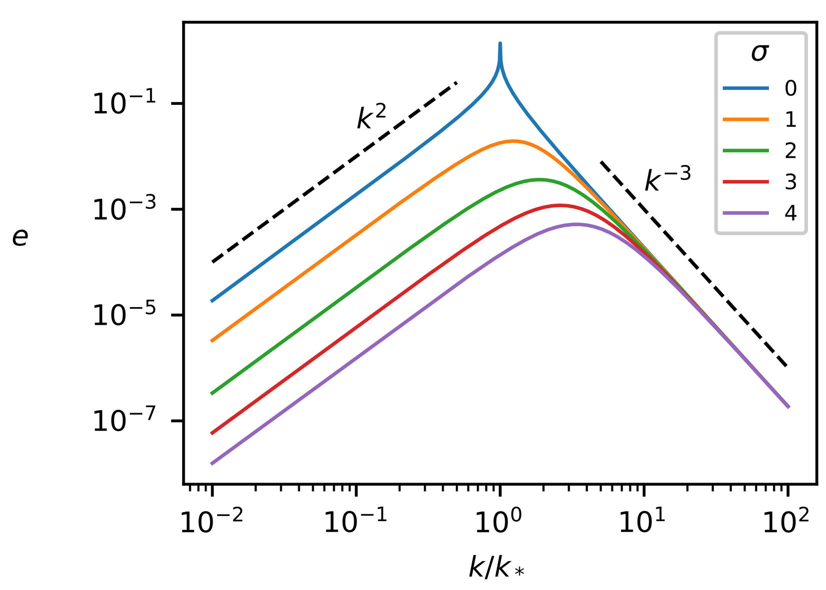

Figure 3. Log-log plot of figure 2.

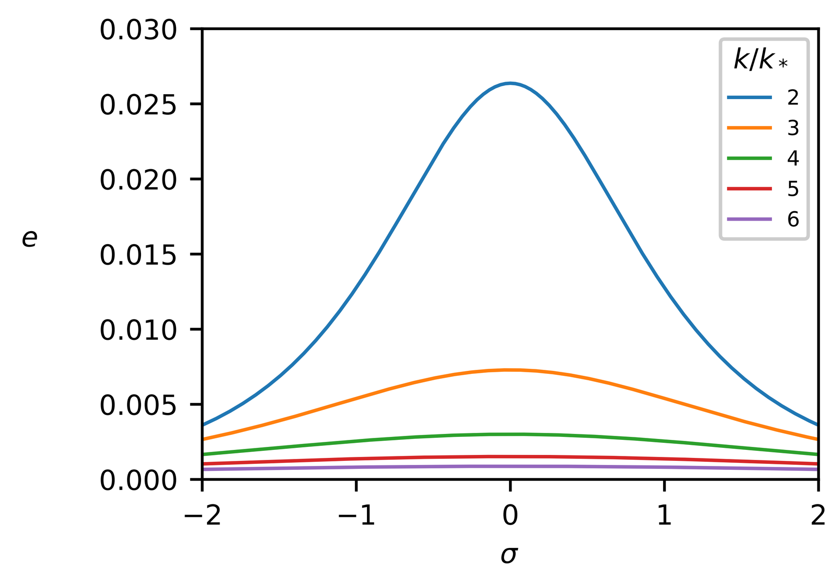

Figure 4. Scaled energy distribution $e(k, \sigma)$ against σ for different values of k/k_*.

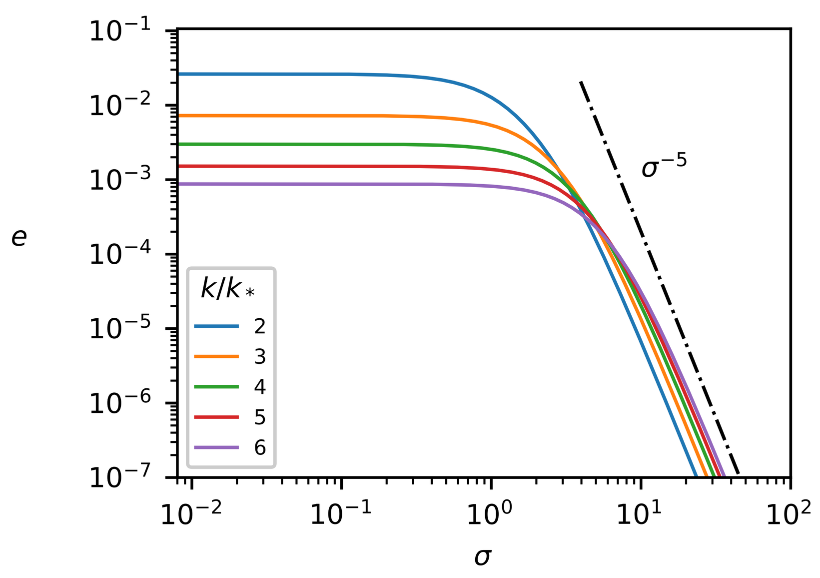

Figure 5. Log-log plot of figure 4.

Comparison with a Boussinesq simulation

Power laws of e with respect to k and σ come from the large argument limit of the Legendre function,

$$e(k, \sigma) \sim \frac{3}{16} k^{-3} \left(1 + \frac{k_*^2 + \sigma^2}{k^2}\right)^{-5/2}, \tag{8}$$

[see 1]. Therefore,

$$e(k, \sigma) \propto \sigma^{-5} \quad \text{as} \quad \sigma \to \infty, \tag{9}$$

$$e(k, \sigma) \propto k^{-3} \quad \text{as} \quad k \to \infty, \tag{10}$$

$$e(k, \sigma) \propto k^{2} \quad \text{as} \quad k \to 0. \tag{11}$$

The σ limit (9) characterises the angular localisation of the energy in wavevector space and appears in the exact solution plot, figure 5, and the Boussinesq simulation, figure 7. Both k limits (10) and (11) can be seen in figure 3 and the Boussinesq simulation, figure 6.

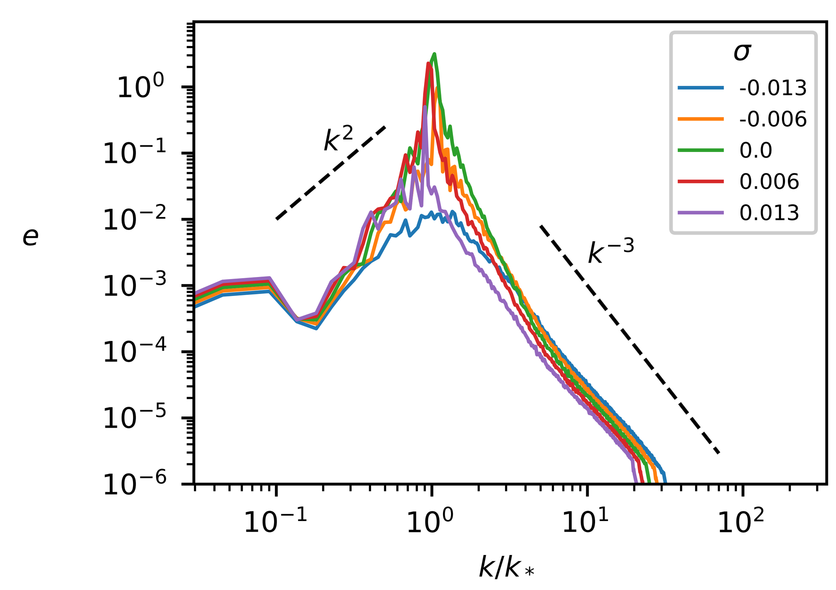

Figure 6. Log-log plot of energy $e(k, \sigma)$ against k/k_* for different σ values from a low-Rossby number quasigeostrophic flow Boussinesq simulation.

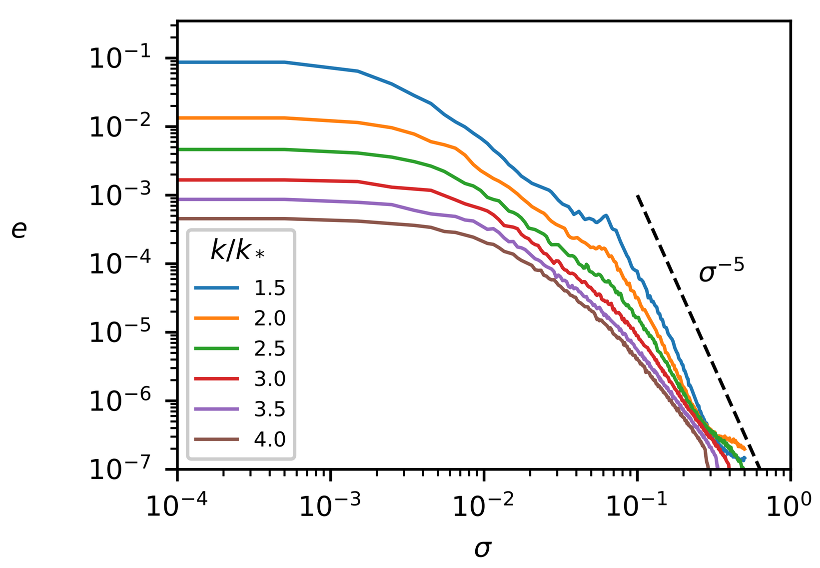

Figure 7. As in figure 6 but with $e(k, \sigma)$ against σ for different k/k_* values.

References

[1] *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.1.4 of 2022-01-15, 2022. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

[2] W. Dong, O. Bühler, and K. S. Smith, *Frequency diffusion of waves by unsteady flows*, J. Fluid Mech., 905 (2020), p. R3.

[3] H. A. Kafiabad, M. A. C. Savva, and J. Vanneste, *Diffusion of inertia-gravity waves by geostrophic turbulence*, J. Fluid Mech., 869 (2019), p. R7.

[4] C. H. McComas and F. P. Bretherton, *Resonant interaction of oceanic internal waves*, J. Geophys. Res., 82 (1977), p. 1397–1412.

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Live presentation slides

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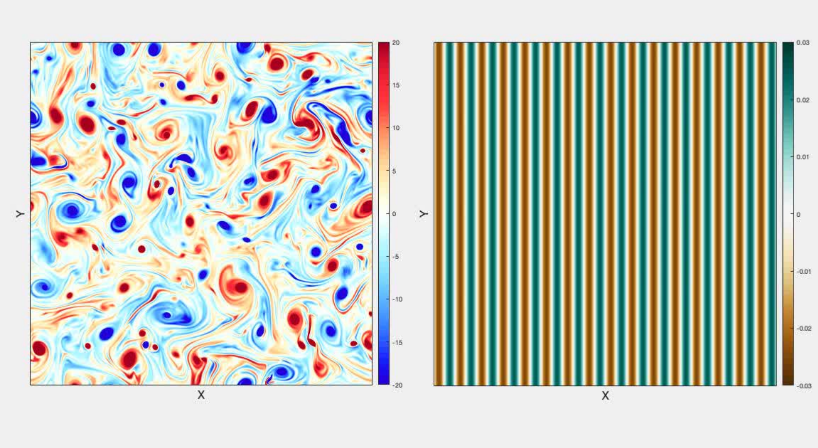
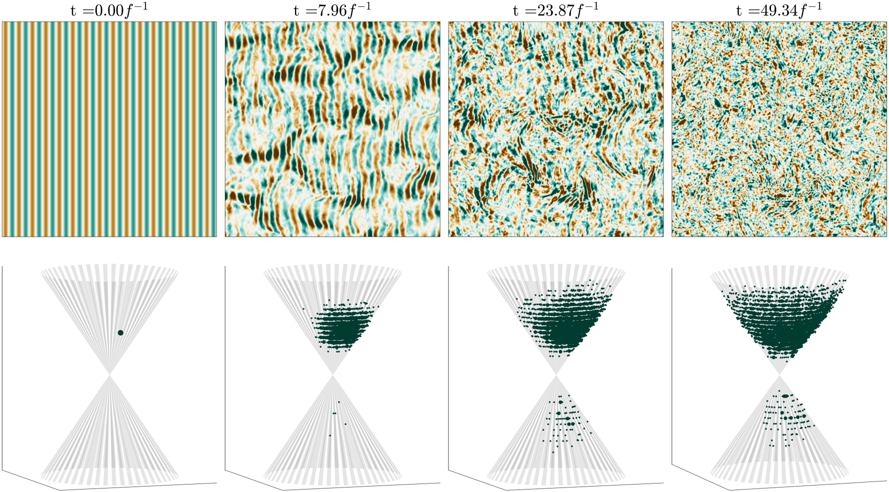
IGW scattering by geostrophic turbulence



IGW scattering by geostrophic turbulence



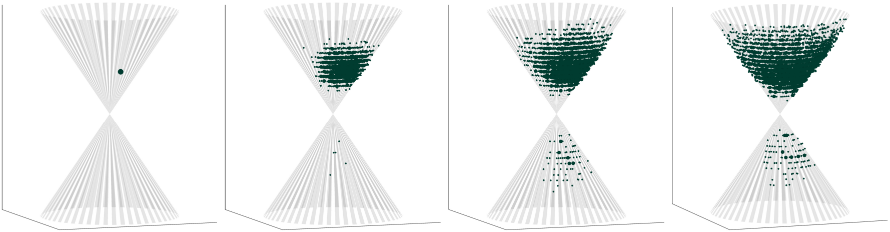
Diffusion approximation for time-independent flow

$$\partial_t a + \boldsymbol{c} \cdot \nabla_{\boldsymbol{x}} a = \nabla_{\boldsymbol{k}} \cdot (\mathbf{D} \cdot \nabla_{\boldsymbol{k}} a) + F \tag{1}$$

Assume flow is time-independent

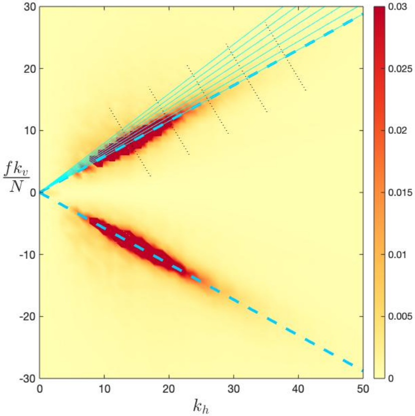
$\Downarrow$

$$\boldsymbol{e}_\theta \cdot \mathbf{D} = 0 \tag{2}$$



Equilibrium spectrum with time-dependent flow

$$\boldsymbol{e}_\theta \cdot \mathbf{D} = O(\epsilon^2) \neq 0, \quad \text{where} \quad \epsilon = \frac{\text{flow frequency}}{\text{wave frequency}} \tag{3}$$



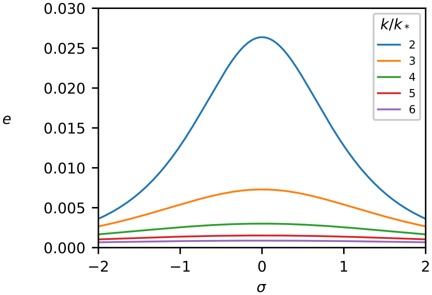
Equilibrium spectrum with time-dependent flow

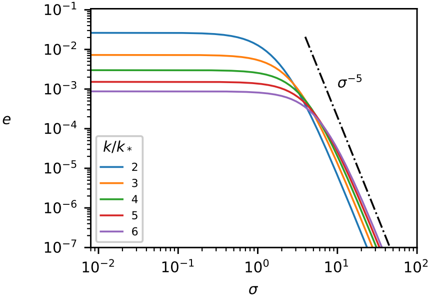
Forced, steady state wave energy density

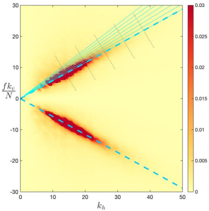
$$\boxed{e(k,\sigma) = \frac{1}{2k_*^{5/2}k^{1/2}} \mathrm{Q}_{3/2}\left(\frac{k_*^2 + k^2 + \sigma^2}{2k_* k}\right)} \tag{4}$$

$$\sigma = \frac{\theta - \theta_*}{\epsilon} = O(1) \tag{5}$$

Equilibrium spectrum with time-dependent flow

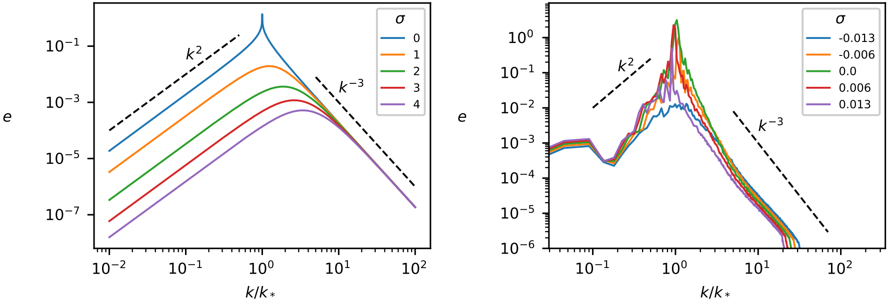






Energy is localised within an $O(\epsilon)$ layer around the constant frequency cone $\theta = \theta_*$.

Comparison to a Boussinesq simulation



See poster online for more detail